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# ON THE LIMIT RELIABILITY FUNCTIONS OF REGULAR HOMOGENEOUS CONSECUTIVE M-OUT-OF-L SERIES AND PARALLEL SYSTEMS

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ABSTRACT. This paper focuses on analyzing the potential limit reliability functions of consecutive m-out-of-l:G systems arranged in series and parallel configurations. We begin by deriving explicit reliability formulas for these systems and then, investigate their asymptotic behavior using tools from extreme value theory and system reliability analysis.

*Keywords.* Consecutive, series, parallel, limit reliability function
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## 1. Introduction

Improving techniques for computing systems' reliability is one of the main objectives of reliability theory researchers. This could be feasible in cases where a system has a minimal number of simple components. However, as technology and engineering advance at a rapid pace, systems get more complex, making traditional approaches of assuming their reliability ineffective and posing the challenge of the complexity of their reliability functions. This problem seems to be well suited for the extreme value theory, which looks at the behavior of the reliability function when the number of system components' tends to infinity.

Gnedenko's paper [2] was the first to fix some limit reliability functions of series and parallel systems. Chernoff and Teicher [4] provided three classes of limit reliability functions for series-parallel and parallel-series systems. Later, Kolowrocki in [5]-[10] extended these three classes to ten classes. Milczek [3] obtained ten classes for homogeneous k-out-of-n series systems, which are similar to those in [7]. In [11], Kolowrocki summarized all possible limit reliability functions for series, parallel, k-out-of-n, series-parallel, and parallel-series with independent, homogeneous or non-homogeneous components ; besides, concrete applications are discussed. Chichocki [1] generalized the results in [7] to series-parallel and parallel-series systems of higher-order.

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The present work elaborates on this generalization, investigating consecutive m-out-of-l series and parallel systems due to their wide range of applications. As a first example, a multi-segment pipeline monitoring with local redundancy is a practical example of a system composed of r blocks in series, each following a consecutive m-out-of-l:G structure, arises in long-distance oil or gas pipelines. These pipelines are typically divided into r independent monitoring zones corresponding to different geographic or risk-prone segments (e.g., urban areas, river crossings, mountainous terrain). In each zone, l sensors are linearly deployed to detect pressure anomalies, leakage, or structural vibration. To ensure local fault tolerance, each block is considered operational if at least m consecutive sensors are functioning, thus modeling a consecutive m-out-of-l:G system. However, because the safety of the entire pipeline depends on the correct operation of all segments, the global system is modeled as a series of r such independent blocks. The system as a whole fails if any one segment fails to meet its operational threshold. This structure reflects a common engineering principle; local redundancy within segments, combined with global fragility across segments, and is particularly important for safety-critical infrastructure where early detection and continuous monitoring are essential. A redundant communication links in a high-speed rail network is another relevant real-world example of a system composed of r blocks arranged in parallel, each modeled as a consecutive m-out-of-l:G system, can be found in high-speed railway communication networks. Such systems rely on multiple independent communication paths such as fiber-optic lines or wireless channels to relay control signals between central command units and trackside devices (e.g., switches, sensors, and signals). Each path consists of a linear sequence of l communication nodes and is considered operational if at least m consecutive nodes are functioning, there by satisfying the consecutive m-out-of-l:G criterion. The entire communication system is deemed operational if at least one path is operational, forming a parallel configuration. This structure effectively combines local robustness tolerance to partial node failures within a path with system-level redundancy, as long as one full path remains functional. It captures real operational needs, where localized disturbances such as environmental damage or signal interference may affect only segments of a path, and reliable communication is ensured by maintaining functional continuity over a minimum segment. This modeling approach allows engineers to design communication infrastructures that are both resilient and fault-tolerant, particularly suited to mission-critical applications such as high-speed rail systems.

The paper is organized as follows, in Section 2, essential concepts and definitions needed for the results are provided, while Section 3 develops explicit formulas for the reliability of the systems under investigation. Section 4 delves into the asymptotic behavior of these formulas where an illustrative example is given.

## 2. Essential concepts and definitions

The document requires the following assumptions, notations, definitions and lemmas.

**Notations:**

$m$: minimum number of consecutive components required for a consecutive $m$-out-of-$l$:G system to function.
$r$: number of blocks in a consecutive $m_i$-out-of-$l_i$:G series or parallel system.
$l_i$: number of components in block $i$, $i = \overline{1,r}$.
$m_i$: minimum number of consecutive components required for a consecutive $m_i$-out-of-$l_i$:G system to function.
$X_j$: random variable represents lifetime of component $j$.
$F_j$: distribution function of $X_j$, $F_j(x) = P(X_j \le x)$.
$R_j$: reliability function of component $j$, $R_j(x) = P(X_j > x)$.
$R^c$: reliability function of a consecutive $m$-out-of-$l$:G system.
$R^{cs}$: reliability function of a consecutive $m_i$-out-of-$l_i$:G series system.
$R^{cp}$: reliability function of a consecutive $m_i$-out-of-$l_i$:G parallel system.

**Assumptions:**

The components are independent.
The blocks are independent.

