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INCLUSION PROPERTIES OF RABOTNOV FUNCTION FOR INCLUSIVE SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. This paper examines some inclusion properties, sufficient requirements for the Rabotnov function to be in a certain qualitative subclass of analytic functions. Our findings will also indicate some corollaries.

Keywords. Analytic function, Rabotnov function, Geometric functions
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1. Introduction and preliminaries

An analytical model known as the Rabotnov function is used in the research of creep mechanics to describe the time-dependent deformation behavior of materials under continuous load. Foretelling the progressive deformation of materials under continuous stresses requires the use of this function, which Yuri Rabotnov invented. Reference time, Young's modulus, applied stress, initial strain, and the material-specific creep exponent all play crucial roles in determining the Rabotnov function. These properties are derived through fitting of experimental data, enabling accurate material performance predictions for long-term applications in industries such as aerospace.

Let $\digamma$ be the family of univalent and analytic functions of the form:

$$\mathbf{q}(\kappa)=\kappa+\sum_{\varpi=2}^{\infty}\alpha_{\varpi}\kappa^{\varpi},\quad \kappa\in\mathbb{C},\ |\kappa|<1, \tag{1.1}$$

which are normalized by $\mathbf{q}(0)=\mathbf{q}'(0)-1=0$.

Let $\mathcal{V}(\hbar_1,\hbar_2)$ to be a subclass of $\digamma$ due to Kamali et al. [11] including the functions of the form (1.1) that meet the analytical requirements

$$\mathrm{Re}\left(\frac{\hbar_2\kappa^3\mathbf{q}'''(\kappa)+(2\hbar_2+1)\kappa^2\mathbf{q}''(\kappa)+\kappa\mathbf{q}'(\kappa)}{\hbar_2\kappa^2\mathbf{q}''(\kappa)+\kappa\mathbf{q}'(\kappa)}\right)>\hbar_1,\ (\hbar_1,\hbar_2\in[0,1)),\ \ |\kappa|<1).$$

Example 1.1. [13] For some $\hbar_1\in[0,1)$ and choosing $\hbar_2=0$, and $\mathbf{q}(\kappa)$ of the form (1.1), let the subclass $\mathcal{V}(\hbar_1,0)\equiv\mathcal{K}(\hbar_1)$ consists of functions in $\digamma$ satisfying the inequality

$$\mathrm{Re}\left(\frac{\kappa\mathbf{q}''(\kappa)}{\mathbf{q}'(\kappa)}+1\right)>\hbar_1,\ \ |\kappa|<1.$$

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Also, if $\hbar_1 = \hbar_2 = 0$, the subclass $\mathcal{V}(0,0) \equiv \mathcal{K}(0) \equiv \mathcal{K}$ consists of functions in $\digamma$ satisfying the inequality

$$\operatorname{Re}\left(\frac{\kappa \mathbf{q}''(\kappa)}{\mathbf{q}'(\kappa)}+1\right) > 0, \quad |\kappa| < 1.$$

Both subclasses $\mathcal{K}(\hbar_1)$ and $\mathcal{K}$ are well known subclasses of convex functions of order $\hbar_1$ and convex functions (convex functions of order zero), respectively (see [13]).

We need the following sufficient and necessary requirements for functions in the aforementioned subclass.

Lemma 1.2. [11] *A function* $\mathbf{q} \in \mathcal{V}(\hbar_1, \hbar_2)$ *if and only if*

$$\sum_{\varpi=2}^{\infty} \varpi(\varpi-\hbar_1)\left(\hbar_2(\varpi-1)+1\right)|\alpha_{\varpi}| \leq 1-\hbar_1 \tag{1.2}$$

Special functions are widely recognized as important in geometric function theory. Many authors discuss the geometric characteristics of several types of special functions. (see [1], [4], [7]-[10]).

