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SUM OF SMALL ARITHMETIC FUNCTIONS IN A FLOOR FUNCTION SET

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ABSTRACT. In this paper, we derive asymptotic formulas for the summation of small arithmetic functions over a floor function set. Our results are obtained under assumptions on the size of the floor function set and its alignment with a given arithmetic progression.

Keywords. Floor function set, Small arithmetic function
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1. INTRODUCTION AND RESULTS

As usual, denote by $\tau(n)$ the divisor function, $\mu(n)$ the Möbius function, $\sigma(n)$ the sum-of-divisors function, $\beta(n)$ the alternative sum-of-divisors function, $\phi(n)$ the Euler totient function, $\Omega(n)$ the number of prime factors of n and $\Psi(n)$ the Dedekind function. Let $\lfloor t \rfloor$ be the integral part of the real number t. The study of sums of the form, for an arithmetic function f,

$$\sum_{n\le x} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right), \tag{1.1}$$

is an intensive problem. It was first studied in [1] with f satisfying various conditions. Specifically, Bordellés-Dai-Heyman-Pan-Shparlinski [1, Corollary 2.4] showed that

$$\sum_{n\le x} \frac{\phi\left(\left\lfloor \frac{x}{n}\right\rfloor\right)}{\left\lfloor \frac{x}{n}\right\rfloor} = x\sum_{n=1}^{\infty}\frac{\phi(n)}{n^2(n+1)} + O(x^{1/2}). \tag{1.2}$$

Later, Wu [5] improved the error term in (1.2) to $O(x^{1/3}\log x)$. Subsequently, Stucky [4] generalized Wu's results by letting a class of functions f, when f is "close to one" in a suitable sense. In this approach, Stucky proved that

$$\sum_{n\le x} f\left(\left\lfloor \frac{x}{n}\right\rfloor\right) = x\sum_{n=1}^{\infty}\frac{f(n)}{n(n+1)} + O(x^{(1+\alpha)/(3-\alpha)}(\log x)^{\theta}), \tag{1.3}$$

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where $f(n)=\sum_{d|n} g(d)$ such that $\sum_{d\le x}|g(d)| \ll x^{\alpha}(\log x)^{\theta}$, for some $\alpha\in[0,1)$, $\theta\ge 0$ and the implied constant depends only on α. In the case $\alpha=0$, we can replace θ by $\max(1,\theta)$. Utilizing the following well-known results

$$\frac{\phi(n)}{n}=\sum_{d|n}\frac{\mu(d)}{d} \quad \text{and} \quad \sum_{d\le x}\left|\frac{\mu(d)}{d}\right| \le \sum_{d\le x}\frac{1}{d} \ll x^{\epsilon},$$

in (1.3), Stucky enabled to show that

$$\sum_{n\le x}\frac{\phi\left(\left\lfloor \frac{x}{n}\right\rfloor\right)}{\left\lfloor \frac{x}{n}\right\rfloor}=x\sum_{n=1}^{\infty}\frac{\phi(n)}{n^2(n+1)}+O(x^{1/3+\epsilon}).$$

Recently, Heyman [2] introduced a set $S(x):=\{\lfloor \frac{x}{n}\rfloor \ : 1\le n\le x\}$ called a floor function set and also studied the number of primes in such a set. Heyman proved that, as $x\to\infty$,

$$\sum_{\substack{p\le x\\ \exists\, n\in\mathbb{N} \text{ such that } \lfloor \frac{x}{n}\rfloor=p}} 1=\frac{4x^{1/2}}{\log x}+O\Big(\frac{x^{1/2}}{(\log x)^2}\Big). \tag{1.4}$$

