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EXISTENCE OF EXTREMAL SOLUTIONS FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

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Abstract. This paper investigates the existence of both minimal and maximal solutions for a certain class of fractional differential equations involving the p-Laplacian operator, subject to boundary conditions. The nonlinear term is assumed to be a continuous function. The method of lower and upper solutions, combined with a monotone iterative technique, is employed to establish the existence of extremal solutions. An example is provided to illustrate the applicability of the main results.

Keywords. Fractional differential equations, upper and lower solutions method, monotone iterative technique, extremal Solution, p-Laplacian
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1. Introduction

Fractional differential equations have emerged as powerful tools for modeling complex phenomena across various disciplines of science and engineering. Their applications span areas, such as viscoelasticity, physics, biology, chemistry, biotechnology, polymer rheology, electrodynamics of complex medium, electromagnetic, population dynamics, economic, showcasing their versatility and importance, etc. For more details, see [7, 8, 13, 16, 18]. The field of FDEs has been extensively developed and is well-documented in many monographs, see [4, 10, 15]. More recently, increasing attention has been given to various classes of (FDEs) due to their ability to model memory and hereditary phenomena in numerous scientific fields such as physics, biology, and control theory. In particular, fractional problems involving nonlinearities and boundary conditions have become a subject of active research. Many existence results have been achieved through the use of several techniques such as fixed point theorems, degree theory, and the Leray-Schauder theorem (see [2, 3, 5, 6, 11, 12, 14, 17, 20, 21]).

Despite this progress, the application of the upper and lower solution method in the context of (FDEs) involving the p-Laplacian operator remains relatively limited in the literature. Moreover, problems involving nonlinear, nonlocal boundary

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conditions-especially those expressed as functionals-present additional analytical challenges that merit further investigation.

In this work, we study the existence of extremal solutions, namely minimal and maximal solutions, to the following FDE:

$$\begin{cases} -D_{0^+}^{\varrho}(\varphi_p(\mathfrak{g}')) = f(\varsigma, \mathfrak{g}),\ \varsigma \in [0,1], \\ \mathfrak{g}(0) = 0, \\ a_1\mathfrak{g}(1) + a_2\mathfrak{g}'(1) = L(\mathfrak{g}), \end{cases} \tag{1.1}$$

where $\varphi_p(\mathfrak{g}) = |\mathfrak{g}|^{p-2}.\mathfrak{g}$, such that $p > 1$ and $D_{0^+}^{\varrho}$ is the Riemann-Liouville fractional derivative such that $1 < \varrho \leq 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $L : \mathcal{C}([0,1]) \to \mathbb{R}$ is continuous and increasing function. Lastly, let $a_1, a_2 \in \mathbb{R}^+$. An additional advantage of considering boundary conditions of the form

$$a_1\mathfrak{g}(1) + a_2\mathfrak{g}'(1) = L(\mathfrak{g}),$$

where L has been previously defined is that they encompass a broad range of classical and nonlocal boundary conditions. For instance multi-point boundary conditions, integral boundary conditions and weighted integral boundary conditions.

Our analysis is based on the method of lower and upper solutions. Under appropriate conditions on the nonlinearity $f(\varsigma, \mathfrak{g})$, we establish the existence of extremal solutions. An illustrative example is also provided to validate the theoretical results.

The upper and lower solutions method plays a key role in establishing existence results for various BVPs, especially for proving the existence of solutions. This technique has been widely applied to various classes of differential equations, including multi-point and nonlinear problems involving integer-order derivatives (see, [9, 11, 12, 19, 21]).

The remainder of the paper is organized as follows. In Section 2, we recall some necessary preliminaries and definitions from fractional calculus. Section 3 is devoted to the main existence results. In Section 4, we provide an example to demonstrate the applicability of our results.

2. Preliminary results

In this section, we recall some fundamental definitions and results that will be used throughout this paper.

Definition 2.1. [15] Let $\varrho > 0$. The Riemann-Liouville fractional integral of order ϱ corresponding to a function $\Psi : (0, +\infty) \longrightarrow \mathbb{R}$ is expressed as

$$I_{0^+}^{\varrho}\Psi(\varsigma) = \frac{1}{\Gamma(\varrho)} \int_0^{\varsigma} (\varsigma - s)^{\varrho-1}\Psi(s)\, ds,$$

Definition 2.2. [15] Let $\varrho > 0$. The Riemann-Liouville fractional derivative of order ϱ corresponding to a function $\Psi : (0, +\infty) \longrightarrow \mathbb{R}$ is formulated as

$$D_{0^+}^{\varrho}\Psi(\varsigma) = \frac{1}{\Gamma(n-\varrho)} \Big(\frac{d}{d\varsigma}\Big)^n \int_0^{\varsigma} (\varsigma - s)^{n-\varrho-1}\Psi(s)\, ds.$$

Here $n=[\varrho]+1$, where $[\varrho]$ denotes the integer part of ϱ, the expression on the right-hand side is assumed to be well-defined pointwise on the interval $(0,+\infty)$

Lemma 2.3. [15] *We have*

$$I^{\varrho}_{0^+}D^{\varrho}_{0^+}\Psi(\varsigma)=\Psi(\varsigma)+c_1\varsigma^{\varrho-1}+c_2\varsigma^{\varrho-2}+\cdots+c_n\varsigma^{\varrho-n},$$

with $c_i\in\mathbb{R}$; $i=1,2,...,n$ *such that* $n-1<\varrho\leq n$.

