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DEGREE EXPONENT SUM ENERGY OF A GRAPH

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Abstract. In this paper, we introduce a matrix associated with a graph called *degree exponent sum matrix*. We compute the degree exponent sum polynomial of graph operations, cycle related graphs, product related graphs and transformation graphs. We give bounds for the largest degree exponent sum eigenvalue and degree exponent sum energy of a graph.

1. Introduction

The *energy* of a graph is the sum of absolute values of eigenvalues of its adjacency matrix. It has a correlation with the total π-electron energy of a molecule in the quantum chemistry as calculated with the *Hückel molecular orbital method* ([16]).

Let G be a simple, finite, undirected, nontrivial graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be a vertex set and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be an edge set of G. The *degree* $d_G(v_i)$ (or simply d_i) of a vertex v_i is the number of vertices adjacent to it in G. The graph G is an *r-regular graph* if the degree of every vertex in G is r.

The *adjacency matrix* $A(G) = [a_{ij}]$ of a graph G ([18]) is a matrix of order $n \times n$, whose elements are defined as

$$a_{ij} = \begin{cases} 1, & if\ v_i\ is\ adjacent\ to\ v_j, \\ 0, & otherwise. \end{cases}$$

The *characteristic polynomial* of $A(G)$ is given by $\phi(G : \lambda)$=det$(\lambda I_n - A(G))$, where I_n is an identity matrix of order n. The roots of an equation $\phi(G : \lambda) = 0$ are called the *eigenvalues* of G and they are labeled as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The collection of eigenvalues of G is called the *spectrum of G* denoted by $Spec(G)$, refer to ([11]). The two nonisomorphic graphs are *cospectral* if they have the same spectra. The *energy* $\epsilon(G)$ of a graph G with n vertices is defined as $\epsilon(G) = \sum_{i=1}^{n} |\lambda_i|$, refer to ([15]). For undefined graph theoretic terminologies and notations, one can refer to ([18]) or ([19]).

Followed by the adjacency matrix, many other graph matrices are defined in literature such as distance matrix ([1]), Laplacian matrix ([20]), signless Laplacian

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matrix ([12, 22]), degree sum matrix ([23]), Seidel matrix ([7]), degree square sum matrix ([2, 3]), minimum degree matrix ([4]) etc.

We introduce a new matrix associated with a graph G called *degree exponent sum matrix* of order $n \times n$ given by $DES(G) = [des_{ij}]$ and whose elements are defined as

$$des_{ij} = \begin{cases} {d_i}^{d_j} + {d_j}^{d_i}, & if\ i \neq j, \\ 0, & otherwise. \end{cases}$$

Here, d_i is degree of a vertex v_i and d_j is degree of a vertex v_j. The *degree exponent sum polynomial* of a graph G is defined as

$$P_{DES(G)}(\mu) = det(\mu I_n - DES(G)),$$

where I_n is an identity matrix. The eigenvalues of $DES(G)$ are called the *degree exponent sum eigenvalues* of G, denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and their collection is called the *degree exponent sum spectra* of G. The *degree exponent sum energy* $E_{DES}(G)$ of a graph G is defined as $E_{DES}(G) = \sum\limits_{i=1}^{n} |\mu_i|$.

For an r-regular graph G, $DES(G) = 2r^r J_n - 2r^r I_n$, where J_n is a matrix of order $n \times n$ whose all entries are equal to 1. Therefore, for an r-regular graph G of order n,

$$P_{DES(G)}(\mu) = \Big(\mu - 2r^r(n-1)\Big)\Big(\mu + 2r^r\Big)^{n-1}. \tag{1.1}$$

Example 1.1. Let $H = C_4$ be a cycle. The degree exponent sum matrix, degree

H :

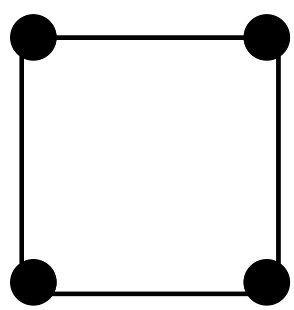

FIGURE 1.

exponent sum polynomial and degree exponent sum energy of H are as follows:

$$DES(H) = \begin{bmatrix} 0 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 \\ 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 0 \end{bmatrix},$$

$P_{DES(H)}(\mu) = \mu^4 - 384\mu^2 - 4096\mu - 12288$,
$E_{DES}(H) = 48$.

Example 1.2. Let $G = K_4 - e$, where e is an edge of K_4. The degree exponent sum matrix, degree exponent sum polynomial and degree exponent sum energy of G are as follows:

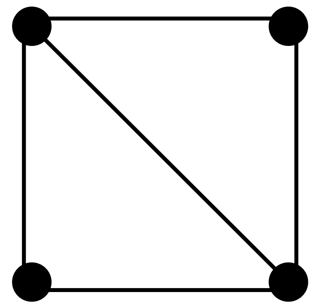

FIGURE 2.

$$DES(G) = \begin{bmatrix} 0 & 17 & 54 & 17 \\ 17 & 0 & 17 & 8 \\ 54 & 17 & 0 & 17 \\ 17 & 8 & 17 & 0 \end{bmatrix},$$

$P_{DES(G)}(\mu) = \mu^4 - 4136\mu^2 - 71672\mu - 312768,$
$E_{DES}(G) \approx 144.097503.$

2. Preliminaries

Definition 2.1. ([18]) The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is a graph whose vertex set is $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and an edge set is $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is a graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2. The *cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1(G_1), E_1(G_1))$ and $G_2 = (V_2(G_2), E_2(G_2))$ is defined as follows: Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 \times G_2) = V_1(G_1) \times V_2(G_2)$. The two vertices u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and $u_2v_2 \in E(G_2)$) or ($u_2 = v_2$ and $u_1v_1 \in E(G_1)$). The *composition* $G_1[G_2]$ of two graphs G_1 and G_2 is defined as follows: Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1[G_2]) = V_1(G_1) \times V_2(G_2)$. The two vertices u and v are adjacent in $G_1[G_2]$ whenever ($u_1v_1 \in E(G_1)$) or ($u_1 = v_1$ and $u_2v_2 \in E(G_2)$). The *corona* $G_1 \circ G_2$ of graphs G_1 and G_2 is a graph obtained from G_1 and G_2 by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining by an edge each vertex of the i^{th} copy of G_2 is named (G_2, i) with the i^{th} vertex of G_1.