Rabotnov introduced the Rabotnov function (Rabotnov fractional exponential function) as follows (see [12]):

$$\Upsilon_{\xi,\gamma}(\kappa) = \kappa^{\xi} \sum_{\varpi=0}^{\infty} \frac{(\gamma)^{\varpi} \kappa^{\varpi(1+\xi)}}{\Gamma((\varpi+1)(1+\xi))}, \quad (\xi, \gamma, \kappa \in \mathbb{C}\ ). \tag{1.3}$$

Clearly, $\Upsilon_{\xi,\gamma}(\kappa) \notin \digamma$. So, we consider the Rabotnov functions as:

$$\begin{aligned}\mathcal{R}_{\xi,\gamma}(\kappa) &= \kappa^{\frac{1}{1+\xi}} \Gamma(1+\xi)\, \Upsilon_{\xi,\gamma}(\kappa^{\frac{1}{1+\xi}}) \\ &= \kappa + \sum_{\varpi=2}^{\infty} \frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\varpi)} \kappa^{\varpi}, \ |\kappa|<1. \end{aligned} \tag{1.4}$$

Now, by the convolution $(*)$, consider a linear operator $\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q} : \digamma \to \digamma$ as:

$$\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q} = \mathcal{R}_{\xi,\gamma}(\kappa) * \mathbf{q}(\kappa) = \kappa + \sum_{\varpi=2}^{\infty} \frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\varpi)} \alpha_{\varpi} \kappa^{\varpi}.$$

The present work establish some relations between Rabotnov function and geometric function theory. Inspired by numerous literary works (see for example [2, 3, 5, 6]).

2. Main Results

We shall provide some requirements in this section for the normalized Rabotnov function to belong to the previously indicated subclass.

The following Lemma given by Sümer Eker [15] it is necessary for our main results.

Lemma 2.1. *For* $\varpi \in \mathbb{N}$ *and* $\xi \geq 0$, *then*

$$(\xi+1)^{\varpi-1}(\varpi-1)!\,\Gamma(\xi+1) \leq \Gamma((\xi+1)\varpi). \tag{2.1}$$

And from (2.1), it is possible to write

$$\frac{\Gamma(\xi+1)}{\Gamma((\xi+1)\varpi)} \leq \frac{1}{(\xi+1)^{\varpi-1}(\varpi-1)!}. \tag{2.2}$$

Theorem 2.2. *If* $\xi \geq 0$ *and* $\gamma > 0$, *then* $\mathcal{R}_{\xi,\gamma}(\kappa) \in \mathcal{V}(\hbar_1, \hbar_2)$ *if the following inequality is met:*

$$\begin{aligned} & \frac{\hbar_2\gamma^3}{(\xi+1)^3} e^{\frac{\gamma}{\xi+1}} + \frac{(\hbar_2(5-\hbar_1)+1)\,\gamma^2}{(\xi+1)^2} e^{\frac{\gamma}{\xi+1}} \\ & + \frac{(2\hbar_2(2-\hbar_1)+3-\hbar_1)\,\gamma}{\xi+1} e^{\frac{\gamma}{\xi+1}} + (1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}} - 1\right) \\ \leq\ & 1-\hbar_1. \end{aligned}$$

Proof. By the function $\mathcal{R}_{\xi,\gamma}(\kappa)$ given by (1.4) and inequality (1.2), it suffices to show that

$$\begin{aligned} \mathcal{P}_1(\gamma,\xi;\hbar_1,\hbar_2) &= \sum_{\varpi=2}^{\infty} \varpi(\varpi-\hbar_1)\,(\hbar_2(\varpi-1)+1)\,\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)} \\ &\equiv \sum_{\varpi=2}^{\infty} \left[\hbar_2\varpi^3 + (1-\hbar_2(\hbar_1+1))\,\varpi^2 + \hbar_1(\hbar_2-1)\varpi\right] \frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)} \leq 1-\hbar_1. \end{aligned}$$