Definition 2.4. Let $\mathcal{C}(I,\mathbb{R})$ denote the Banach space consisting of all continuous real-valued functions defined on $I:=[0,1]$ endowed with the supremum norm

$$\|\mathfrak{g}\|_\infty=\max_{0\leq\varsigma\leq1}|\mathfrak{g}(\varsigma)|.$$

We proceed by examining the following problem:

$$\begin{cases}-D^{\varrho}_{0^+}(\varphi_p(\mathfrak{g}'))=-\mathfrak{R}(\varsigma,\mathfrak{g}),\ \varsigma\in[0,1]\,,\\ \mathfrak{g}(0)=0,\\ \mathfrak{g}(1)+a\mathfrak{g}^{'}(1)=\mu,\end{cases}\tag{2.1}$$

where $\mathfrak{R}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and exhibits strict monotonicity in its second variable, with a is a positive constant, $\mu\in\mathbb{R}$.

Lemma 2.5. *(Weak comparison principle)*

Let $\mathfrak{g}_1,\mathfrak{g}_2\in\mathcal{C}^1([0,1])$, $\varphi_p(\mathfrak{g}_1^{'}),\varphi_p(\mathfrak{g}_2^{'})\in\mathcal{C}([0,1])$, *so*

$$\begin{cases}-D^{\varrho}_{0^+}(\varphi_p(\mathfrak{g}_1^{'}))+\mathfrak{R}(\varsigma,\mathfrak{g}_1)\leq-D^{\varrho}_{0^+}(\varphi_p(\mathfrak{g}_2^{'}))+\mathfrak{R}(\varsigma,\mathfrak{g}_2),\ \varsigma\in[0,1]\,,\\ \mathfrak{g}_1(0)\leq\mathfrak{g}_2(0),\\ \mathfrak{g}_1(1)+a\mathfrak{g}_1^{'}(1)\leq\mathfrak{g}_2(1)+a\mathfrak{g}_2^{'}(1),\end{cases}\tag{2.2}$$

then $\mathfrak{g}_1(\varsigma)\leq\mathfrak{g}_2(\varsigma)$, *for all* $\varsigma\in[0,1]$.

Proof. Suppose that there exists $\varsigma_0\in[0,1]$ for which

$$\mathfrak{X}(\varsigma_0):=\mathfrak{g}_2(\varsigma_0)-\mathfrak{g}_1(\varsigma_0)=\min\{\mathfrak{X}(\varsigma):0\leq\varsigma\leq1\}<0,$$

and

$$\mathfrak{X}(\varsigma)>\mathfrak{X}(\varsigma_0),\text{ for all }\varsigma\in(\varsigma_0,1].\tag{2.3}$$

The analysis is divided into the following cases:

Case 01:

If $\varsigma_0=0$, this leads to a contradiction

$$0>\mathfrak{g}_2(0)-\mathfrak{g}_1(0)=0.$$

Case 02:

If $\varsigma_0=1$, gives this contradiction

$$0>\mathfrak{g}_2(1)-\mathfrak{g}_1(1)\geq-a(\mathfrak{g}_2^{'}(1)-\mathfrak{g}_1^{'}(1))=0.$$

Case 03: If $\varsigma_0\in(0,1)$, we have three subcases.

(i) If $\mathfrak{X}^{'}(\varsigma_0)>0$, and since $\mathfrak{X}^{'}\in\mathcal{C}^1([0,1])$, it can be concluded that there exists $\delta>0$ with $\mathfrak{X}^{'}(\varsigma)>0$ on $(\varsigma_0-\delta,\varsigma_0]$, then $\mathfrak{X}$ is increasing on $(\varsigma_0-\delta,\varsigma_0]$. This outcome conflicts with the definition of ς_0.

(ii) If $\mathfrak{X}^{'}(\varsigma_0)<0$, and since $\mathfrak{X}^{'}\in\mathcal{C}^1([0,1])$, it follows that there exists $\delta>0$ such that $\mathfrak{X}^{'}(\varsigma)<0$ on $[\varsigma_0,\varsigma_0+\delta)$, which means that, $\mathfrak{X}$ is decreasing on $[\varsigma_0,\varsigma_0+\delta)$. This outcome conflicts with the definition of ς_0.

(iii) If $\mathfrak{X}'(\varsigma_0) = 0$, so $\mathfrak{g}_1'(\varsigma_0) = \mathfrak{g}_2'(\varsigma_0)$ and then $\varphi_p(\mathfrak{g}_1'(\varsigma_0)) = (\varphi_p(\mathfrak{g}_2'(\varsigma_0))$. The strict monotonicity of φ_p leads to

$$D_0^{\varrho}[-(\varphi_p(\mathfrak{g}_2')) + (\varphi_p(\mathfrak{g}_1'))](\varsigma_0) \quad = \quad 0.$$

Here, we observe that

$$-D_0^{\varrho}(\varphi_p(\mathfrak{g}_2'(\varsigma_0))) + \mathfrak{R}(\varsigma_0, \mathfrak{g}_2) + D_0^{\varrho}(\varphi_p(\mathfrak{g}_1'(\varsigma_0))) - \mathfrak{R}(\varsigma_0, \mathfrak{g}_1) \geq 0.$$

This indicates that

$$-D_0^{\varrho}(\varphi_p(\mathfrak{g}_2'(\varsigma_0))) + D_0^{\varrho}(\varphi_p(\mathfrak{g}_1')(\varsigma_0)) \geq \mathfrak{R}(\varsigma_0, \mathfrak{g}_1) - \mathfrak{R}(\varsigma_0, \mathfrak{g}_2).$$

Since $\mathfrak{g}_2(\varsigma_0) < \mathfrak{g}_1(\varsigma_0)$ with the strict increase of $\mathfrak{R}$ in its second parameter, and since $\mathfrak{g}_1'(\varsigma_0) = \mathfrak{g}_2'(\varsigma_0)$, we obtain

$$\begin{aligned} -D_a^{\varrho}(\varphi_p(\mathfrak{g}_2'(\varsigma_0))) + D_a^{\varrho}(\varphi_p(\mathfrak{g}_1'(\varsigma_0))) \quad &= \quad D_0^{\varrho}[-(\varphi_p(\mathfrak{g}_2')) + (\varphi_p(\mathfrak{g}_1'))](\varsigma_0) \\ &> \quad 0. \end{aligned}$$

This leads to a contradiction.