**Definition 2.1.** *A consecutive $m$-out-of-$l$:G system consists of $l$ components that are arranged linearly. The system works if and only if at least $m$ consecutive components work ($m \le l$). Its reliability function is given by*

$$R^c(x) = P\left(\max_{1\le p\le l-m+1}\left(\min_{p\le j\le p+m-1} X_j\right) > x\right) \tag{2.1}$$

**Definition 2.2.** *A consecutive $m_i$-out-of-$l_i$:G series system consists of $r$ blocks that are arranged linearly and labeled $B_1, B_2, \ldots, B_r$. The system works if all the blocks work and each block works if and only if at least $m_i$ consecutive components work ($m_i \le l_i$, $i = \overline{1,r}$). Its reliability function is given by*

$$R^{cs}(x) = P\left(\min_{1\le i\le r}\left[\max_{(i-1)l_i+1\le s\le il_i-m_i+1}\left(\min_{s\le j\le s+m_i-1} X_j\right)\right] > x\right) \tag{2.2}$$

**Definition 2.3.** *A consecutive $m_i$-out-of-$l_i$:G parallel system consists of $r$ blocks that are arranged in parallel and labeled $B_1, B_2, \ldots, B_r$. The system works if at least one block works and each block works if and only if at least $m_i$ consecutive components work ($m_i \le l_i$, $i = \overline{1,r}$). Its reliability function is given by*

$$R^{cp}(x) = P\left(\max_{1\le i\le r}\left[\max_{(i-1)l_i+1\le s\le il_i-m_i+1}\left(\min_{s\le j\le s+m_i-1} X_j\right)\right] > x\right) \tag{2.3}$$

**Definition 2.4.** *A consecutive $m_i$-out-of-$l_i$:G series (resp. parallel) system is said to be regular if $l_1 = l_2 = \ldots = l_r = l$, $m_1 = m_2 = \ldots = m_r = m$ where $l$ and $m$ are natural numbers.*

**Definition 2.5.** *A regular consecutive $m$-out-of-$l$:G series (resp. parallel) is said to be homogeneous if all components of the system have the same reliability function*

$$R(x) = 1 - F(x),\ x \in (-\infty, +\infty). \tag{2.4}$$

**Remark 2.1.** *In this work, we do not assume that the lifetime distribution is necessarily supported on the non-negative real line. Accordingly, the reliability function $R(x)$ is not required to satisfy the classical condition $R(x) = 1$ for all $x \leq 0$. It is an extension of the well accepted notion of a reliability function. In theoretical terms, this generalization makes sense. Concurrently, the obtained outcomes pertaining to the generalized reliability function exhibit the identical characteristics as the widely employed reliability functions.*

Based on this assumption, an explicit correspondence exists between a reliability function $R(x)$ and a distribution function $F(x)$, as shown by

$$F(x) + R(x) = 1,\, x \in (-\infty, +\infty)\,. \tag{2.5}$$

Given the properties of a distribution function, the following lemmas become clear.

**Lemma 2.1.** *A reliability function $R(x)$ is non-increasing, right-continuous, $R(-\infty) = 1$ and $R(+\infty) = 0$.*

**Definition 2.6.** *A reliability function $R(x)$ is degenerate if there exists $x_0 \in (-\infty, +\infty)$ such that*

$$R(x) = \begin{cases} 1, & x < x_0 \\ 0, & x \geq x_0. \end{cases} \tag{2.6}$$

**Lemma 2.2.** *A function*

$$\Re(x) = 1 - \exp\left[-V(x)\right],\, x \in (-\infty, +\infty) \tag{2.7}$$

*is a reliability function if and only if $V(x)$ is a non-negative, non-increasing, right-continuous function, $V(-\infty) = \infty$, $V(+\infty) = 0$ and besides $V(x)$ may be identically equal to $\infty$ within a certain interval.*

**Agreement 2.1.** *In the remainder of this work, whenever we refer to a function $V(x)$, we assume it satisfies the properties stated in Lemma 2.2. If $V(x) \equiv \infty$ over an interval, we adopt the convention that $\exp(-\infty) = 0$. The terms "non-negative, non-increasing, and right-continuous" are understood to apply over the domain where $V(x) \neq \infty$. Additionally, we denote by $C_{\Re}$ the set of continuity points of the reliability function $\Re(x)$, and by $C_V$ the set of continuity points of $V(x)$, including points where $V(x) = \infty$.*

**Definition 2.7.** *A function $V(x)$ defined for $x \in (-\infty, +\infty)$, non-negative, non-increasing, right-continuous and such that $V(-\infty) = \infty$, $V(+\infty) = 0$ is degenerate if there exists $x_0 \in (-\infty, +\infty)$ such that*

$$V(x) = \begin{cases} \infty, & x < x_0 \\ 0, & x \geq x_0. \end{cases} \tag{2.8}$$

**Lemma 2.3.** [6] *A reliability function $\Re(x)$ given by (2.7) is degenerate if and only if the function $V(x)$ is degenerate.*

We are interested in studying the limit distributions of the standardized lifetime of the system, represented by the random variable $(X - b_t)/a_t$ where $X$ denotes the lifetime of the system and $a_t = a(t) > 0$, $b_t = b(t) \in (-\infty, +\infty)$ are appropriately

chosen norming functions.
Since

$$P\left(\frac{X-b_t}{a_t}>x\right)=P\left(X>a_tx+b_t\right)=R_t\left(a_tx+b_t\right) \tag{2.9}$$

the following definition emerges naturally and provides the foundation for analyzing the limiting behavior of system reliability.