Definition 2.2. ([18]) For $n \geq 4$, the graph $W_n = C_{n-1} + K_1$ is called a *wheel graph.* In W_n, a vertex of degree $n-1$ is called a central vertex and the vertices on the cycle C_{n-1} are called rim vertices.

Definition 2.3. ([14]) Let $C_n^{(t)}$ denote the one-point union of $t \geq 2$ cycles of length n. The graph $C_3^{(t)}$ is called a *friendship graph.* The *helm* H_n is a graph obtained from a wheel W_n by attaching a pendant edge at each vertex of a cycle C_{n-1}. The *closed helm* $H_n^{'}$ is a graph obtained from a helm H_n by joining each pendant vertex to form a cycle. The *sunflower graph* SF_n is a graph obtained from a wheel with central vertex v_0, $(n-1)$-cycle $v_1, v_2, \ldots, v_{n-1}, v_1$ and additional

$n-1$ vertices $w_1, w_2, \ldots, w_{n-1}$, where w_i is joined by edges to v_i, v_{i+1} for $i = 1, 2, \ldots, n-1$, where $i+1$ is taken modulo $n-1$. The *double cone* DC_n is a graph $C_n + 2K_1$. The *book graph* B_b is a graph $K_{1,b} \times P_2$. A *book with triangular pages* B_t is a graph $P_2 + tK_1$ where $t \geq 1$. The graph $P_n \times P_2$ is called a *ladder graph* L_n. The *prism* Π_n is a graph $C_n \times P_2$. The *triangular snake* T_n is obtained from the path P_n by replacing each edge of a path by a triangle C_3. The *quadrilateral snake* Q_n is obtained from the path P_n by replacing each edge of the path by a cycle C_4.

Definition 2.4. The *complement* $\overline{G}$ of a graph G ([18]) is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are nonadjacent in G. The (*line graph*) $L(G)$ of a graph G ([18]) is defined as a graph with vertex set as $E(G)$ where the two vertices of $L(G)$ are adjacent if and only if they correspond to two adjacent edges of G. The k^{th} *iterated line graph* $L^k(G)$ of G ([8, 9, 18]) is a graph defined as $L(L^{k-1}(G))$, $k = 1, 2, \ldots$, where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$. The *jump graph* $J(G)$ of a graph G ([10]) is defined as a graph with vertex set as $E(G)$ where the two vertices of $J(G)$ are adjacent if and only if they correspond to two nonadjacent edges of G. The *subdivision graph* $S(G)$ of a graph G ([18]) is defined as a graph with vertex set $V(G) \cup E(G)$ and is obtained by inserting a new vertex of degree 2 into each edge of G. The *semitotal point graph* $T_2(G)$ of a graph G ([25]) is defined as a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_2(G)$ are adjacent if and only if they correspond to two adjacent vertices of G or one is a vertex of G and another is an edge G incident with it in G. The *semitotal line graph* $T_1(G)$ ([25]) or *middle graph* $M(G)$ ([17]) of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices are adjacent if and only if they correspond to two adjacent edges of G or one is a vertex of G and another is an edge G incident with it in G. The *total graph* $T(G)$ of a graph G ([18]) is defined as a graph with vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements (vertices and edges) of G are either adjacent or incident.

The following lemma is useful for computing degree exponent sum polynomial of graphs.

Lemma 2.5. *([24]) If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\mu + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\mu + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \tag{2.1}$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\mu + a)^{n_1 - 1}(\mu + b)^{n_2 - 1}\bigg((\mu - (n_1 - 1)a)\,(\mu - (n_2 - 1)b) - n_1 n_2 cd \bigg).$$

We use the below mentioned relations in computing the bounds for degree exponent sum energy.

The Cauchy-Schwarz inequality ([5]) states that if $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are n real vectors, then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2.2}$$

Theorem 2.6. *([21]) If a_i and b_i are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2, \tag{2.3}$$

where $M_1 = max_{1\leq i\leq n}(a_i)$; $M_2 = max_{1\leq i\leq n}(b_i)$; $m_1 = min_{1\leq i\leq n}(a_i)$; $m_2 = min_{1\leq i\leq n}(b_i)$.

Theorem 2.7. *([6]) If a_i and b_i are nonnegative real numbers, then*

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \alpha(n)(A-a)(B-b), \tag{2.4}$$

where a, b, A and B are real constants such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for each $i, 1 \leq i \leq n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Theorem 2.8. *([13]) If a_i and b_i are nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + C_1C_2\sum_{i=1}^{n} a_i^2 \leq (C_1 + C_2)\left(\sum_{i=1}^{n} a_i b_i\right), \tag{2.5}$$

where C_1 and C_2 are real constants such that $C_1 a_i \leq b_i \leq C_2 a_i$ for each $i, 1 \leq i \leq n$.