□

Definition 2.6. ϕ is lower solution of (2.1) if

(i) $\phi \in \mathcal{C}^1([0,1])$, $\varphi_p(\phi') \in \mathcal{C}^1([0,1])$.

(ii) $\begin{cases} -D_{0^+}^{\varrho}(\varphi_p(\phi')) \leq -\mathfrak{R}(\varsigma, \phi),\ \varsigma \in [0,1]\,, \\ \phi(0) \leq 0, \\ \phi(1) + a\phi'(1) \leq \mu. \end{cases}$

Definition 2.7. ψ is upper solution of (2.1) if

(i) $\psi \in \mathcal{C}^1([0,1])$, $\varphi_p(\psi') \in \mathcal{C}^1([0,1])$.

(ii) $\begin{cases} -D_{0^+}^{\varrho}(\varphi_p(\psi')) \geq -\mathfrak{R}(\varsigma, \psi),\ \varsigma \in [0,1]\,, \\ \psi(0) \geq 0, \\ \psi(1) + a\psi'(1) \geq \mu. \end{cases}$

Theorem 2.8. *Assume that ϕ and ψ are, respectively, lower and upper solutions of (2.1) such that $\phi(\varsigma) \leq \psi(\varsigma)$, $\forall \varsigma \in [0,1]$, so the problem (2.1) admits unique solution $\mathfrak{g} \in \mathcal{C}^1([0,1])$ with $\varphi_p(\mathfrak{g}') \in \mathcal{C}^1([0,1])$ such that*

$$\phi(\varsigma) \leq \mathfrak{g}(\varsigma) \leq \psi(\varsigma),\ \forall \varsigma \in [0,1]\,.$$

Proof. Employing a method analogous to that used in Theorem 3.1 of [1], we establish the existence of at least one solution to problem (2.1). Furthermore, by applying Lemma 2.5, we conclude that this solution is unique

□

3. Main results

In this section, we need to present some definitions to demonstrate our results.

Definition 3.1. $\underline{\mathfrak{g}}$ is a lower solution of (1.1) if

(i) $\underline{\mathfrak{g}} \in \mathcal{C}^1([0,1])$, $\varphi_p(\underline{\mathfrak{g}}^{'}) \in \mathcal{C}^1([0,1])$.

(ii) $\begin{cases} -D^{\varrho}_{0^+}(\varphi_p(\underline{\mathfrak{g}}^{'})) \leq f(\varsigma, \underline{\mathfrak{g}}), \, \varsigma \in [0,1], \\ \underline{\mathfrak{g}}(0) \leq 0, \\ a_1\underline{\mathfrak{g}}(1) + a_2\underline{\mathfrak{g}}^{'}(1) \leq L(\underline{\mathfrak{g}}). \end{cases}$

Definition 3.2. $\overline{\mathfrak{g}}$ is an upper solution of (1.1) if

(i) $\overline{\mathfrak{g}} \in \mathcal{C}^1([0,1])$, $\varphi_p(\overline{\mathfrak{g}}^{'}) \in \mathcal{C}^1([0,1])$.

(ii) $\begin{cases} -D^{\varrho}_{0^+}(\varphi_p(\overline{\mathfrak{g}}^{'})) \geq f(\varsigma, \overline{\mathfrak{g}}), \, \varsigma \in [0,1], \\ \overline{\mathfrak{g}}(0) \geq 0, \\ a_1\overline{\mathfrak{g}}(1) + a_2\overline{\mathfrak{g}}^{'}(1) \geq L(\overline{\mathfrak{g}}). \end{cases}$

In light of the nonlinearity of f, we consider the following assumption:
$(\mathcal{H})$ There exists a continuous and strictly increasing function $h : \mathcal{C}([0,1]) \to \mathbb{R}$ such that for every $\varsigma \in [0,1]$ the mapping $\mathfrak{g} \longmapsto f(\varsigma, \mathfrak{g}) + h(\mathfrak{g})$ is increasing with respect to the second argument of f.

The principal result established in this study is presented in the following theorem.

Theorem 3.3. *Suppose that hypothesis $(\mathcal{H})$ holds and $\underline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}$ are the lower and upper solutions respectively of problem (1.1) satisfying $\underline{\mathfrak{g}} \leq \overline{\mathfrak{g}}$ on $[0,1]$. Thus problem (1.1) possesses a minimal solution $\mathfrak{g}_{\min}$ and maximal solution $\mathfrak{g}_{\max}$ such that for any solution $\mathfrak{g}$ of (1.1) with $\underline{\mathfrak{g}} \leq \mathfrak{g} \leq \overline{\mathfrak{g}}$ on $[0,1]$, the following holds:*

$$\underline{\mathfrak{g}} \leq \mathfrak{g}_{\min} \leq \mathfrak{g} \leq \mathfrak{g}_{\max} \leq \overline{\mathfrak{g}} \quad on \quad [0,1].$$

To establish the main theorem, we first present the following principal lemma. Let $\underline{\mathfrak{X}}$ and $\overline{\mathfrak{X}} \in \mathcal{C}^1([0,1])$ fixed such that

(i) $\varphi_p(\underline{\mathfrak{X}}^{'})$, $\varphi_p\left(\overline{\mathfrak{X}}^{'}\right) \in C^1([0,1])$,

(ii) $\underline{\mathfrak{g}} \leq \underline{\mathfrak{X}} \leq \overline{\mathfrak{X}} \leq \overline{\mathfrak{g}}$ in $[0,1]$.