Very recently, Rong and Wu [3] gave a better result of (1.4). They showed that, as $x\to\infty$,

$$\sum_{\substack{p\le x\\ \exists\, n\in\mathbb{N} \text{ such that } \lfloor \frac{x}{n}\rfloor=p}} 1=\int_2^{x^{1/2}}\frac{dt}{\log t}+\int_2^{x^{1/2}}\frac{dt}{\log(x/t)}+O\Big(\frac{x^{1/2}\exp(-c(\log x)^{3/5}}{(\log\log x)^{1/5})}\Big),$$

where $c>0$ is a positive constant. These results are rather interesting since $S(x)$ is a very sparse subset of $[1,x]\cap\mathbb{N}$. It seems natural and fascinating to study an analogue of (1.3). Namely, we shall derive the following sum

$$T_f(x):=\sum_{n\le x}\mathbb{1}_{S(x)}(n)f(n),$$

where $\mathbb{1}_{S(x)}$ is the characteristic function of the set $S(x)$ and

$$f(n)=\sum_{d|n} g(d) \text{ such that } \sum_{d\le x}|g(d)| \ll x^{\alpha}(\log x)^{\theta}, \tag{1.5}$$

for some $\alpha\in[0,1)$ and $\theta\ge 0$. Interestingly, Heyman [2] considered the set $S_d(x):=\{m\in S(x): d\mid m\}$ for $d\le x$ and assumed that

$$|S_d(x)|\sim\frac{2x^{\frac{1}{2}}}{d}. \tag{1.6}$$

Under the assumption (1.6), we can establish the next theorem.

Theorem 1.1. *Under the assumption* (1.6), *we have*

$$T_f(x)=2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d}+O\Big(x^{(\alpha+1)/4}(\log x)^{3/2-3\alpha/2+\theta}\Big),$$

where

$$f(n) = \sum_{d|n} g(d) \text{ such that } \sum_{d\le x} |g(d)| \ll x^{\alpha}(\log x)^{\theta},$$

for some $\alpha \in [0,1)$ *and* $\theta \ge 0$.

Using the following well-known relations:

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}, \qquad \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d},$$

$$\frac{\beta(n)}{n} = \sum_{d|n} \frac{(-1)^{\Omega(d)}}{d}, \qquad \frac{\Psi(n)}{n} = \sum_{d|n} \frac{\mu^2(d)}{d}$$

with Theorem 1.1, we obtain the following results.

Corollary 1.2. *Under the assumption* (1.6), *we have*

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)\frac{\sigma(n)}{n} = 2\zeta(2)x^{1/2} + O\left(F(x)\right), \tag{1.7}$$

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)\frac{\phi(n)}{n} = \frac{2}{\zeta(2)}x^{1/2} + O\left(F(x)\right), \tag{1.8}$$

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)\frac{\beta(n)}{n} = \frac{2\zeta(4)}{\zeta(2)}x^{1/2} + O\left(F(x)\right), \tag{1.9}$$

and

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)\frac{\Psi(n)}{n} = \frac{2\zeta(2)}{\zeta(4)}x^{1/2} + O\left(F(x)\right), \tag{1.10}$$

where $F(x) = x^{1/4}(\log x)^{5/2}$.

Proof. To prove (1.7)-(1.10), we set $f(n) = \frac{\sigma(n)}{n}, \frac{\phi(n)}{n}, \frac{\beta(n)}{n}$ and $\frac{\Psi(n)}{n}$, respectively. Then, we have $g(n) = \frac{1}{n}, \frac{\mu(n)}{n}, \frac{(-1)^{\Omega(n)}}{n}$ and $\frac{\mu^2(n)}{n}$, respectively and $\sum_{n\le x}|g(n)| \ll \log x$. Thus, $\alpha = 0$ and $\theta = 1$. Then, the main term of (1.7)-(1.10) follows from the following Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2), \qquad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)},$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{\Omega(n)}}{n^2} = \frac{\zeta(4)}{\zeta(2)}, \qquad \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^2} = \frac{\zeta(2)}{\zeta(4)}.$$

□

Moreover, we will extend this problem to include additional classes of arithmetic functions. In 2019, the following three categories of assumptions on g were presented by the authors in [1]. Namely,

$$\textit{Type I:}\quad |g(n)| \ll \tau_k(n) \quad (n \in \mathbb{N}), \tag{1.11}$$

$$\textit{Type II:}\quad |g(n)| \ll n^{\varphi-1}(\log(en))^{-A} \quad (n \in \mathbb{N}, \varphi = \tfrac{1+\sqrt{5}}{2}, A > 0), \tag{1.12}$$

$$\textit{Type III:}\quad \sum_{n\le x} |g(n)|^2 \ll x^{\theta} \quad (x \ge 1, n \in \mathbb{N}, 0 < \theta < 2). \tag{1.13}$$

Here, the asymptotic behaviour of a multiplicative function

$$h(n) := \sum_{d^2|n} g(d)$$

will be examined where $g(d)$ is a multiplicative function satisfying (1.11)–(1.13) in the floor function set $S(x)$.