3. Degree exponent sum polynomial of graph operations

Theorem 3.1. *If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then*

$$P_{DES(G\cup H)}(\mu) = (\mu + 2r_1^{r_1})^{n_1-1}(\mu + 2r_2^{r_2})^{n_2-1}\Big((\mu - 2(n_1-1)r_1^{r_1})(\mu - 2(n_2-1)r_2^{r_2}) - n_1n_2(r_1^{r_2} + r_2^{r_1})^2\Big).$$

Proof. The graph $G \cup H$ has two types of vertices, the n_1 vertices of degree r_1 and the remaining n_2 vertices are of degree r_2. Hence,

$$DES(G\cup H) = \begin{bmatrix} 2r_1^{r_1}(J_{n_1} - I_{n_1}) & (r_1^{r_2} + r_2^{r_1})J_{n_1\times n_2} \\ (r_1^{r_2} + r_2^{r_1})J_{n_2\times n_1} & 2r_2^{r_2}(J_{n_2} - I_{n_2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(G\cup H)}(\mu) &= |\mu I - DES(G\cup H)| \\ &= \begin{vmatrix} (\mu + 2r_1^{r_1})I_{n_1} - 2r_1^{r_1}J_{n_1} & -(r_1^{r_2} + r_2^{r_1})J_{n_1\times n_2} \\ -(r_1^{r_2} + r_2^{r_1})_{n_2\times n_1} & (\mu + 2r_2^{r_2})I_{n_2} - 2r_2^{r_2}J_{n_2}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the required result. $\square$

Theorem 3.2. *If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then*

$$P_{DES(G+H)}(\mu) = (\mu + 2R_1^{R_1})^{n_1-1}(\mu + 2R_2^{R_2})^{n_2-1}\Big((\mu - 2(n_1-1)R_1^{R_1})(\mu - 2(n_2-1)R_2^{R_2}) - n_1n_2(R_1^{R_2} + R_2^{R_1})^2\Big).$$

Proof. The graph $G+H$ has two types of vertices, the n_1 vertices of degree $R_1 = r_1 + n_2$ and the remaining n_2 vertices are of degree $R_2 = r_2 + n_1$. Hence,

$$DES(G+H) = \begin{bmatrix} 2R_1^{R_1}(J_{n_1} - I_{n_1}) & (R_1^{R_2} + R_2^{R_1})J_{n_1\times n_2} \\ (R_1^{R_2} + R_2^{R_1})J_{n_2\times n_1} & 2R_2^{R_2}(J_{n_2} - I_{n_2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(G+H)}(\mu) &= |\mu I - DES(G+H)| \\ &= \begin{vmatrix} (\mu + 2R_1^{R_1})I_{n_1} - 2R_1^{R_1}J_{n_1} & -(R_1^{R_2} + R_2^{R_1})J_{n_1\times n_2} \\ -(R_1^{R_2} + R_2^{R_1})J_{n_2\times n_1} & (\mu + 2R_2^{R_2})I_{n_2} - 2R_2^{R_2}J_{n_2} \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the required result. □

Theorem 3.3. *If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then*

$$P_{DES(G\times H)}(\mu) = \Big(\mu - 2(n_1n_2 - 1)(r_1 + r_2)^{r_1+r_2}\Big)\Big(\mu + 2(r_1 + r_2)^{r_1+r_2}\Big)^{n_1n_2-1}.$$

Proof. The graph $G \times H$ is an $(r_1 + r_2)$-regular graph with n_1n_2 vertices. Hence, the result follows from Eq. (1.1). □

Theorem 3.4. *If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then*

$$P_{DES(G[H])}(\mu) = \Big(\mu - 2(n_1n_2 - 1)(n_2r_1 + r_2)^{(n_2r_1+r_2)}\Big)\Big(\mu + 2(n_2r_1 + r_2)^{(n_2r_1+r_2)}\Big)^{(n_1n_2-1)}.$$

Proof. The graph $G[H]$ is an $(n_2r_1 + r_2)$-regular graph with n_1n_2 vertices. Hence, the result follows from Eq. (1.1). □

Theorem 3.5. *If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then*

$$P_{DES(G\circ H)}(\mu) = \Big(\mu + 2R_1^{R_1}\Big)^{n_1-1}\Big(\mu + 2R_2^{R_2}\Big)^{n_1n_2-1}\Big((\mu - 2(n_1-1)R_1^{R_1})(\mu - 2(n_1n_2-1)R_2^{R_2}) - n_1^2n_2(R_1^{R_2} + R_2^{R_1})^2\Big).$$

Proof. The graph $G \circ H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and the remaining n_1n_2 vertices are of degree $R_2 = r_2 + 1$. Hence,

$$DES(G\circ H) = \begin{bmatrix} 2R_1^{R_1}(J_{n_1} - I_{n_1}) & (R_1^{R_2} + R_2^{R_1})J_{n_1\times n_1n_2} \\ (R_1^{R_2} + R_2^{R_1})J_{n_1n_2\times n_1} & 2R_2^{R_2}(J_{n_1n_2} - I_{n_1n_2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(G\circ H)}(\mu) &= |\mu I - DES(G\circ H)| \\ &= \begin{vmatrix} (\mu+2R_1^{R_1})I_{n_1} - 2R_1^{R_1}J_{n_1} & -(R_1^{R_2}+R_2^{R_1})J_{n_1\times n_1 n_2} \\ -(R_1^{R_2}+R_2^{R_1})J_{n_1 n_2\times n_1} & (\mu+2R_2^{R_2})I_{n_1 n_2} - 2R_2^{R_2}J_{n_1 n_2}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the required result. □

4. Degree exponent sum polynomial of cycle related graphs and product related graphs

Theorem 4.1. *If W_n is a wheel graph, then*

$$P_{DES(W_n)}(\mu) = (\mu+54)^{n-2}\Big(\mu\,(\mu-54(n-2)) - (n-1)\left(3^{n-1}+(n-1)^3\right)^2\Big).$$