We consider the following problems:

$$\begin{cases} -D^{\varrho}_{0^+}(\varphi_p(\mathfrak{g}^{'})) + h(\mathfrak{g}) = f(\varsigma, \overline{\mathfrak{X}}) + h(\overline{\mathfrak{X}}), \, \varsigma \in [0,1], \\ \mathfrak{g}(0) = 0, \\ a_1\mathfrak{g}(1) + a_2\mathfrak{g}^{'}(1) = L(\overline{\mathfrak{X}}), \end{cases} \tag{3.1}$$

and

$$\begin{cases} -D^{\varrho}_{0^+}(\varphi_p(\mathfrak{g}^{'})) + h(\mathfrak{g}) = f(\varsigma, \underline{\mathfrak{X}}) + h(\underline{\mathfrak{X}}), \, \varsigma \in [0,1], \\ \mathfrak{g}(0) = 0, \\ a_1\mathfrak{g}(1) + a_2\mathfrak{g}^{'}(1) = L(\underline{\mathfrak{X}}). \end{cases} \tag{3.2}$$

Lemma 3.4. *Consider $\underline{\mathfrak{X}}$ and $\overline{\mathfrak{X}}$ be a lower and upper solutions respectively of problem (1.1), and suppose that $(\mathcal{H})$ is fulfilled. Therefore, there exist unique solution $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}}$ to (3.1) and (3.2) respectively, such that*

$$\underline{\mathfrak{g}} \leq \underline{\mathfrak{X}} \leq \widetilde{\mathfrak{g}} \leq \widehat{\mathfrak{g}} \leq \overline{\mathfrak{X}} \leq \overline{\mathfrak{g}} \quad on \quad [0,1] .$$

Proof. We present the proof through a sequence of steps.

Step 01: $\underline{\mathfrak{X}}$ is a lower solution of (3.1).

If $\varsigma \in (0,1)$, we get

$$\begin{aligned} -D^{\varrho}_{0^+}(\varphi_p(\underline{\mathfrak{X}}')) + h(\underline{\mathfrak{X}}) &\leq f(\varsigma, \underline{\mathfrak{X}}) + h(\underline{\mathfrak{X}}), \\ &\leq f(\varsigma, \overline{\mathfrak{X}}) + h(\overline{\mathfrak{X}}). \end{aligned}$$

It follows that

$$\forall \varsigma \in (0,1){:} -D^{\varrho}_{0^+}(\varphi_p(\underline{\mathfrak{X}}')) + h(\underline{\mathfrak{X}}) \leq f(\varsigma, \overline{\mathfrak{X}}) + h(\overline{\mathfrak{X}}). \tag{3.3}$$

Moreover, it holds that

$$\underline{\mathfrak{X}}(0) \leq 0, \tag{3.4}$$

and we have

$$a_1\underline{\mathfrak{X}}(1) + a_2\underline{\mathfrak{X}}'(1) \leq L(\overline{\mathfrak{X}}). \tag{3.5}$$

By (3.3), (3.4) and (3.5) it follows that $\underline{\mathfrak{X}}$ is a lower solution of (3.1).

Step 02: $\overline{\mathfrak{X}}$ is an upper solution of problem (3.1).

Let $\varsigma \in (0,1)$ then

$$-D^{\varrho}_{0^+}(\varphi_p(\overline{\mathfrak{X}}')) + h(\overline{\mathfrak{X}}) \geq f(\varsigma, \overline{\mathfrak{X}}) + h(\overline{\mathfrak{X}}).$$

we get

$$\forall \varsigma \in (0,1) : -D^{\varrho}_{0^+}(\varphi_p(\overline{\mathfrak{X}}')) + h(\overline{\mathfrak{X}}) \geq f(\varsigma, \overline{\mathfrak{X}}) + h(\overline{\mathfrak{X}}). \tag{3.6}$$

Furthermore, we have

$$\overline{\mathfrak{X}}(0) \geq 0. \tag{3.7}$$

and

$$a_1\overline{\mathfrak{X}}(1) + a_2\overline{\mathfrak{X}}'(1) \geq L(\overline{\mathfrak{X}}). \tag{3.8}$$

Then, by (3.6), (3.7) and (3.8), it follows that $\overline{\mathfrak{X}}$ is an upper solution of (3.1).

From **Steps** 01 and 02, and given that the mapping $(\varsigma, \mathfrak{g}) \longmapsto f(\varsigma, \mathfrak{g}) + h(\mathfrak{g})$ is continuous, bounded, and strictly increasing, we apply Theorem 2.8, to conclude that problem (3.1) admits a unique solution $\widehat{\mathfrak{g}}$ satisfying

$$\underline{\mathfrak{X}} \leq \widehat{\mathfrak{g}} \leq \overline{\mathfrak{X}}.$$

Employing the same reasoning, we can conclude the existence and uniqueness of a solution to problem (3.2), denoted by $\widetilde{\mathfrak{g}}$ with

$$\underline{\mathfrak{X}} \leq \widetilde{\mathfrak{g}} \leq \overline{\mathfrak{X}}.$$

Lastly, by applying a demonstration analogous to that of Lemma 2.5, we conclude that $\widetilde{\mathfrak{g}} \leq \widehat{\mathfrak{g}}$ on $[0,1]$. $\square$

Proof. The demonstration of Theorem 3.3 is structured in the following steps.