Theorem 1.3. *Let g be a multiplicative function satisfying (1.11). Under the assumption (1.6), we obtain*

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)h(n) = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(x^{3/8}(\log x)^{k-1/4}\Big).$$

Theorem 1.4. *Let φ be the Golden Ratio and g a multiplicative function satisfying (1.12). Under the assumption (1.6), we get*

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)h(n) = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(x^{1/4+\varphi/8}(\log x)^{(6-3\varphi)/4}\Big).$$

Theorem 1.5. *Let g be a multiplicative function satisfying (1.13). Under the assumption (1.6), we get*

$$\sum_{n\le x} \mathbb{1}_{S(x)}(n)h(n) = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(x^{5/16+\theta/16}\log^{9/8-3\theta/8} x\Big).$$

The proof of Theorems makes use of the following estimate, due originally to [6].

Lemma 1.6 ([6])**.** *Let q be an integer and x a positive real number such that $0 < q \le x^{1/4}\log^{-3/2} x$. Then*

$$\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ q)}} \mathbb{1}_{S(x)}(n) = \frac{2x^{1/2}}{q} + O\Big(\frac{x^{1/3}}{q^{1/3}}\log x\Big).$$

2. Proofs

Proof of Theorem 1.1. For $x > 1$, we have

$$\begin{aligned}
T_f(x) &= \sum_{n \le x} \mathbb{1}_{S(x)}(n) f(n) \\
&= \sum_{n \le x} \mathbb{1}_{S(x)}(n) \sum_{d \mid n} g(d) \\
&= \sum_{d \le x} g(d) \sum_{\substack{n \le x \\ n \equiv 0 \, (\mathrm{mod}\ d)}} \mathbb{1}_{S(x)}(n) \\
&= \sum_{d \le x^{1/4} \log^{-3/2} x} g(d) \sum_{\substack{n \le x \\ n \equiv 0 \, (\mathrm{mod}\ d)}} \mathbb{1}_{S(x)}(n) \\
&\quad + \sum_{x^{1/4} \log^{-3/2} x < d \le x} g(d) \sum_{\substack{n \le x \\ n \equiv 0 \, (\mathrm{mod}\ d)}} \mathbb{1}_{S(x)}(n). \qquad (2.1)
\end{aligned}$$

We employ Lemma 1.6 to evaluate the first sum in (2.1). It follows that

$$\begin{aligned}
&\sum_{d \le x^{1/4} \log^{-3/2} x} g(d) \sum_{\substack{n \le x \\ n \equiv 0 \, (\mathrm{mod}\ d)}} \mathbb{1}_{S(x)}(n) \\
&\qquad = \sum_{d \le x^{1/4} \log^{-3/2} x} g(d) \Big(\frac{2x^{1/2}}{d} + O\Big(\frac{x^{1/3}}{d^{1/3}} \log x \Big) \Big) \\
&\qquad = 2x^{1/2} \sum_{d \le x^{1/4} \log^{-3/2} x} \frac{g(d)}{d} \\
&\qquad\quad + O\Big(x^{1/3} \log x \sum_{d \le x^{1/4} \log^{-3/2} x} \frac{|g(d)|}{d^{1/3}} \Big) \\
&\qquad =: \Sigma_1 + \Sigma_2.
\end{aligned}$$

From the assumption (1.5), we have the series $\sum_{d=1}^{\infty} \frac{g(d)}{d^s}$, is absolutely convergent for $\Re(s) > \alpha$. Then by the partial summation and (1.5), we have

$$\begin{aligned}
\sum_{d \le x^{1/4} \log^{-3/2} x} \frac{g(d)}{d} &= \sum_{d=1}^{\infty} \frac{g(d)}{d} - \sum_{d > x^{1/4} \log^{-3/2} x} \frac{g(d)}{d} \\
&= \sum_{d=1}^{\infty} \frac{g(d)}{d} + O\Big(\sum_{d > x^{1/4} \log^{-3/2} x} \frac{|g(d)|}{d} \Big) \\
&= \sum_{d=1}^{\infty} \frac{g(d)}{d} + O\Big(x^{(\alpha-1)/4} (\log x)^{3/2 - 3\alpha/2 + \theta} \Big).
\end{aligned}$$