Proof. The graph W_n of order n has two types of verices namely, $n-1$ rim vertices are of degree 3 and central vertex has degree $n-1$. Hence,

$$DES(W_n) = \begin{bmatrix} 54(J_{n-1}-I_{n-1}) & (3^{n-1}+(n-1)^3)J_{(n-1)\times 1} \\ (3^{n-1}+(n-1)^3)J_{1\times(n-1)} & 2(n-1)^2(J_1-I_1) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(W_n)}(\mu) &= |\mu I - DES(W_n)| \\ &= \begin{vmatrix} (\mu+54)I_{n-1} - 54J_{n-1} & -(3^{n-1}+(n-1)^3)J_{(n-1)\times 1} \\ -(3^{n-1}+(n-1)^3)J_{1\times(n-1)} & (\mu+2(n-1)^2)I_1 - 2(n-1)^2 J_1 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.2. *If $C_3^{(t)}$ is a friendship graph, then*

$$P_{DES(C_3^{(t)})}(\mu) = (\mu+8)^{2t-1}\Big(\mu\,(\mu-8(2t-1)) - (2t-1)\left(2^{2t}+(2t)^2\right)^2\Big).$$

Proof. The graph $C_3^{(t)}$ of order $2t+1$ has two types of verices namely, $2t$ vertices of degree 2 and 1 vertex of degree $2t$. Hence,

$$DES(C_3^{(t)}) = \begin{bmatrix} 8(J_{2t}-I_{2t}) & (2^{2t}+(2t)^2)J_{(2t)\times 1} \\ (2^{2t}+(2t)^2)J_{1\times(2t)} & 2(2t)^{2t}(J_1-I_1) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(C_3^{(t)})}(\mu) &= |\mu I - DES(C_3^{(t)})| \\ &= \begin{vmatrix} (\mu+8)I_{2t} - 8J_{2t} & -(2^{2t}+(2t)^2)J_{(n-1)\times 1} \\ -(2^{2t}+(2t)^2)J_{1\times(2t)} & 0 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.3. *If $H_n - c$ is a helm without central vertex, then*

$$P_{DES(H_n-c)}(\mu) = (\mu+54)^{n-2}(\mu+2)^{n-2}\Big(\,(\mu-54(n-2))\,(\mu-2(n-2)) - 16(n-1^2\Big).$$

Proof. The helm $H_n - c$ without central vertex is a graph of order $2(n-1)$, which has two types of vertices. The $n-1$ vertices have degree 3 and the remaining $n-1$ vertices have degree 1. Hence,

$$DES(H_n - c) = \begin{bmatrix} 54(J_{n-1} - I_{n-1}) & 4J_{(n-1)\times(n-1)} \\ 4J_{(n-1)\times(n-1)} & 2(J_{n-1} - I_{n-1}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(H_n-c)}(\mu) &= |\mu I - DES(H_n - c)| \\ &= \begin{vmatrix} (\mu+54)I_{n-1} - 54J_{n-1} & -4J_{(n-1)\times(n-1)} \\ -4J_{(n-1)\times(n-1)} & (\mu+2)I_{n-1} - 2J_{n-1} \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.4. *If $H'_n - c$ is a closed helm without central vertex, then*

$$P_{DES(H'_n-c)}(\mu) = (\mu - 54(2n-3))(\mu+54)^{2n-3}.$$

Proof. The closed helm without central vertex $H'_n - c$ is 3-regular graph with $2(n-1)$ vertices. Hence, the result follows from Eq. (1.1). □

Theorem 4.5. *If $SF_n - c$ is a sunflower graph without central vertex, then*

$$P_{DES(SF_n-c)}(\mu) = (\mu+54)^{n-2}(\mu+8)^{n-2}\Big((\mu - 54(n-2))\,(\mu - 8(n-2)) - 289(n-1)^2 \Big).$$

Proof. The sunflower graph $SF_n - c$ without central vertex is a graph of order $2(n-1)$, which has two types of vertices. The $n-1$ vertices have degree 3 and the remaining $n-1$ vertices have degree 2. Hence,

$$DES(SF_n - c) = \begin{bmatrix} 54(J_{n-1} - I_{n-1}) & 17J_{(n-1)\times(n-1)} \\ 17J_{(n-1)\times(n-1)} & 8(J_{n-1} - I_{n-1}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(SF_n-c)}(\mu) &= |\mu I - DES(SF_n - c)| \\ &= \begin{vmatrix} (\mu+54)I_{n-1} - 54J_{n-1} & -17J_{(n-1)\times(n-1)} \\ -17J_{(n-1)\times(n-1)} & (\mu+8)I_{n-1} - 8J_{n-1} \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.6. *If DC_n is a double cone, then*

$$P_{DES(DC_n)}(\mu) = (\mu+512)^{n-1}(\mu+2n^n)\Big((\mu - 512(n-1))\,(\mu - 2n^n) - 2n\,(4^n + n^4)^2 \Big).$$

Proof. The double cone is a graph of of order $n+2$ has two types of vertices. The n vertices have degree 4 and the remaining 2 vertices have degree n. Hence,

$$DES(DC_n) = \begin{bmatrix} 512(J_n - I_n) & (4^n + n^4)J_{n\times 2} \\ (4^n + n^4)J_{2\times n} & 2n^n(J_2 - I_2) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(DC_n)}(\mu) &= |\mu I - DES(DC_n)| \\ &= \begin{vmatrix} (\mu+512)I_n - 512J_n & -(4^n+n^4)J_{n\times 2} \\ -(4^n+n^4)J_{2\times n} & (\mu+2n^n)I_2 - 2n^nJ_2 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the expected result. □

Theorem 4.7. *If B_b is a book graph, then*

$$P_{DES(B_b)}(\mu) = (\mu+8)^{2b-1}(\mu+2(b+1)^{b+1})\Big((\mu - 8(2b-1))\left(\mu - 2(b+1)^{b+1}\right) - 4b\left(2^{b+1}+(b+1)^2\right)^2 \Big).$$