We have $\underline{\mathfrak{g}}_0 = \underline{\mathfrak{g}}$, $\overline{\mathfrak{g}}_0 = \overline{\mathfrak{g}}$ and let's present the sequences $(\underline{\mathfrak{g}}_n)_{n\in\mathbb{N}^*}$ and $(\overline{\mathfrak{g}}_n)_{n\in\mathbb{N}^*}$ by

$$\begin{cases} -D^{\varrho}_{0+}(\varphi_p(\underline{\mathfrak{g}}'_{n+1})) + h(\underline{\mathfrak{g}}_{n+1}) = f(\varsigma, \underline{\mathfrak{g}}_n) + h(\underline{\mathfrak{g}}_n),\ \varsigma \in [0,1], \\ \underline{\mathfrak{g}}_{n+1}(0) = 0, \\ a_1\underline{\mathfrak{g}}_{n+1}(1) + a_2\underline{\mathfrak{g}}'_{n+1}(1) = L(\underline{\mathfrak{g}}_n), \end{cases} \tag{3.9}$$

and

$$\begin{cases} -D^{\varrho}_{0+}(\varphi_p(\overline{\mathfrak{g}}'_{n+1})) + h(\overline{\mathfrak{g}}_{n+1}) = f(\varsigma, \overline{\mathfrak{g}}_n) + h(\overline{\mathfrak{g}}_n),\ \varsigma \in [0,1], \\ \overline{\mathfrak{g}}_{n+1}(0) = 0, \\ a_1\overline{\mathfrak{g}}_{n+1}(1) + a_2\overline{\mathfrak{g}}'_{n+1}(1) = L(\overline{\mathfrak{g}}_n). \end{cases} \tag{3.10}$$

Step 1. We show that

$$\forall n \in \mathbb{N}: \underline{\mathfrak{g}} \le \underline{\mathfrak{g}}_n \le \underline{\mathfrak{g}}_{n+1} \le \overline{\mathfrak{g}}_{n+1} \le \overline{\mathfrak{g}}_n \le \overline{\mathfrak{g}}, \text{ on } [0,1].$$

(i) Firstly, for $n = 0$:

$$\begin{cases} -D^{\varrho}_{0+}(\varphi_p(\underline{\mathfrak{g}}'_1)) + h(\underline{\mathfrak{g}}_1) = f(\varsigma, \underline{\mathfrak{g}}_0) + h(\underline{\mathfrak{g}}_0), \varsigma \in [0,1], \\ \underline{\mathfrak{g}}_1(0) = 0, \\ a_1\underline{\mathfrak{g}}_1(1) + a_2\underline{\mathfrak{g}}'_1(1) = L(\underline{\mathfrak{g}}_0), \end{cases} \tag{3.11}$$

and

$$\begin{cases} -D^{\varrho}_{0+}(\varphi_p(\overline{\mathfrak{g}}'_1)) + h(\overline{\mathfrak{g}}_1) = f(\varsigma, \overline{\mathfrak{g}}_0) + h(\overline{\mathfrak{g}}_0),\ \varsigma \in [0,1], \\ \overline{\mathfrak{g}}_1(0) = 0, \\ a_1\overline{\mathfrak{g}}_1(1) + a_2\overline{\mathfrak{g}}'_1(1) = L(\overline{\mathfrak{g}}_0). \end{cases} \tag{3.12}$$

Since $\underline{\mathfrak{g}}$, $\overline{\mathfrak{g}}$ are a lower and upper solutions for problem (1.1), and with the employ of Lemma 3.4, we get

$$\underline{\mathfrak{g}} \le \underline{\mathfrak{g}}_0 \le \underline{\mathfrak{g}}_1 \le \overline{\mathfrak{g}}_1 \le \overline{\mathfrak{g}}_0 \le \overline{\mathfrak{g}}, \text{ in } [0,1].$$

(ii) Suppose $n > 1$ fixed, then

$$\underline{\mathfrak{g}} \le \underline{\mathfrak{g}}_{n-1} \le \underline{\mathfrak{g}}_n \le \overline{\mathfrak{g}}_n \le \overline{\mathfrak{g}}_{n-1} \le \overline{\mathfrak{g}}, \text{ in } [0,1],$$

and we prove that

$$\underline{\mathfrak{g}} \le \underline{\mathfrak{g}}_n \le \underline{\mathfrak{g}}_{n+1} \le \overline{\mathfrak{g}}_{n+1} \le \overline{\mathfrak{g}}_n \le \overline{\mathfrak{g}}, \text{ in } [0,1].$$

Let $\varsigma \in (0,1)$, we have

$$-D^{\varrho}_{0+}(\varphi_p(\overline{\mathfrak{g}}'_{n+1})) + h(\overline{\mathfrak{g}}_{n+1}) = f(\varsigma, \overline{\mathfrak{g}}_n) + h(\overline{\mathfrak{g}}_n), \tag{3.13}$$

such as $\overline{\mathfrak{g}}_{n-1} > \overline{\mathfrak{g}}_n$, and by employing the hypothesis $(\mathcal{H})$, we get

$$f(\varsigma, \overline{\mathfrak{g}}_{n-1}) + h(\overline{\mathfrak{g}}_{n-1}) \ge f(\varsigma, \overline{\mathfrak{g}}_n) + h(\overline{\mathfrak{g}}_n), \tag{3.14}$$

so, by (3.13) and (3.14), it follows that

$$\forall \varsigma \in [0,1]: \ -D^{\varrho}_{0+}(\varphi_p(\overline{\mathfrak{g}}'_n)) \ge f(\varsigma, \overline{\mathfrak{g}}_n).$$

On the other hand, we have

$$\begin{aligned} \overline{\mathfrak{g}}_n(0) &= 0, \\ &\ge 0. \end{aligned} \tag{3.15}$$

and

$$a_1\overline{\mathfrak{g}}_n(1)+a_2\overline{\mathfrak{g}}_n^{'}(1) = L(\overline{\mathfrak{g}}_{n-1}), \tag{3.16}$$
$$\geq L(\overline{\mathfrak{g}}_n).$$

By (3.14), (3.15) and (3.16), it follows that $\overline{\mathfrak{g}}_n$ is an upper solution of problem (1.1). With the same method, it can be shown that $\underline{\mathfrak{g}}_n$ is a lower solution of (1.1). Hence, by applying Lemma 3.4, we deduce that there exist unique solutions $\underline{\mathfrak{g}}_{n+1}$ and $\overline{\mathfrak{g}}_{n+1}$ to problem (3.9) and (3.10) such that

$$\underline{\mathfrak{g}} \leq \underline{\mathfrak{g}}_n \leq \underline{\mathfrak{g}}_{n+1} \leq \overline{\mathfrak{g}}_{n+1} \leq \overline{\mathfrak{g}}_n \leq \overline{\mathfrak{g}}, \text{ on } [0,1].$$