**Definition 2.8.** *A reliability function $\Re(x)$ is called a limit reliability function of the regular homogeneous consecutive $m$-out-of-$l$:G parallel (or series with necessary changes) system if there exist functions $a_t = a(t) > 0$ and $b_t = b(t) \in (-\infty, +\infty)$ such that*

$$\lim_{t\to\infty} R_t^{cp}(a_tx+b_t)=\Re(x),\ \ x\in C_{\mathrm{R}}. \tag{2.10}$$

*A pair $(a_t, b_t)$ is called a norming functions pair.*

From the condition $\lim\limits_{t\to\infty} R_t^{cp}(a_tx+b_t) = \Re(x)$ and if we assume that $\alpha_t = aa_t$, $\beta_t = ba_t + b_t$ where $a > 0$, $b \in (-\infty, +\infty)$, we have

$$\lim_{t\to\infty} R_t^{cp}(\alpha_tx+\beta_t)=\lim_{t\to\infty} R_t^{cp}(a_t(ax+b)+b_t)=\Re(ax+b),\ \ x\in C_{\Re}. \tag{2.11}$$

As a result, if $\Re(x)$ is a limit reliability function, then any affine transformation of the form $\Re(ax + b)$, with $a > 0$ and $b \in (-\infty, +\infty)$, also qualifies as a limit reliability function. This property of stability under linear transformation allows us to introduce the following definitions, which formalize the characterization of such functions.

**Definition 2.9.** *The reliability functions $\Re_0(x)$ and $\Re(x)$ are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, +\infty)$ such that*

$$\Re_0(x)=\Re(ax+b) \tag{2.12}$$

*for all $x \in (-\infty, +\infty)$.*

**Definition 2.10.** *The functions $V_0(x)$ and $V(x)$ with properties mentioned in Definition 2.7 are said to be of the same type if there exist numbers $a > 0$ and $b \in (-\infty, +\infty)$ such that*

$$V_0(x)=V(ax+b) \tag{2.13}$$

*for all $x \in (-\infty, +\infty)$.*

## 3. Reliability of Consecutive $m_i$-out-of-$l_i$:G Series and Parallel Systems

Before studying the asymptotic behavior of the systems under investigation, let first present the explicit formulas used to compute their reliability.

**Proposition 3.1.** *The reliability functions for a consecutive $m_i$-out-of-$l_i$:G series and parallel systems under the condition $m_i \geq l_i - m_i$ for $i = \overline{1, r}$ are respectively given by*

$$R^{cs}(x)=\prod_{i=1}^{r}\left[1-\prod_{s=(i-1)l_i+1}^{il_i-m_i+1}\left[1-\prod_{j=s}^{s+m_i-1}\left(R_j(x)\right)\right]\right] \tag{3.1}$$

$$R^{cp}(x) = 1 - \prod_{i=1}^{r} \left[ \prod_{s=(i-1)l_i+1}^{il_i - m_i + 1} \left[ 1 - \prod_{j=s}^{s+m_i-1} (R_j(x)) \right] \right]. \tag{3.2}$$

*Proof.* Under the condition $m \geq l - m$ and by setting

$$Y_p = \min_{p \leq j \leq p+m-1} X_j$$

in Definition 2.1, then

$$\begin{aligned} R^c(x) &= P\left( \max_{1 \leq p \leq l-m+1} Y_p > x \right) \\ &= 1 - P\left(Y_1 \leq x, Y_2 \leq x, \ldots, Y_{l-m+1} \leq x\right) \end{aligned}$$

but

$$\begin{aligned} P\left(Y_1 \leq x, Y_2 \leq x, \ldots, Y_{l-m+1} \leq x\right) = P\left(Y_1 \leq x\right) \times P\left(Y_2 \leq x / Y_1 \leq x\right) \times \ldots \\ \times P\left(Y_{l-m+1} \leq x / Y_1 \leq x, \ldots, Y_{l-m} \leq x\right) \end{aligned}$$

with

$$\begin{aligned} P\left(Y_1 \leq x\right) &= P\left( \min_{1 \leq j \leq m} X_j \leq x \right) \\ &= 1 - \prod_{j=1}^{m} R_j(x) \end{aligned}$$

and

$$\begin{aligned} P\left(Y_2 \leq x \mid Y_1 \leq x\right) &= \frac{P\left(Y_2 \leq x,\, Y_1 \leq x\right)}{P\left(Y_1 \leq x\right)} \\ &= \frac{P\left( \left( \min_{1 \leq j \leq m} X_j \leq x \right) \cap \left( \min_{2 \leq j \leq m+1} X_j \leq x \right) \right)}{P\left( \min_{1 \leq j \leq m} X_j \leq x \right)} \\ &= \frac{1 - \left( \prod_{j=1}^{m} R_j(x) + \prod_{j=2}^{m+1} R_j(x) - \prod_{j=1}^{m+1} R_j(x) \right)}{1 - \prod_{j=1}^{m} R_j(x)} \end{aligned}$$

as a result

$$\mathrm{P}(Y_1 \leq x, Y_2 \leq x, \ldots, Y_{l-m+1} \leq x) = 1 - \left[ \sum_{p=1}^{l-m+1} \left( \prod_{j=p}^{m+p-1} R_j(x) \right) - \sum_{p=1}^{l-m} \left( \prod_{j=p}^{m+p} R_j(x) \right) \right].$$