Proof. The Book graph B_b has two types of vertices. The $2b$ vertices with degree 2 and 2 vertices are with degree $b+1$. Hence,

$$DES(B_b) = \begin{bmatrix} 8(J_{2b}-I_{2b}) & (2^{b+1}+(b+1)^2)J_{2b\times 2} \\ (2^{b+1}+(b+1)^2)J_{2\times 2b} & 2(b+1)^{b+1}(J_2-I_2) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(B_b)}(\mu) &= |\mu I - DES(B_b)| \\ &= \begin{vmatrix} (\mu+8)I_{2b} - 8J_{2b} & -(2^{b+1}+(b+1)^2)J_{2b\times 2} \\ -(2^{b+1}+(b+1)^2)J_{2\times 2b} & (\mu+2(b+1)^{b+1})I_2 - 2(b+1)^{b+1}J_2 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.8. *If B_t is a book with triangular pages, then*

$$P_{DES(B_t)}(\mu) = (\mu+8)^{t-1}(\mu+2(t+1)^{t+1})\Big((\mu - 8(t-1))(\mu - 2(t+1)^{t+1}) - 2t(2^{t+1}+(t+1)^2)^2 \Big).$$

Proof. The book B_t with triangular pages of order $t+2$ has two types of vertices. The t vertices have degree 2 and the remaining 2 vertices have degree $t+1$. Hence,

$$DES(B_t) = \begin{bmatrix} 8(J_t-I_t) & (2^{t+1}+(t+1)^2)J_{t\times 2} \\ (2^{t+1}+(t+1)^2)J_{2\times t} & 2(t+1)^{t+1}(J_2-I_2) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(B_t)}(\mu) &= |\mu I - DES(B_t)| \\ &= \begin{vmatrix} (\mu+8)I_t - 8J_t & -(2^{t+1}+(t+1)^2)J_{t\times 2} \\ -(2^{t+1}+(t+1)^2)J_{2\times t} & (\mu+2(t+1)^{t+1})I_2 - 2(t+1)^{t+1}J_2 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the required result. □

Theorem 4.9. *If L_n is a ladder graph, then*

$$P_{DES(L_n)}(\mu) = (\mu+54)^{2n-5}(\mu+8)^3\Big((\mu - 54(2n-5))(\mu-24) - 1156(2n-4)\Big).$$

Proof. The ladder graph L_n is a graph of order $2n$ and has two types of vertices. The 4 vertices have degree 2 and $2n-4$ vertices have degree 3. Hence,

$$DES(L_n) = \begin{bmatrix} 54(J_{2n-4} - I_{2n-4}) & 17J_{(2n-4)\times 4} \\ 17J_{4\times(2n-4)} & 8(J_4 - I_4) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(L_n)}(\mu) &= |\mu I - DES(L_n)| \\ &= \begin{vmatrix} (\mu+54)I_{2n-4} - 54J_{2n-4} & -17J_{(2n-4)\times 4} \\ -17J_{4\times(2n-4)} & (\mu+8)I_4 - 8J_4 \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the expected result. □

Theorem 4.10. *If Π_n is a prism graph, then*

$$P_{DES(\Pi_n)}(\mu) = (\mu - 54(2n-1))\,(\mu+54)^{2n-1}.$$

Proof. The prism Π_n is 3-regular graph with $2n$ vertices. Hence, the result follows from Eq. (1.1). □

Theorem 4.11. *If T_n is a triangular snake, then*

$$P_{DES(T_n)}(\mu) = (\mu+8)^n(\mu+512)^{n-3}\Big((\mu-8n)\,(\mu-512(n-3)) - 1024(n+1)(n-2)\Big).$$

Proof. The triangular snake T_n has two types of vertices. The $n+1$ vertices have degree 2 and the remaining $n-2$ vertices have degree 4. Hence,

$$DES(T_n) = \begin{bmatrix} 8(J_{n+1} - I_{n+1}) & 32J_{(n+1)\times(n-2)} \\ 32J_{(n-2)\times(n+1)} & 512(J_{n-2} - I_{n-2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(T_n)}(\mu) &= |\mu I - DES(T_n)| \\ &= \begin{vmatrix} (\mu+8)I_{n+1} - 8J_{n+1} & -32J_{(n+1)\times(n-2)} \\ -32J_{(n-2)\times(n+1)} & (\mu+512)I_{n-2} - 512J_{n-2} \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 4.12. *If Q_n is a quadrilateral snake, then*

$$\begin{aligned} P_{DES(Q_n)}(\mu) = (\mu+8)^{2n-1}(\mu+512)^{n-3}\Big(&(\mu-8(2n-1))\,(\mu-512(n-3)) \\ &- 2048n(n-2)\Big). \end{aligned}$$

Proof. The quadrilateral snake Q_n is a graph of order $3n-2$, which has two types of vertices. The $2n$ vertices have degree 2 and the remaining $n-2$ vertices have degree 4. Hence,

$$DES(Q_n) = \begin{bmatrix} 8(J_{2n} - I_{2n}) & 32J_{2n\times(n-2)} \\ 32J_{(n-2)\times 2n} & 512(J_{n-2} - I_{n-2}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(Q_n)}(\mu) &= |\mu I - DES(Q_n)| \\ &= \begin{vmatrix} (\mu+8)I_{2n} - 8J_{2n} & -32J_{2n\times(n-2)} \\ -32J_{(n-2)\times 2n} & (\mu+512)I_{n-2} - 512J_{n-2} \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

5. Degree exponent sum polynomial of some transformation graphs

Theorem 5.1. *If G is an r-regular graph of order n, then*

$$P_{DES(\overline{G})}(\mu) = \left(\mu - 2(n-1)(n-1-r)^{(n-1-r)}\right)\left(\mu + 2(n-1-r)^{(n-1-r)}\right)^{n-1}.$$

Proof. The complement of an r-regular graph is an $(n-1-r)$-regular graph with n vertices. Hence, the result follows from Eq. (1.1). □

Theorem 5.2. *If G is an r-regular graph of order n and n_k is the order of $L^k(G)$ $(k = 1, 2, \dots)$, then*

$$P_{DES(L^k(G))}(\mu) = \left(\mu + 2(2^k r - 2^{k+1} + 2)^{2^k r - 2^{k+1}+2}\right)^{n_k - 1} \left(\mu - 2(n_k - 1)(2^k r - 2^{k+1} + 2)^{2^k r - 2^{k+1}+2}\right).$$