Thus, we have

$$\Sigma_1 = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d} + O\Big(x^{(\alpha+1)/4}(\log x)^{3/2-3\alpha/2+\theta}\Big).$$

Using (1.5) and the partial summation again, we have

$$\sum_{d\le x^{1/4}\log^{-3/2}x}\frac{|g(d)|}{d^{1/3}} \ll x^{-1/12+\alpha/4}(\log x)^{1/2-3\alpha/2+\theta}.$$

Thus, we have

$$\Sigma_2 = O\Big(x^{(\alpha+1)/4}(\log x)^{3/2-3\alpha/2+\theta}\Big).$$

Then,

$$\begin{aligned}\sum_{d\le x^{1/4}\log^{-3/2}x} g(d)\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ d)}}\mathbb{1}_{S(x)}(n)\\ &= 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d} + O\Big(x^{(\alpha+1)/4}(\log x)^{3/2-3\alpha/2+\theta}\Big). \qquad (2.2)\end{aligned}$$

Next, the bound of the second sum is determined. Under the assumption (1.6), we have

$$\sum_{x^{1/4}\log^{-3/2}x<d\le x} g(d)\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ d)}}\mathbb{1}_{S(x)}(n) = O\Big(x^{1/2}\sum_{x^{1/4}\log^{-3/2}x<d\le x}\frac{|g(d)|}{d}\Big).$$

Then by the partial summation and (1.5), we have

$$x^{1/2}\sum_{x^{1/4}\log^{-3/2}x<d\le x}\frac{|g(d)|}{d} \ll x^{(\alpha+1)/4}(\log x)^{3/2-3\alpha/2+\theta}. \qquad (2.3)$$

Then, the proof follows from (2.2) and (2.3). □

Proof of Theorem 1.3. For $x>1$, it follows that

$$\begin{aligned}\sum_{n\le x}\mathbb{1}_{S(x)}(n)h(n) &= \sum_{n\le x}\mathbb{1}_{S(x)}(n)\sum_{d^2|n}g(d)\\ &= \sum_{d\le x^{1/2}} g(d)\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ d^2)}}\mathbb{1}_{S(x)}(n)\\ &= \sum_{d\le x^{1/8}\log^{-3/4}x} g(d)\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ d^2)}}\mathbb{1}_{S(x)}(n)\\ &\quad + \sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}} g(d)\sum_{\substack{n\le x\\ n\equiv 0\ (\mathrm{mod}\ d^2)}}\mathbb{1}_{S(x)}(n)\\ &=: A_1 + A_2. \qquad (2.4)\end{aligned}$$

Utilizing Lemma 1.6, we get

$$\begin{aligned} A_1 &= \sum_{d\le x^{1/8}\log^{-3/4}x} g(d)\Big(\frac{2x^{1/2}}{d^2}+O\Big(\frac{x^{1/3}}{d^{2/3}}\log x\Big)\Big) \\ &= 2x^{1/2}\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{g(d)}{d^2}+O\Big(x^{1/3}\log x\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^{2/3}}\Big). \end{aligned} \tag{2.5}$$

In view of the hypothesis (1.11) and the well-known formula

$$\sum_{n\le x}\tau_k(n)\ll x\log^{k-1}x, \tag{2.6}$$

by the partial summation, we have

$$\sum_{d>x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^2}\ll\sum_{d>x^{1/8}\log^{-3/4}x}\frac{\tau_k(d)}{d^2}\ll\frac{\log^{k-1}x}{x^{1/8}\log^{-3/4}x}=\frac{\log^{k-1/4}x}{x^{1/8}}.$$

Thus,

$$\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{g(d)}{d^2}=\sum_{d=1}^{\infty}\frac{g(d)}{d^2}+O\Big(\frac{\log^{k-1/4}x}{x^{1/8}}\Big). \tag{2.7}$$