As a result, we get

$$\forall n\in\mathbb{N}:\ \underline{\mathfrak{g}} \leq \underline{\mathfrak{g}}_n \leq \underline{\mathfrak{g}}_{n+1} \leq \overline{\mathfrak{g}}_{n+1} \leq \overline{\mathfrak{g}}_n \leq \overline{\mathfrak{g}}, \text{ on } [0,1].$$

Step 2. Let $C_1>0$ be a constant independent of the index $n\in\mathbb{N}$, and it verifies:

$$\|\overline{\mathfrak{g}}_n^{'}\|_\infty \leq C_1, \forall n\in\mathbb{N}.$$

Let $n\in\mathbb{N}$ and $\varsigma\in[0,1]$, by the lemma 2.3, we have:

$$-I_0^{\varrho}D_{0^+}^{\varrho}(\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'}(\varsigma))) = I_0^{\varrho}\Big[f(\varsigma,\overline{\mathfrak{g}}_n(\varsigma))+h(\overline{\mathfrak{g}}_n(\varsigma))-h(\overline{\mathfrak{g}}_{n+1}(\varsigma))\Big],$$

so

$$-\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'})(\varsigma) = c_1\varsigma^{\varrho-1}+c_2\varsigma^{\varrho-2}+\frac{1}{\Gamma(\varrho)}\int_0^{\varsigma}(\varsigma-s)^{\varrho-1}\Big[f(s,\overline{\mathfrak{g}}_n(s))+h(\overline{\mathfrak{g}}_n(s))-h(\overline{\mathfrak{g}}_{n+1}(s))\Big]ds.$$

The condition $\overline{\mathfrak{g}}_{n+1}(0)=0$ implies that $c_2=0$, therefore

$$\begin{aligned}
\Big|\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'})(\varsigma)\Big| &= \left|c_1\varsigma^{\varrho-1}+\frac{1}{\Gamma(\varrho)}\int_0^{\varsigma}(\varsigma-s)^{\varrho-1}\Big[f(s,\overline{\mathfrak{g}}_n(s))+h(\overline{\mathfrak{g}}_n(s))-h(\overline{\mathfrak{g}}_{n+1}(s))\Big]ds\right| \\
&\leq \left|c_1\varsigma^{\varrho-1}\right| + \left|\frac{1}{\Gamma(\varrho)}\int_0^{\varsigma}(\varsigma-s)^{\varrho-1}\Big[f(s,\overline{\mathfrak{g}}_n(s))+h(\overline{\mathfrak{g}}_n(s))-h(\overline{\mathfrak{g}}_{n+1}(s))\Big]ds\right| \\
&\leq |c_1| + \frac{M_1(f)+2M_2(h)}{\Gamma(\varrho)}\left|\int_0^{\varsigma}(\varsigma-s)^{\varrho-1}ds\right| \\
&\leq |c_1| + \frac{M_1(f)+2M_2(h)}{\Gamma(\varrho+1)},
\end{aligned}$$

where

$$M_1(f) = \max\left\{|f(\varsigma,\mathfrak{g})|:\ \varsigma\in[0,1],\ \underline{\mathfrak{g}}\leq\mathfrak{g}\leq\overline{\mathfrak{g}}\right\},$$

and

$$M_2(h) = \max\left\{|h(\mathfrak{g})|:\underline{\mathfrak{g}}\leq\mathfrak{g}\leq\overline{\mathfrak{g}}\right\}.$$

Consequently, by setting

$$C_1 = |c_1| + \frac{M_1(f) + 2M_2(h)}{\Gamma(\varrho+1)},$$

we obtain

$$\|\overline{\mathfrak{g}}'_{n+1}\|_\infty \leq C_1.$$

Step 3. Let $C_2 > 0$ be a constant independent of the index $n \in \mathbb{N}$, with:

$$\|\underline{\mathfrak{g}}'_n\|_\infty \leq C_2, \forall n \in \mathbb{N}.$$

Since the proof follows the same arguments as in **Step 2** , it is omitted here.

Step 4. $(\overline{\mathfrak{g}}_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

For this, let $\epsilon > 0$ such that

$$\eta = \frac{1}{2}\min\left\{ \left(\frac{\epsilon}{2|c_1|}\right)^{\frac{1}{\varrho-1}}, \left(\frac{\epsilon\Gamma(\varrho+1)}{2\big[M_1(f)+2M_2(h)\big]}\right)^{\frac{1}{\varrho}} \right\}.$$

Then, for any $\varsigma_1, \varsigma_2 \in [0,1]$ satisfying $\varsigma_1 < \varsigma_2$ with $\varsigma_2 - \varsigma_1 < \eta$, accordingly