Proof. The line graph of a regular graph is a regular graph. In particular, the line graph of an r-regular graph G of order n is an $r_1 = (2r-2)$-regular graph of order $n_1 = \frac{1}{2}nr$. Thus, $L^k(G)$ is an r_k-regular graph of order n_k given by

$$n_k = \frac{n}{2^k}\prod_{i=0}^{k-1}(2^i r - 2^{i+1} + 2) \quad \text{and} \quad r_k = 2^k r - 2^{k+1} + 2.$$

Hence, the result follows from Eq. (1.1). □

Theorem 5.3. *If G is an r-regular graph of order n, then*

$$P_{DES(J(G))}(\mu) = \left(\mu - 2r_1^{r_1}\left(\tfrac{nr}{2} - 1\right)\right)\left(\mu + 2r_1^{r_1}\right)^{\left(\frac{nr}{2}-1\right)}.$$

Proof. The jump graph of an r-regular graph is $r_1 = \left(\frac{(n-4)r}{2} + 1\right)$-regular graph with $\frac{nr}{2}$ vertices. Hence, the result follows from Eq. (1.1). □

Theorem 5.4. *If G is an r-regular graph of order n, then*

$$P_{DES(S(G))}(\mu) = (\mu + 2r^r)^{n-1}(\mu+8)^{\frac{nr}{2}-1}\Big(\left(\mu - 2r^r(n-1)\right)\left(\mu - 8\left(\tfrac{nr}{2} - 1\right)\right) - \tfrac{n^2 r}{2}\left(r^2 + 2^r\right)^2\Big).$$

Proof. The subdivision graph of an r-regular graph has two types of vertices. The n vertices with degree r and $\frac{nr}{2}$ vertices with degree 2. Hence,

$$DES(S(G)) = \begin{bmatrix} 2r^r(J_n - I_n) & (r^2+2^r)J_{n\times\frac{nr}{2}} \\ (2^r+r^2)J_{\frac{nr}{2}\times n} & 8(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(S(G))}(\mu) &= |\mu I - DES(S(G))| \\ &= \begin{vmatrix} (\mu+2r^r)I_n - 2r^rJ_n & -(r^2+2^r)J_{n\times\frac{nr}{2}} \\ -(2^r+r^2)J_{\frac{nr}{2}\times n} & (\mu+8)I_{\frac{nr}{2}} - 8J_{\frac{nr}{2}}) \end{vmatrix}. \end{aligned}$$

Using Lemma 2.5, we get the desired result. □

Theorem 5.5. *If G an an r-regular graph of order n, then*

$$\begin{aligned} P_{DES(T_2(G))}(\mu) &= \left(\mu + 2(2r)^{2r}\right)^{n-1}(\mu+8)^{\frac{nr}{2}-1}\Big((\mu - 2(n-1)(2r)^{2r})(\mu \\ &\quad -8\left(\frac{nr}{2}-1\right)\Big) - \frac{n^2r}{2}\left(4r^2+2^{2r}\right)^2\Big). \end{aligned}$$

Proof. The semitotal point graph of an r-regular graph has two types of vertices. The n vertices with degree $2r$ and $\frac{nr}{2}$ vertices have degree 2. Hence,

$$DES(T_2(G)) = \begin{bmatrix} 2(2r)^{2r}(J_n - I_n) & (4r^2+2^{2r})J_{n\times\frac{nr}{2}} \\ (4r^2+2^{2r})J_{\frac{nr}{2}\times n} & 8(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(T_2(G))}(\mu) &= |\mu I - DES(T_2(G))| \\ &= \begin{vmatrix} (\mu+2(2r)^{2r})I_n - 2(2r)^{2r}J_n & -(4r^2+2^{2r})J_{n\times\frac{nr}{2}} \\ -(4r^2+2^{2r})J_{\frac{nr}{2}\times n} & (\mu+8)I_{\frac{nr}{2}} - 8J_{\frac{nr}{2}} \end{vmatrix}. \end{aligned}$$

The result follows from Lemma 2.5. □

Theorem 5.6. *If G is an r-regular graph of order n, then*

$$\begin{aligned} P_{DES(T_1(G))}(\mu) = (\mu+2r^r)^{n-1}(\mu+2(2r)^{2r})^{\frac{nr}{2}-1}\Big((\mu-2(n-1)r^r)\left(\mu - 2(\tfrac{nr}{2}-1)(2r)^{2r}\right) \\ - \tfrac{n^2r}{2}\left(r^{2r}+(2r)^r\right)^2\Big). \end{aligned}$$

Proof. The semitotal line graph of an r-regular graph has two types of vertices. The n vertices with degree r and the remaining $\frac{nr}{2}$ vertices are of degree $2r$. Hence,

$$DES(T_1(G)) = \begin{bmatrix} 2r^r(J_n - I_n) & (r^{2r}+(2r)^r)J_{n\times\frac{nr}{2}} \\ (r^{2r}+(2r)^r)J_{\frac{nr}{2}\times n} & 2(2r)^{2r}(J_{\frac{nr}{2}} - I_{\frac{nr}{2}}) \end{bmatrix}$$

and

$$\begin{aligned} P_{DES(T_1(G))}(\mu) &= |\mu I - DES(T_1(G))| \\ &= \begin{vmatrix} (\mu+2r^r)I_n - 2r^rJ_n & -(r^{2r}+(2r)^r)J_{n\times\frac{nr}{2}} \\ -(r^{2r}+(2r)^r)J_{\frac{nr}{2}\times n} & (\mu+2(2r)^{2r})I_{\frac{nr}{2}} - 2(2r)^{2r}J_{\frac{nr}{2}}) \end{vmatrix}. \end{aligned}$$