By substitution, we obtain this formula which can be verified by recurrence

$$R^c(x) = \sum_{p=1}^{l-m+1} \left( \prod_{j=p}^{m+p-1} R_j(x) \right) - \sum_{p=1}^{l-m} \left( \prod_{j=p}^{m+p} R_j(x) \right)$$

Taking into account that all the blocks in the both consecutive $m_i$-out-of-$l_i$:G series and parallel systems are independent where each block $i$ for $i = \overline{1, r}$ has a consecutive $m_i$-out-of-$l_i$:G configuration, then using Definitions 2.2 and 2.3, the proof is completed. □

**Corollary 3.1.** *If the consecutive $m_i$-out-of-$l_i$:G series and parallel systems are regular with $m_1 = m_2 = \ldots = m_r = m$, $l_1 = l_2 = \ldots = l_r = l$, $m$ and $l$ are natural numbers with $m \geq l - m$, and if all the components are identical with the same reliability function $R_j(x) = R(x)$ for $j = 1, 2, \ldots, l$ then*

$$R^{cs}(x) = \left[1 - (1 - (R(x))^m)^{l-m+1}\right]^r \tag{3.3}$$

$$R^{cp}(x) = 1 - \left[(1 - (R(x))^m)^{l-m+1}\right]^r. \tag{3.4}$$

Now, assume in Corollary 3.1 that $m = m_n$, $l = l_n$ and $r = r_n$, where $n$ is a natural number, and the sequences $m_n$, $l_n$ and $r_n$ are sequences of natural numbers such that at least one of them tends to infinity as $n$ tends to infinity. Under these assumptions, we obtain a sequence of regular homogeneous consecutive $m_n$-out-of-$l_n$:G arranged either in series or in parallel, indexed by $n$. Each of these systems is associated with a corresponding reliability function, yielding a sequence of reliability functions $R_n(x)$.
The following result is naturally motivated by this setup.

**Fact 3.1.** *The sequences of reliability functions of the regular homogeneous consecutive $m$-out-of-$l$:G series and parallel systems are respectively given by*

$$R_n^{cs}(x) = \left[1 - (1 - (R_n(x))^{m_n})^{l_n-m_n+1}\right]^{r_n} \tag{3.5}$$

$$R_n^{cp}(x) = 1 - \left[(1 - (R_n(x))^{m_n})^{l_n-m_n+1}\right]^{r_n}. \tag{3.6}$$

Next, by substituting $n$ with a positive real number $t$ and assuming that $m_t$, $l_t$ and $r_t$ are positive real numbers, we obtain a family of systems that corresponds to sequences $m_t$; $l_t$ and $r_t$. This family of systems has a family of reliability functions. The next fact is a straightforward extension of Fact 3.1.

**Fact 3.2.** *The family of reliability functions of the regular homogeneous consecutive $m$-out-of-$l$:G series and parallel systems are respectively given by*

$$R_t^{cs}(x) = \left[1 - (1 - (R_t(x))^{m_t})^{l_t-m_t+1}\right]^{r_t} \tag{3.7}$$

$$R_t^{cp}(x) = 1 - \left[(1 - (R_t(x))^{m_t})^{l_t-m_t+1}\right]^{r_t} \tag{3.8}$$

## 4. Limit Reliability Functions of Regular Homogeneous Consecutive $m$-out-of-$l$:G Series and Parallel Systems

The problem of characterizing the closed class of possible nondegenerate limit reliability functions for regular homogeneous consecutive $m$-out-of-$l$:G systems (in both series and parallel configurations) reduces to identifying the potential limit functions associated with the family of reliability functions $R_t^{cs}(x)$ and $R_t^{cp}(x)$ as defined by equations (3.7) and (3.8).

In lihgt of Lemma 2.3, our analysis focuses on assigning a class of admissible nondegenerate functions to the norming function $V(x)$, which governs the asymptotic form of these reliability functions.

**Agreement 4.1.** [1]

(1) $x(t) \sim y(t) \Longleftrightarrow \lim\limits_{t\to\infty} \frac{x(t)}{y(t)} = 1$,

(2) $x(t) \ll y(t) \Longleftrightarrow \lim\limits_{t\to\infty} \frac{x(t)}{y(t)} = 0$,

(3) $x(t) \gg y(t) \Longleftrightarrow \lim\limits_{t\to\infty} \frac{x(t)}{y(t)} = \infty$,

(4) $o(1)$ *denotes a positive function of* $t$ *such that* $\lim\limits_{t\to\infty} o(t) = 0$.