By writing

$$\begin{aligned} \varpi &= (\varpi-1)+1, \\ \varpi^2 &= (\varpi-1)(\varpi-2)+3(\varpi-1)+1, \\ \varpi^3 &= (\varpi-1)(\varpi-2)(\varpi-3)+6(\varpi-1)(\varpi-2)+7(\varpi-1)+1, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{P}_1(\gamma,\xi;\hbar_1,\hbar_2) &= \hbar_2 \sum_{\varpi=2}^{\infty} (\varpi-1)(\varpi-2)(\varpi-3)\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)} \\ &\quad + (\hbar_2(5-\hbar_1)+1) \sum_{\varpi=2}^{\infty} (\varpi-1)(\varpi-2)\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)} \\ &\quad + (2\hbar_2(2-\hbar_1)+3-\hbar_1) \sum_{\varpi=2}^{\infty} (\varpi-1)\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)} \\ &\quad + (1-\hbar_1) \sum_{\varpi=2}^{\infty} \frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\,\varpi)}. \end{aligned}$$

Unsing the inequality (2.2), we get

$$\begin{aligned}\mathcal{P}_1(\gamma,\xi;\hbar_1,\hbar_2) &\leq \hbar_2\sum_{\varpi=4}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-4)!}\\ &\quad+(\hbar_2(5-\hbar_1)+1)\sum_{\varpi=3}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-3)!}\\ &\quad+(2\hbar_2(2-\hbar_1)+3-\hbar_1)\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-2)!}\\ &\quad+(1-\hbar_1)\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-1)!}\\ &= \frac{\hbar_2\gamma^3}{(\xi+1)^3}e^{\frac{\gamma}{\xi+1}}+\frac{(\hbar_2(5-\hbar_1)+1)\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}\\ &\quad+\frac{(2\hbar_2(2-\hbar_1)+3-\hbar_1)\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right). \quad (2.3)\end{aligned}$$

But (2.3) is bounded above by $1-\hbar_1$, thus

$$\begin{aligned}&\frac{\hbar_2\gamma^3}{(\xi+1)^3}e^{\frac{\gamma}{\xi+1}}+\frac{(\hbar_2(5-\hbar_1)+1)\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}\\ &+\frac{(2\hbar_2(2-\hbar_1)+3-\hbar_1)\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\\ \leq\ & 1-\hbar_1.\end{aligned}$$

□

Corollary 2.3. *If $\xi\geq 0$ and $\gamma>0$, then $\mathcal{R}_{\xi,\gamma}(\kappa)\in\mathcal{K}(\hbar_1)$ if*

$$\frac{\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}-\frac{(\hbar_1-3)\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\leq 1-\hbar_1.$$

Corollary 2.4. *If $\xi\geq 0$ and $\gamma>0$, then $\mathcal{R}_{\xi,\gamma}(\kappa)\in\mathcal{K}$ if*

$$\frac{\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}+\frac{3\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+\left(e^{\frac{\gamma}{\xi+1}}-1\right)\leq 1.$$

Remark 2.5. The results in Corollary *2.3 and Corollary 2.4 were obtained by Eker et al.* [16] and Eker and Ece [15], respectively.

3. Inclusion Properties

For $\mu_1\in(0,1]$, $\mu_2<1$ and $\mu_3\in\mathbb{C}-\{0\}$, a function $\mathbf{q}\in\digamma$ is said to be in the subclass $\mathcal{B}^{\mu_3}(\mu_1,\mu_2)$ if it satisfies the following relation:

$$\left|\frac{(1-\mu_1)\frac{\mathbf{q}(\kappa)}{\kappa}+\mu_1\mathbf{q}'(\kappa)-1}{2\mu_3(1-\mu_2)+(1-\mu_1)\frac{\mathbf{q}(\kappa)}{\kappa}+\mu_1\mathbf{q}'(\kappa)-1}\right|<1,\ \ |\kappa|<1.$$

Lemma 3.1. [14] *If $\mathbf{q}\in\mathcal{B}^{\mu_3}(\mu_1,\mu_2)$ of the form (1.1), then*

$$|\alpha_{\mathbf{j}}|\leq\frac{2\,|\mu_3|\,(1-\mu_2)}{\mu_1(\mathbf{j}-1)+1},\quad \mathbf{j}\in\mathbb{N}-\{1\}.$$

Using Lemma 3.1, we will examine the behavior of Rabotnov function $\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q}$ on the subclass $\mathcal{B}^{\mu_3}(\mu_1,\mu_2)$.