The Lemma 2.5 gives the required result. □

Theorem 5.7. *If G is an r-regular graph of order n, then*

$$P_{DES(T(G))}(\mu) = \left(\mu - 2\left(n + \frac{nr}{2} - 1\right)(2r)^{2r}\right)\left(\mu + 2(2r)^{2r}\right)^{n+\frac{nr}{2}-1}.$$

Proof. The total graph of an r-regular graph is a regular graph of degree $2r$ with $n + \frac{nr}{2}$ vertices. Hence, the result follows from Eq. (1.1). □

6. Bounds for the largest degree exponent sum eigenvalue and degree exponent sum energy

Theorem 6.1. *The degree exponent sum eigenvalues of $DES(G)$ satisfies the following relations:*

(1) $\sum_{i=1}^{n} \mu_i = 0$,
(2) $\sum_{i=1}^{n} \mu_i^2 = 2\omega$, where $\omega = \sum_{i<j}(d_i^{d_j} + d_j^{d_i})^2$.

Proof. By the definition of $DES(G)$,

$$\sum_{i=1}^{n} \mu_i = 0. \tag{6.1}$$

Further,

$$\begin{aligned} \sum_{i=1}^{n} \mu_i^2 &= trace\left((DES(G))^2\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 \\ &= 2\sum_{i<j}(d_i^{d_j} + d_j^{d_i})^2 \end{aligned}$$

$$= 2\omega, \ where\ \omega = \sum_{i<j}(d_i^{d_j} + d_j^{d_i})^2. \tag{6.2}$$

□

Theorem 6.2. *If G is an r-regular graph of order n, then G has only one positive degree exponent sum eigenvalue $\mu = 2r^r(n-1)$.*

Proof. Let G be an r-regular graph of order n and $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. If $d_i = r$ is the degree of $v_i, i = 1, 2, \ldots, n$, then

$$d_{ij} = \begin{cases} d_i^{d_j} + d_j^{d_i} = 2r^r, & if\ \ i \neq j, \\ 0, & otherwise. \end{cases}$$

The degree exponent sum polynomial of $DES(G)$ is

$$\begin{aligned} P_{DES(G)}(\mu) &= det(\mu I - DES(G)) \\ &= det(\mu I - 2r^r A(K_n)) \\ &= (2r^r)^n \left| \frac{\mu}{2r^r} I - A(K_n) \right| \\ &= (2r^r)^n \left(\frac{\mu}{2r^r} - n + 1 \right) \left(\frac{\mu}{2r^r} + 1 \right)^{n-1} \\ &= (\mu - 2r^r(n-1))(\mu + 2r^r)^{n-1}. \end{aligned}$$

The characteristic equation of $DES(G)$ is $(\mu - 2r^r(n-1))(\mu + 2r^r)^{n-1} = 0$.

This implies, $\mu = \begin{cases} 2r^r(n-1), & once, \\ -2r^r, & (n-1)\ times. \end{cases}$ □

Theorem 6.3. *If G is any graph of order n and μ_1 is the largest degree exponent sum eigenvalue, then*

$$\mu_1 \leq \sqrt{\frac{2\omega(n-1)}{n}}. \tag{6.3}$$

Proof. Substituting $a_i = 1$ and $b_i = \mu_i$ for $i = 2, 3, \ldots, n$ in inequality (2.2), we get

$$\left(\sum_{i=1}^{n} \mu_i \right)^2 \leq (n-1) \left(\sum_{i=1}^{n} \mu_i^2 \right). \tag{6.4}$$

From Eqs. (6.1) and (6.2), $\sum_{i=2}^{n} \mu_i = -\mu_1$ *and* $\sum_{i=2}^{n} \mu_i^2 = 2\omega - \mu_1^2$.
Inequality (6.4) is written as $(-\mu_1)^2 \leq (n-1)(2\omega - \mu_1^2)$.
This implies, $\mu_1 \leq \sqrt{\frac{2\omega(n-1)}{n}}$.
Equality holds if G is a regular graph. □

Theorem 6.4. *If G is an r-regular graph of order n, then $-2r^r$ and $2r^r(n-1)$ are degree exponent sum eigenvalues of G with multiplicities $(n-1)$ and 1, respectively and $E_{DES}(G) = 4r^r(n-1)$.*

Proof.

$$\begin{aligned} |\mu I - DES(G)| &= \begin{vmatrix} \mu & -2r^r & -2r^r & \ldots & -2r^r \\ -2r^r & \mu & -2r^r & \ldots & -2r^r \\ -2r^r & -2r^r & \mu & \ldots & -2r^r \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -2r^r & -2r^r & -2r^r & \ldots & \mu \end{vmatrix} \\ &= (\mu + 2r^r)^{n-1} \begin{vmatrix} \mu & -2r^r & \ldots & -2r^r \\ -1 & 1 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & \ldots & 1 \end{vmatrix} \\ &= (\mu - 2r^r(n-1))(\mu + 2r^r)^{n-1}. \end{aligned}$$

Thus, $E_{DES}(G) = 4r^r(n-1)$. □

Theorem 6.5. *If G is a graph of order n, then*

$$E_{DES}(G) \geq \sqrt{2n\omega - \frac{n^2}{4}\Big(|\mu_1| - |\mu_n|\Big)^2}, \tag{6.5}$$

where $|\mu_1|$ and $|\mu_n|$ are maximum and minimum of the absolute value of μ_i's.