We study, in a first step, the limit reliability functions of the regular homogeneous consecutive $m$-out-of-$l$:G parallel system, then, due to the famous relationship between a series and parallel configuration, those of the series case will be natural results.

### 4.1. Limit Reliability Functions of the Regular Homogeneous Consecutive $m$-out-of-$l$:G Parallel System.

**Lemma 4.1.** *If*

- *a nondegenerate function* $\Re(x)$ *is given by (2.7),*
- $R_t^c(x) = 1 - \left[ (1 - (R(x))^{m_t})^{l_t - m_t + 1} \right]$ *is a family of reliability functions of the regular homogeneous consecutive* $m$*-out-of-*$l$*:G system,*
- $\lim\limits_{t\to\infty} (l_t - m_t + 1) = \infty$,
- $a_t > 0$, $b_t \in (-\infty, +\infty)$ *are some functions,*

*then the assertion*

$$\lim_{t\to\infty} R_t^c(a_t x + b_t) = \Re(x),\ x \in C_\Re$$

*is equivalent to the assertion*

$$\lim_{t\to\infty} (l_t - m_t + 1) \left[ R(a_t x + b_t) \right]^{m_t} = V(x),\ x \in C_V.$$

*Proof.* The proof follows a similar reasoning to that of [[7], Lemma 1], based on the observation that a regular homogeneous consecutive $m$-out-of-$l$:G system can be reorganized into $l - m + 1$ paths, each consisting of $m$ components in series. In other words, this system may be equivalently viewed as a series-parallel system. □

**Lemma 4.2.** *Assume that*

- *a nondegenerate function* $\Re(x)$ *is given by (2.7),*
- $R_t^{cp}(x)$ *is given by (3.8),*
- $\lim\limits_{t\to\infty} (l_t - m_t + 1) = \infty$ *and* $m_t \gg \ln(l_t - m_t + 1)$,
- $a_t > 0$, $b_t \in (-\infty, +\infty)$ *are some functions,*

*then the assertion*

$$\lim_{t\to\infty} R_t^{cp}(a_t x + b_t) = \Re(x),\ x \in C_\Re$$

*is equivalent to the assertion*

$$\lim_{t\to\infty} \eta_t \left[R(a_t x + b_t)\right]^{\xi_t} = V(x),\ x \in C_V,$$

*where*

$$\xi_t = m_t(l_t - m_t + 1)\ln(l_t - m_t + 1)\left[\left(1 - (l_t - m_t + 1)^{-1/m_t}\right)^{\frac{-1}{l_t - m_t + 1}} - 1\right] \quad (4.1)$$

$$\eta_t = \left\{1 - \left[1 - (l_t - m_t + 1)^{-1/m_t}\right]^{\frac{1}{l_t - m_t + 1}}\right\}^{-\xi_t/m_t} \quad (4.2)$$

*Proof.* Each block in the regular homogeneous consecutive $m$-out-of-$l$:G parallel system can be decomposed into $l-m+1$ parallel paths, where each path consists of $m$ components arranged in series. In other words, each block can be viewed as a series-parallel subsystem.

This structural interpretation allows us to conclude that the regular homogeneous consecutive $m$-out-of-$l$:G parallel system is, in fact, a series-parallel system of order 2, as discussed in [1].

On the other hand, we have

$$\begin{aligned}
\lim_{t\to\infty} \ln \eta_t &= \lim_{t\to\infty}(l_t - m_t + 1)\ln(l_t - m_t + 1)\left[\left(1 - (l_t - m_t + 1)^{-1/m_t}\right)^{\frac{-1}{l_t - m_t + 1}} - 1\right] \\
&\times \left\{-\ln\left[1 - \left[1 - (l_t - m_t + 1)^{-1/m_t}\right]^{\frac{1}{l_t - m_t + 1}}\right]\right\} \\
&\geq \lim_{t\to\infty}(l_t - m_t + 1)\ln(l_t - m_t + 1)\left[\left(1 - (l_t - m_t + 1)^{-1/m_t}\right)^{\frac{-1}{l_t - m_t + 1}} - 1\right] \\
&\times \left[1 - (l_t - m_t + 1)^{-1/m_t}\right]^{\frac{1}{l_t - m_t + 1}} \\
&= \lim_{t\to\infty}(l_t - m_t + 1)\ln(l_t - m_t + 1)\left[1 - \exp\left(\tfrac{\ln(1 - \exp(-\ln(l_t - m_t + 1)/m_t))}{l_t - m_t + 1}\right))\right] \\
&\geq \lim_{t\to\infty}(l_t - m_t + 1)\ln(l_t - m_t + 1)\left[1 - \exp(\tfrac{-1}{l_t - m_t + 1})\right] \\
&= \lim_{t\to\infty}(l_t - m_t + 1)\ln(l_t - m_t + 1)(\tfrac{1}{l_t - m_t + 1})(1 - o(1)) \\
&= \lim_{t\to\infty}\ln(l_t - m_t + 1)(1 - o(1)) \\
&= \infty.
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty}\frac{\ln \eta_t}{\xi_t} &= \lim_{t\to\infty}\frac{1}{m_t}\left\{-\ln\left[1 - \exp\left(\frac{\ln\left(1 - \exp\left(-\ln(l_t - m_t + 1)/m_t\right)\right)}{l_t - m_t + 1}\right)\right]\right\} \\
&\leq \lim_{t\to\infty}\frac{1}{m_t}\left\{-\ln\left[1 - \exp\left(-\frac{1}{l_t - m_t + 1}\right)\right]\right\} \\
&= \lim_{t\to\infty}\frac{1}{m_t}\left[-\ln\left(\frac{1}{l_t - m_t + 1}(1 - o(1))\right)\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}\frac{\xi_t}{m_t} &= \lim_{t\to\infty}(l_t-m_t+1)\ln(l_t-m_t+1)\left[\exp\left(\frac{-\ln(1-\exp(-\ln(l_t-m_t+1)/m_t)))}{l_t-m_t+1}\right)-1\right]\\ &\geq \lim_{t\to\infty}(l_t-m_t+1)\ln(l_t-m_t+1)\left[\exp\left(\frac{1}{l_t-m_t+1}\right)-1\right]\\ &= \lim_{t\to\infty}(l_t-m_t+1)\ln(l_t-m_t+1)\frac{1}{l_t-m_t+1}\left(1+o(1)\right)\\ &= \infty.\end{aligned}$$