$$\begin{aligned}
&\left|\varphi_p(\overline{\mathfrak{g}}'_{n+1}(\varsigma_2)) - \varphi_p(\overline{\mathfrak{g}}'_{n+1}(\varsigma_1))\right| \\
= & \left| c_1\varsigma_2^{\varrho-1} - c_1\varsigma_1^{\varrho-1} + \frac{1}{\Gamma(\varrho)}\int_0^{\varsigma_2} (\varsigma_2 - s)^{\varrho-1}\left[f(s,\overline{\mathfrak{g}}_n(s)) + h(\overline{\mathfrak{g}}_n(s)) - h(\overline{\mathfrak{g}}_{n+1}(s))\right]\, ds \right. \\
& \left. -\frac{1}{\Gamma(\varrho)}\int_0^{\varsigma_1} (\varsigma_1 - s)^{\varrho-1}\left[f(s,\overline{\mathfrak{g}}_n(s)) + h(\overline{\mathfrak{g}}_n(s)) - h(\overline{\mathfrak{g}}_{n+1}(s))\right]\, ds \right| \\
\leq & \ |c_1|(\varsigma_2^{\varrho-1} - \varsigma_1^{\varrho-1}) + \left| \int_0^{\varsigma_1} \frac{\left[(\varsigma_2 - s)^{\varrho-1} - (\varsigma_1 - s)^{\varrho-1}\right]}{\Gamma(\varrho)}\left[f(s,\overline{\mathfrak{g}}_n(s)) + h(\overline{\mathfrak{g}}_n(s)) - h(\overline{\mathfrak{g}}_{n+1}(s))\right]\, ds \right. \\
& \left. -\frac{1}{\Gamma(\varrho)}\int_{\varsigma_1}^{\varsigma_2} (\varsigma_2 - s)^{\varrho-1}\left[f(s,\overline{\mathfrak{g}}_n(s)) + h(\overline{\mathfrak{g}}_n(s)) - h(\overline{\mathfrak{g}}_{n+1}(s))\right]\, ds \right| \\
\leq & \ |c_1|(\varsigma_2^{\varrho-1} - \varsigma_1^{\varrho-1}) + \frac{M_1(f) + 2M_2(h)}{\Gamma(\varrho)}\int_0^{\varsigma_1} \left[(\varsigma_2 - s)^{\varrho-1} - (\varsigma_1 - s)^{\varrho-1}\right]\, ds \\
& + \frac{M_1(f) + 2M_2(h)}{\Gamma(\varrho)}\int_{\varsigma_1}^{\varsigma_2} (\varsigma_2 - s)^{\varrho-1}\, ds \\
\leq & \ |c_1|(\varsigma_2^{\varrho-1} - \varsigma_1^{\varrho-1}) + \frac{M_1(f) + 2M_2(h)}{\Gamma(\varrho+1)}(\varsigma_2^{\varrho} - \varsigma_1^{\varrho}).
\end{aligned}$$

To estimate the quantities $\varsigma_2^{\varrho-1}-\varsigma_1^{\varrho-1}$ and $\varsigma_2^{\varrho}-\varsigma_1^{\varrho}$, we employ an approach similar to the one used in [2]. We have

Case 01. $\eta \leq \varsigma_1 < \varsigma_2 < 1$:

$$\begin{aligned}\left|\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'}(\varsigma_2))-\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'}(\varsigma_1))\right| &\leq |c_1|\frac{\varrho-1}{\eta^{2-\varrho}}(\varsigma_2-\varsigma_1)+\frac{M_1(f)+2M_2(h)}{\Gamma(\varrho+1)}\varrho(\varsigma_2-\varsigma_1),\\ &\leq |c_1|(\varrho-1)\eta^{\varrho-1}+\frac{M_1(f)+2M_2(h)}{\Gamma(\varrho)}\eta\\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Case 02. $0 \leq \varsigma_1 < \eta$, $\varsigma_2 < 2\eta$:

$$\begin{aligned}\left|\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'}(\varsigma_2))-\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'}(\varsigma_1))\right| &\leq |c_1|\varsigma_2^{\varrho-1}+\frac{M_1(f)+2M_2(h)}{\Gamma(\varrho+1)}\varsigma_2^{\varrho}\\ &< |c_1|(2\eta)^{\varrho-1}+\frac{M_1(f)+2M_2(h)}{\Gamma(\varrho+1)}(2\eta)^{\varrho}\\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Therefore, the sequence $(\overline{\mathfrak{g}}_n^{'})_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$. Such us φ_p^{-1} is the homeomorphisms from $\mathbb{R}$ into $\mathbb{R}$, we deduce that

$$\left|\overline{\mathfrak{g}}_n^{'}(\varsigma_2)-\overline{\mathfrak{g}}_n^{'}(\varsigma_1)\right|=\left|\varphi_p^{-1}(\varphi_p(\overline{\mathfrak{g}}_n^{'}(\varsigma_2)))-\varphi_p^{-1}(\varphi_p(\overline{\mathfrak{g}}_n^{'}(\varsigma_1)))\right|.$$

Consequently $(\overline{\mathfrak{g}}_n^{'})_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

According to the Arzelà-Ascoli theorem, there exists a convergent subsequence $(\overline{\mathfrak{g}}_{n_j}^{'})_{n_j\in\mathbb{N}}$ in the space $\mathcal{C}^1([0,1])$.

Set

$$\mathfrak{g}:=\lim_{n_j\to+\infty}\overline{\mathfrak{g}}_{n_j},$$

so

$$\mathfrak{g}^{'}=\lim_{n_j\to+\infty}\overline{\mathfrak{g}}_{n_j}^{'}.$$

Using **Step 1**, the sequence $(\overline{\mathfrak{g}}_n)_{n\in\mathbb{N}}$ is monotonically decreasing and has a lower bound. Therefore, it admits a pointwise limit, which we denote by $\mathfrak{g}_{\max}$.

Thus, we have $\mathfrak{g}=\mathfrak{g}_{\max}$ and furthermore, the entire sequence converges to $\mathfrak{g}_{\max}$ in $\mathcal{C}^1([0,1])$.