Proof. Substituting $a_i = 1$ and $b_i = |\mu_i|$ in inequality (2.3), we get

$$\begin{aligned} \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\mu_i|^2 - \left(\sum_{i=1}^{n} |\mu_i|\right)^2 &\leq \frac{n^2}{4}\Big(|\mu_1| - |\mu_n|\Big)^2 \\ 2n\omega - (E_{DES}(G))^2 &\leq \frac{n^2}{4}\Big(|\mu_1| - |\mu_n|\Big)^2 \\ E_{DES}(G) &\geq \sqrt{2n\omega - \frac{n^2}{4}\Big(|\mu_1| - |\mu_n|\Big)^2}. \end{aligned}$$

□

Theorem 6.6. *If G is a graph of order n, then*

$$\sqrt{2\omega} \leq E_{DES}(G) \leq \sqrt{2n\omega}.$$

Proof. Substituting $a_i = 1$ and $b_i = \mu_i$ in inequality (2.2), we get

$$\begin{aligned} \left(\sum_{i=1}^{n} |\mu_i|\right)^2 &\leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\mu_i|^2 \\ (E_{DES}(G))^2 &\leq 2n\omega. \end{aligned}$$

This implies,

$$E_{DES}(G) \leq \sqrt{2n\omega}. \tag{6.6}$$

We have $(E_{DES}(G))^2 = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \geq \sum_{i=1}^{n} |\mu_i|^2 = 2\omega$.
This implies,

$$E_{DES}(G) \geq \sqrt{2\omega}. \tag{6.7}$$

Combining inequalities (6.6) and (6.7), we get the desired result. □

Theorem 6.7. *If G is a graph of order n and Δ' be the absolute value of the determinant of $DES(G)$, then*

$$\sqrt{2\omega + n(n-1)(\Delta')^{2/n}} \leq E_{DES}(G) \leq \sqrt{2n\omega}.$$

Proof. From the definition of degree exponent sum energy,

$$\begin{aligned}(E_{DES}(G))^2 &= \left(\sum_{i=1}^{n}|\mu_i|\right)^2 \\ &= \sum_{i=1}^{n}\mu_i^2 + 2\sum_{i<j}|\mu_i||\mu_j| \\ &= 2\omega + 2\sum_{i<j}|\mu_i||\mu_j| \\ &= 2\omega + \sum_{i\neq j}|\mu_i||\mu_j|. \qquad (6.8)\end{aligned}$$

For nonnegative numbers, the arithmetic mean is always greater than or equal to the geometric mean.

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{i=1}^{n}|\mu_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \prod_{i=1}^{n}|\mu_i|^{2/n} \\ &= (\Delta')^{2/n}.\end{aligned}$$

Therefore,

$$\sum_{i\neq j}|\mu_i||\mu_j| \geq n(n-1)(\Delta')^{2/n}. \qquad (6.9)$$

Combining Eq. (6.8) and inequality (6.9), we get

$$E_{DES}(G) \geq \sqrt{2\omega + n(n-1)(\Delta')^{2/n}}. \qquad (6.10)$$

Consider a nonnegative quantity

$$Y = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|\mu_i| - |\mu_j|\Big)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|\mu_i|^2 - |\mu_j|^2 - 2|\mu_i||\mu_j|\Big).$$

On direct expansion, we get

$$Y = n\sum_{i=1}^{n}|\mu_i|^2 + n\sum_{j=1}^{n}|\mu_j|^2 - 2\left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right).$$

From the definition of degree exponent sum energy of a graph and Eq. (6.2), we have $Y = 4n\omega - 2(E_{DES}(G))^2 \geq 0$, since $Y \geq 0$.
So,

$$E_{DES}(G) \leq \sqrt{2n\omega}. \qquad (6.11)$$

Combining inequalities (6.10) and (6.11), we get the desired result. □

Corollary 6.8. *If G is an r-regular graph of order n, then*

$$E_{DES}(G) \leq 2nr^r\sqrt{n-1}.$$

Theorem 6.9. *If G is a graph of order n and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be a nonincreasing arrangement of degree exponent sum eigenvalues, then*

$$E_{DES}(G) \geq \sqrt{2n\omega - \alpha(n)(|\mu_1| - |\mu_n|)^2}, \tag{6.12}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Substituting $a_i = |\mu_i| = b_i,\ a = |\mu_n| = b$ and $A = |\mu_1| = B$ in inequality (2.4), we get

$$\left| n\sum_{i=1}^{n} |\mu_i|^2 - \left(\sum_{i=1}^{n} |\mu_i| \right)^2 \right| \leq \alpha(n)(|\mu_1| - |\mu_n|)^2. \tag{6.13}$$

Since $E_{DES}(G) = \sum_{i=1}^{n} |\mu_i|,\ \sum_{i=1}^{n} |\mu_i|^2 = 2\omega$, we get the required result from (6.13). □

Remark 6.10. Since $\alpha(n) \leq \frac{n^2}{4}$, the lower bound (6.12) is sharper than the lower bound (6.5).

Theorem 6.11. *If G is a graph of order n and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ is a nonincreasing arrangement of degree exponent sum eigenvalues, then*

$$E_{DES}(G) \geq \frac{|\mu_1||\mu_n|n + 2\omega}{|\mu_1| + |\mu_n|},$$

where $|\mu_1|$ and $|\mu_n|$ are maximum and minimum of the absolute value of μ_i's.

Proof. Substituting $b_i = |\mu_i|, a_i = 1, C_1 = |\mu_n|$ and $C_2 = |\mu_1|$ in inequality (2.5), we get

$$\sum_{i=1}^{n} |\mu_i|^2 + |\mu_1||\mu_n| \sum_{i=1}^{n} 1^2 \leq (|\mu_1| + |\mu_n|) \left(\sum_{i=1}^{n} |\mu_i| \right). \tag{6.14}$$

Since $E_{DES}(G) = \sum_{i=1}^{n} |\mu_i| and\ \sum_{i=1}^{n} |\mu_i|^2 = 2\omega$, (6.14) gives the required result. □

7. Conclusion

In this paper, we have added a new graph matrix to literature by introducing a degree exponent sum matrix. We have computed the degree exponent sum polynomial of graph operations, cycle related graphs, product related graphs and transformation graphs. We have given extension to our work to compute bounds for the largest degree exponent sum eigenvalue and degree exponent sum energy.

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