□

The rightness of this lemma follows immediately, according to [[1], Lemma 2].

**Lemma 4.3.** *Assume that*

- $\xi_t$ *and* $\eta_t$ *are given by (4.1) and (4.2),*
- $\lim\limits_{t\to\infty}(l_t-m_t+1)=\infty$ *and* $\rho(t)=\frac{\ln\xi_t}{\ln c+\ln(\ln\eta_t)}$, $t\in(0,\infty)$, $c>0$,
- $c\ln(l_t-m_t+1)\ll m_t$ *and* $\rho(t)\ll(\ln\eta_t)^{\lambda}$ *for every* $\lambda>0$,
- *for every natural* $\upsilon\geq 2$, $\tau_\upsilon$ *is given by*
$$\tau_\upsilon=\eta_t\upsilon^{1/[1-\rho(t)]},\ t\in(0,\infty);$$
$$|\rho(\tau_\upsilon)-\rho(t)|\ \tilde{\ll}\ \frac{\delta\ln\upsilon}{\ln\eta_t\ln(\ln\eta_t)},\ 0<\delta\neq 0,$$
- *a nondegenerate reliability function* $\Re(x)$ *given by (2.7) is a limit reliability function of a regular homogeneous consecutive* $m$*-out-of-*$l$*:*$G$ *parallel system.*

*Then for every natural* $\upsilon\geq 2$, *there exist numbers* $\alpha_\upsilon$, $\gamma_\upsilon\in(-\infty,+\infty)$ *and* $\mu\neq 0$ *such that*

$$\upsilon^{\mu}V(\alpha_\upsilon x+\gamma_\upsilon)=V(x),x\in(-\infty,+\infty) \tag{4.3}$$

*Proof.* The proof is analogous to that in [[7], Lemma 5], and thus is not reproduced here for brevity. □

**Definition 4.1.** *The function* $A(t)$ *is defined for* $t\in(0,\infty)$ *and*

$$A\left(t\right)\sim\prod_{i=1}^{n}f_i(\rho(t)), \tag{4.4}$$

*where* $f_i(t)$ *is the superposition of the function* $\ln t$ *taken* $i$ *times and* $n$ *is such that*

$$f_{n+1}(\rho(t))\ll A\ln(\ln\eta_t),\ \ A>0$$

**Lemma 4.4.** *Assume that*

- $\xi_t$ *and* $\eta_t$ *are given by (4.1) and (4.2),*
- $\lim\limits_{t\to\infty}(l_t-m_t+1)=\infty$ *and* $\rho(t)=\frac{\ln\xi_t}{\ln c+\ln(\ln\eta_t)}$, $t\in(0,\infty)$, $c>0$,
- $c\ln(l_t-m_t+1)\ll m_t$ *and* $\rho(t)\ \tilde{\gg}\ (\ln\eta_t)^{\lambda}\ \lambda>0$,
- *for every natural* $\upsilon\geq 2$, $\tau_\upsilon$ *is given by*
$$\tau_\upsilon=\eta_t\upsilon^{1/[1-\rho(t)]A(t)},\ t\in(0,\infty);$$
$$|\rho(\tau_\upsilon)-\rho(t)|\sim\frac{\delta\ln\upsilon}{\ln\eta_t\ln(\ln\eta_t)},\ \delta>0,$$
- *a nondegenerate reliability function* $\Re(x)$ *given by (2.7) is a limit reliability function of a regular homogeneous consecutive* $m$*-out-of-*$l$*:*$G$ *parallel system.*

*Then for every natural $\upsilon \geq 2$, there exist numbers $\alpha_\upsilon$, $\gamma_\upsilon \in (-\infty, +\infty)$ and $\mu \neq 0$ such that*

$$\upsilon^\mu V(\alpha_\upsilon x + \gamma_\upsilon) = V(x), \ x \in (-\infty, +\infty). \tag{4.5}$$

*Proof.* The argument proceeds in the same way as in [[7], Lemma 7], and is omitted here for conciseness. □

The resulting functional equation (4.5) enables us to define a closed class of nondegenerate functions $V(x)$, corresponding to the given pair $(l_t - m_t + 1, m_t)$. Based on this class of functions, we can directly derive a corresponding class of limit reliability functions for the regular homogeneous consecutive $m$-out-of-$l$:G parallel system.