Let $\varsigma\in(0,1)$, we have

$$-\varphi_p(\overline{\mathfrak{g}}_{n+1}^{'})(\varsigma)=c_1\varsigma^{\varrho-1}+\frac{1}{\Gamma(\varrho)}\int_0^{\varsigma}(\varsigma-s)^{\varrho-1}(f(s,\overline{\mathfrak{g}}_n(s))+h(\overline{\mathfrak{g}}_n(s))-h(\overline{\mathfrak{g}}_{n+1}(s)))ds,$$

As $n\longrightarrow+\infty$, it follows that

$$(f(s,\overline{\mathfrak{g}}_n(s))+h(\overline{\mathfrak{g}}_n(s))-h(\overline{\mathfrak{g}}_{n+1}(s)))\to f(s,\mathfrak{g}_{\max}(s)).$$

Also, we have

$$\exists W > 0; \forall n \in \mathbb{N}; \forall s \in [0,1] : |(f(s, \overline{\mathfrak{g}}_n(s)) + h(\overline{\mathfrak{g}}_n(s)) - h(\overline{\mathfrak{g}}_{n+1}(s)))| \leq W.$$

By virtue of Lebesgue's dominated convergence theorem, we conclude that

$$-\varphi_p(\mathfrak{g}_{\max}{}'(\varsigma)) = c_1\varsigma^{\varrho-1} + \frac{1}{\Gamma(\varrho)}\int\limits_0^\varsigma (\varsigma - s)^{\varrho-1} f(s, \mathfrak{g}_{\max}(s))ds.$$

Then, we obtain $\forall \varsigma \in (0,1)$

$$-\varphi_p(\mathfrak{g}_{\max}{}'(\varsigma)) = c_1\varsigma^{\varrho-1} + I_{0^+}^{\varrho} f(\varsigma, \mathfrak{g}_{\max}(\varsigma)), \tag{3.17}$$

Also, we get

$$\mathfrak{g}_{\max}(0) = 0, \tag{3.18}$$

and

$$a_1\mathfrak{g}_{\max}(1) + a_2\mathfrak{g}_{\max}{}'(1) = L(\mathfrak{g}_{\max}). \tag{3.19}$$

By (3.17), (3.18) and (3.19), this implies that $\mathfrak{g}_{\max}$ solves the problem given below

$$\begin{cases} -D_0^{\varrho}(\varphi_p(\mathfrak{g}_{\max}{}')) = f(\varsigma, \mathfrak{g}_{\max}),\ \varsigma \in [0,1]\,, \\ \mathfrak{g}_{\max}(0) = 0, \\ a_1\mathfrak{g}_{\max}(1) + a_2\mathfrak{g}_{\max}{}'(1) = L(\mathfrak{g}_{\max}), \end{cases} \tag{3.20}$$

We now show that if $\mathfrak{g}$ is the another solution of (1.1) satisfying $\underline{\mathfrak{g}} \leq \mathfrak{g} \leq \overline{\mathfrak{g}}$ on $[0,1]$, then it follows that $\mathfrak{g} \leq \mathfrak{g}_{\max}$ on $[0,1]$.

Since $\mathfrak{g}$ is a lower solution of (1.1), so by **Step 1,** we obtain

$$\forall n \in \mathbb{N}\text{: } \mathfrak{g} \leq \overline{\mathfrak{g}}_n,$$

Passing to the limit as $n \to +\infty$, we get

$$\mathfrak{g} \leq \lim_{n\to+\infty} \overline{\mathfrak{g}}_n = \mathfrak{g}_{\max},$$

which means that $\mathfrak{g}_{\max}$ is the maximal solution of (1.1).

Step 5. Analogously, we show that the sequence $(\underline{\mathfrak{g}}_n)_{n\in\mathbb{N}}$ converges to a minimal solution $\mathfrak{g}_{\min}$ of (1.1). This finishes the proof. □

4. Application

In what follows, we utilize the precedent result to investigate the problem described below.

$$\begin{cases} -D_0^{3/2}(\varphi_p(\mathfrak{g}')) = \beta_1\mathfrak{g}^{\kappa_1} - \mathfrak{g}^{\kappa_2} + \beta_2\mathfrak{g}^{\kappa_1}\left|\mathfrak{g}'\right|^{\frac{1}{p-1}} + h(\varsigma) \quad \varsigma \in [0,1], \\ \mathfrak{g}(0) = 0,\ \mathfrak{g}(1) + a\mathfrak{g}'(1) = \int\limits_0^1 \Psi(s)\mathfrak{g}(s)ds, \end{cases} \tag{4.1}$$

β_1, β_2, are positive real numbers with $0 < \kappa_1 < p-1$, $\kappa_2 > \kappa_1$, and the function $h : [0,1] \to \mathbb{R}_+$ and $\int\limits_0^1 \Psi(s)\mathfrak{g}(s)ds \leq 1$.

where Ψ is real positive function.

Theorem 4.1. *Suppose that* $\max\limits_{0\leq\varsigma\leq1} h(\varsigma) > p$, *hence problem* (4.1) *admits a maximal solution* $\mathfrak{g}_{\max}$ *and minimal solution* $\mathfrak{g}_{\min}$.

Proof. Set $(\underline{\mathfrak{g}}, \overline{\mathfrak{g}}) = (0, \mathfrak{L})$, where $\mathfrak{L}$ is a positive constant.

Firstly, since $\int_0^1 \Psi(s)ds \leq 1$ we have

$$\begin{cases} 0 \leq 0, \\ \mathfrak{L} \geq \int_0^1 \Psi(s)\mathfrak{L}ds. \end{cases}$$

Now $\underline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}$ are a lower and an upper solution of (4.1), if we have

$$\begin{cases} 0 \leq h(\varsigma) \quad \text{on } [0,1], \\ 0 \geq \beta_1 \mathfrak{L}^{\kappa_1} - \mathfrak{L}^{\kappa_2} + h(\varsigma). \end{cases}$$

That is

$$\begin{cases} h(\varsigma) \geq 0 \text{ on } [0,1], \\ \mathfrak{L}^{\kappa_2} \geq \beta_1 \mathfrak{L}^{\kappa_1} + h(\varsigma). \end{cases}$$

Since $\max\limits_{0 \leq \varsigma \leq 1}(h(\varsigma)) > p$ and $\mathfrak{L}$ is sufficiently large, it follows that $\underline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}$ are respectively a lower and an upper solution of problem (4.1).

Then, by Theorem 3.3, we conclude that problem (4.1) admits both a maximal solution $\mathfrak{g}_{\max}$ and a minimal solution $\mathfrak{g}_{\min}$. □

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