To bound the error term in (2.5), we use again (2.6) and the partial summation, we have

$$x^{1/3}\log x\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^{2/3}}\ll x^{3/8}\log^{k-1/4}x. \tag{2.8}$$

In view of (2.7) and (2.8), we have

$$A_1=2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2}+O\Big(x^{3/8}\log^{k-1/4}x\Big). \tag{2.9}$$

Next, we bound the second sum A_2. Under the assumption (1.6), we have

$$A_2\ll x^{1/2}\sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}}\frac{|g(d)|}{d^2}\ll x^{3/8}\log^{k-1/4}x. \tag{2.10}$$

Inserting (2.9) and (2.10) in to (2.4), Theorem 1.3 follows. □

Proof of Theorem 1.4. By using the similar proof of Theorem 1.3, we have

$$\begin{aligned} \sum_{n\le x}\mathbb{1}_{S(x)}(n)h(n) = 2x^{1/2}&\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{g(d)}{d^2}+O\Big(x^{1/3}\log x\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^{2/3}}\Big) \\ &+\sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}}g(d)\sum_{\substack{n\le x\\ n\equiv 0 \pmod{d^2}}}\mathbb{1}_{S(x)}(n). \end{aligned}$$

Let φ be the Golden Ratio. Since $|g(n)|\ll n^{\varphi-1}(\log(en))^{-A}$, we have

$$\sum_{d\le x}|g(d)|\ll x^{\varphi}\quad\text{and}\quad\sum_{d>x}\frac{|g(d)|}{d^2}\ll x^{\varphi-2}.$$

Thus,

$$\sum_{d\le x^{1/8}\log^{-3/4}x} g(d)\frac{2x^{1/2}}{d^2} = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(x^{1/4+\varphi/8}\log^{3/2-3\varphi/4}x\Big). \tag{2.11}$$

Using again the partial summation with the hypothesis (1.12), we have

$$\begin{aligned} x^{1/3}\log x \sum_{d\le x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^{2/3}} &\ll x^{1/3}\log x\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{d^{\varphi-1}}{d^{2/3}} \\ &\ll x^{1/4+\varphi/8}\log^{3/2-3\varphi/4}x, \end{aligned} \tag{2.12}$$

and under the assumption (1.6), we have

$$\begin{aligned} \sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}} g(d)\sum_{\substack{n\le x\\ n\equiv 0 \pmod{d^2}}}\mathbb{1}_{S(x)}(n) &\ll x^{1/2}\sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}}\frac{|g(d)|}{d^2} \\ &\ll x^{1/4+\varphi/8}\log^{3/2-3\varphi/4}x. \end{aligned} \tag{2.13}$$

Then, Theorem 1.4 follows from (2.11)–(2.13). □

Proof of Theorem 1.5. Let $0<\theta<2$. By the Cauchy inequality and the hypothesis (1.13), we have

$$\sum_{d\le x}|g(d)| \ll x^{(1+\theta)/2}. \tag{2.14}$$

By utilizing the partial summation, we get

$$\sum_{d>x}\frac{|g(d)|}{d^2} \ll x^{(\theta-3)/2}. \tag{2.15}$$

In view of (2.15), we have

$$2x^{1/2}\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{g(d)}{d^2} = 2x^{1/2}\sum_{d=1}^{\infty}\frac{g(d)}{d^2} + O\Big(x^{5/16+\theta/16}\log^{9/8-3\theta/8}x\Big). \tag{2.16}$$

Employing again the partial summation with the hypothesis (2.14), we obtain

$$x^{1/3}\log x\sum_{d\le x^{1/8}\log^{-3/4}x}\frac{|g(d)|}{d^{2/3}} \ll x^{5/16+\theta/16}\log^{9/8-3\theta/8}x \tag{2.17}$$

and under the assumption (1.6), we have

$$x^{1/2}\sum_{x^{1/8}\log^{-3/4}x<d\le x^{1/2}}\frac{|g(d)|}{d^2} \ll x^{5/16+\theta/16}\log^{9/8-3\theta/8}x. \tag{2.18}$$

Then, Theorem 1.5 follows from (2.16)–(2.18). □

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