**Theorem 4.1.** *Assume that*

- *$\xi_t$ and $\eta_t$ are given by (4.1) and (4.2),*
- *$\lim\limits_{t\to\infty}(l_t - m_t + 1) = \infty$ and $\rho(t) = \frac{\ln \xi_t}{\ln c + \ln(\ln \eta_t)}$, $t \in (0, \infty)$, $c > 0$, where*

$$c \ln(l_t - m_t + 1) \ll m_t \text{ and } \rho(t) \ll (\ln \eta_t)^\lambda \text{ for every } \lambda > 0,$$

*and for every natural $\upsilon \geq 2$, $\tau_\upsilon$ is given by*

$$\tau_\upsilon = \eta_t \upsilon^{1/[1-\rho(t)]A(t)}, \ t \in (0, \infty);$$
$$|\rho(\tau_\upsilon) - \rho(t)| \overset{\sim}{\ll} \frac{\delta \ln \upsilon}{\ln \eta_t \ln(\ln \eta_t)}, \ 0 < \delta \neq 0,$$

*or $c \ln(l_t - m_t + 1) \ll m_t$ and $\rho(t) \overset{\sim}{\gg} (\ln \eta_t)^\lambda, \lambda > 0$, and for every natural $\upsilon \geq 2$, $\tau_\upsilon$ is given by*

$$\tau_\upsilon = \eta_t \upsilon^{1/[1-\rho(t)]A(t)}, \ t \in (0, \infty);$$
$$|\rho(\tau_\upsilon) - \rho(t)| \sim \frac{\delta \ln \upsilon}{\ln \eta_t \ln(\ln \eta_t)}, \ 0 < \delta \neq 0,$$

*where $A(t)$ is defined by (4.4).*

*Then the only possible nondegenerate limit reliability functions of the regular homogeneous consecutive $m$-out-of-$l$:G parallel system are*

$$\Re_1(x) = \begin{cases} 1, & x \leq 0, \\ 1 - \exp\left[-x^{-\alpha}\right], & x > 0, \ \alpha > 0, \end{cases}$$

$$\Re_2(x) = \begin{cases} 1 - \exp\left[-(-x)^{\alpha}\right], & x < 0, \ \alpha > 0, \\ 0, & x \geq 0, \end{cases}$$

$$\Re_3(x) = 1 - \exp\left[-\exp(-x)\right], \ x \in (-\infty, +\infty).$$

4.2. **Limit Reliability Functions of the Regular Homogeneous Consecutive $m$-out-of-$l$:G Series System.**

**Lemma 4.5.** *If a reliability function $\Re(x)$ is a limit reliability function of a regular homogeneous consecutive $m$-out-of-$l$:G parallel system, where the reliability function of an individual component is $R(x)$, then the transformed function $\overline{\Re}(x) = 1 - \Re(-x)$ for $x \in C_\Re$ is a limit reliability function for the corresponding consecutive $m$-out-of-$l$:G series system, with component reliability function $\overline{R}(x) = 1 - R(-x)$ for $x \in C_R$.*
*Moreover, if $(a_t, b_t)$ is a valid norming pair in the parallel case, then $(a_t, -b_t)$ serves as the corresponding norming pair in the series case.*

*Proof.* We have

$$\begin{aligned} R_t^{cs}(x) &= \left[1-\left(1-\left(\overline{F}(x)\right)^{m_t}\right)^{l_t-m_t+1}\right]^r \\ &= 1-R_t^{cp}(-x), \end{aligned}$$

where

$$\overline{F}(x) = 1-\overline{R}(-x).$$

As a result, if

$$\lim_{t\to\infty} R_t^{cp}(a_t x+b_t) = \Re(x),\ for\ x \in C_{\Re},$$

then

$$\begin{aligned} \lim_{t\to\infty} R_t^{cs}(a_t x+b_t) &= \lim_{t\to\infty}\left[1-R_t^{cp}(a_t(-x)+b_t)\right] \\ &= 1\text{-}\Re(-x) \\ &= \overline{\Re}(x) \end{aligned}$$

for $x \in C_{\overline{\Re}.}$ □

We can derive the following theorem using Lemma 4.2 and theorem 4.1.

**Theorem 4.2.** *Assume that*

- $\xi_t$ *and* $\eta_t$ *are given by (4.1) and (4.2),*
- $\lim\limits_{t\to\infty}(l_t-m_t+1)=\infty$ *and* $\rho(t)=\frac{\ln\xi_t}{\ln c+\ln(\ln\eta_t)}$, $t\in(0,\infty)$, $c>0$,

*where*

$$c\ln(l_t-m_t+1) \ll m_t \text{ and } \rho(t) \ll (\ln\eta_t)^{\lambda} \text{ for every } \lambda>0,$$

*and for every natural* $\upsilon \geq 2$, $\tau_\upsilon$ *is given by*

$$\tau_\upsilon = \eta_t \upsilon^{1/1-\rho(t)},\ t\in(0,\infty),$$
$$|\rho(\tau_\upsilon)-\rho(t)| \overset{\sim}{\ll} \frac{\delta\ln\upsilon}{\ln\eta_t\ln(\ln\eta_t)},\ 0<\delta\neq 0,$$

*or* $c\ln(l_t-m_t+1) \ll m_t$ *and* $\rho(t) \overset{\sim}{\gg} (\ln\eta_t)^{\lambda}$, $\lambda>0$, *and for every natural* $\upsilon\geq 2$, $\tau_\upsilon$ *is given by*

$$\tau_\upsilon = \eta_t \upsilon^{1/[1-\rho(t)]A(t)},\ t\in(0,\infty),$$
$$|\rho(\tau_\upsilon)-\rho(t)| \sim \frac{\delta\ln\upsilon}{\ln\eta_t\ln(\ln\eta_t)},\ \delta>0,$$

*where* $A(t)$ *is defined by (4.4).*
*Then the only possible nondegenerate limit reliability functions of the regular homogeneous consecutive* $m$*-out-of-*$l$*:G series system are those of the form*

$$\overline{\Re}_1(x) = \begin{cases} \exp\left[-(-x)^{-\alpha}\right], & x<0,\ \alpha>0, \\ 0, & x\geq 0, \end{cases}$$

$$\overline{\Re}_2(x) = \begin{cases} 1, & x<0, \\ \exp\left[-x^{\alpha}\right], & x\geq 0,\ \alpha>0, \end{cases}$$

$$\overline{\{}\Re_3(x) = \exp\left[-\exp(x)\right],\ x \in (-\infty,+\infty).$$

**An example**

Let a regular homogeneous consecutive $m$-out-of-$l$:G parallel system be such that

$$R(x) = \begin{cases} 1, & x<0 \\ \exp(-\lambda x), & x\geq 0,\ \lambda>0. \end{cases}$$

If pairs $(l_t,\ m_t)$ and $(a_t,\ b_t)$ satisfy the conditions

- $m_t = t$, $l_t - m_t + 1 = \ln t$, $t \in (e, \infty)$,
- $a_t = \frac{1}{\lambda \xi_t}$, $b_t = -\frac{\ln\left\{1-\left[1-(\ln t)^{-\frac{1}{t}}\right]^{-\frac{1}{\ln t}}\right\}}{\lambda m_t}$,

then

$$\Re_3(x) = \begin{cases} 1, & x < 0, \\ 1 - \exp\left[-\exp(-x)\right], & x \geq 0 \end{cases}$$

is the limit reliability function of the system.

*Proof.* Since $\xi_t > 0$ for $t \in (e, \infty)$, then $a_t > 0$.
We have

$$R(a_t x + b_t) = \begin{cases} 1, \ x < 0 \\ \exp\left[-\frac{x}{\xi_t} + \frac{\ln\left\{1-\left[1-(\ln t)^{-\frac{1}{t}}\right]^{-\frac{1}{\ln t}}\right\}}{m_t}\right], \ x \geq 0 \end{cases}$$

Therefore, for $x < 0$

$$\lim_{t\to\infty} \eta_t \left[R(a_t x + b_t)\right]^{\xi_t} = \infty,$$

and for $x \geq 0$

$$\begin{aligned} \lim_{t\to\infty} \eta_t \left[R(a_t x + b_t)\right]^{\xi_t} &= \lim_{t\to\infty} \eta_t \left[\exp\left(-\frac{x}{\xi_t} + \frac{\ln\left\{1-\left[1-(\ln t)^{-\frac{1}{t}}\right]^{-\frac{1}{\ln t}}\right\}}{m_t}\right)\right]^{\xi_t} \\ &= \lim_{t\to\infty} \eta_t \left[\exp\left(-x + \frac{\xi_t}{m_t} \ln\left\{1 - \left[1 - (\ln t)^{-\frac{1}{t}}\right]^{-\frac{1}{\ln t}}\right\}\right)\right] \\ &= \lim_{t\to\infty} \eta_t \exp\left[-x - \ln \eta_t\right] \\ &= \exp(-x) \end{aligned}$$

Hence, by Lemma 4.2, $\Re_3(x)$ is the limit reliability function of the system. □

## 5. Conclusion

This paper has examined the limit reliability functions of a specific class of consecutive systems, namely regular homogeneous consecutive $m$-out-of-$l$:G series and parallel systems. These functions are derived by modeling the systems as series-parallel structures of order 2 and applying tools from extreme value theory. The methodology developed herein diverges from classical approaches typically used for analyzing the limit behavior of series-parallel and parallel-series systems, offering a novel perspective that may be generalized to a wider class of reliability models. As part of our future work, we aim to investigate the asymptotic behavior of these systems in the non-homogeneous case, and the multi-state systems case.

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