Theorem 3.2. *For* $\mathbf{q} \in \mathcal{B}^{\mu_3}(\mu_1, \mu_2)$, *if the inequality*

$$\frac{\hbar_2 \gamma^2}{(\xi+1)^2} e^{\frac{\gamma}{\xi+1}} + \frac{(\hbar_2(2-\hbar_1)+1)\,\gamma}{\xi+1} e^{\frac{\gamma}{\xi+1}} + (1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}} - 1\right) \le \frac{\mu_1\,(1-\hbar_1)}{2\,|\mu_3|\,(1-\mu_2)}$$

is satisfied, then $\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q} \in \mathcal{V}(\hbar_1, \hbar_2)$.

Proof. Let $\mathbf{q} \in \mathcal{B}^{\mu_3}(\mu_1, \mu_2)$ and given by (1.1), By Lemma 1.2 it suffices to show that

$$\mathcal{P}_2(\gamma, \xi; \hbar_1, \hbar_2) = \sum_{\varpi=2}^{\infty} \varpi(\varpi-\hbar_1)\,(\hbar_2(\varpi-1)+1)\,\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)}\,|\alpha_\varpi| \le 1-\hbar_1.$$

Since $\mathbf{q} \in \mathcal{B}^{\mu_3}(\mu_1, \mu_2)$, then by Lemma 3.1

$$\mathcal{P}_2(\gamma, \xi; \hbar_1, \hbar_2) \le 2\,|\mu_3|\,(1-\mu_2) \sum_{\varpi=2}^{\infty} \frac{\varpi(\varpi-\hbar_1)\,(\hbar_2(\varpi-1)+1)}{\mu_1(\varpi-1)+1}\,\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)}.$$

Also, since $\mu_1(\varpi-1)+1 \ge \varpi\mu_1$, it is possible to write

$$\begin{aligned}
&\mathcal{P}_2(\gamma, \xi; \hbar_1, \hbar_2) \\
&\le \frac{2\,|\mu_3|\,(1-\mu_2)}{\mu_1} \sum_{\varpi=2}^{\infty} (\varpi-\hbar_1)\,(\hbar_2(\varpi-1)+1)\,\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)} \\
&\equiv \frac{2\,|\mu_3|\,(1-\mu_2)}{\mu_1} \sum_{\varpi=2}^{\infty} \left[\hbar_2\varpi^2 + (1-\hbar_2(\hbar_1+1))\,\varpi + \hbar_1(\hbar_2-1)\right]\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)}.
\end{aligned}$$

By writing

$$\begin{aligned}
\varpi &= (\varpi-1)+1, \\
\varpi^2 &= (\varpi-1)(\varpi-2)+3(\varpi-1)+1,
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{P}_2(\gamma, \xi; \hbar_1, \hbar_2) \le\ & \frac{2\,|\mu_3|\,(1-\mu_2)}{\mu_1}\Bigg\{\hbar_2 \sum_{\varpi=2}^{\infty}(\varpi-1)(\varpi-2)\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)} \\
&+ (\hbar_2(2-\hbar_1)+1)\sum_{\varpi=2}^{\infty}(\varpi-1)\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)} \\
&+ (1-\hbar_1)\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}\Gamma\,(1+\xi)}{\Gamma\,((1+\xi)\,\varpi)}\Bigg\}.
\end{aligned}$$

Under the inequality (2.2), we get

$$\begin{aligned}\mathcal{P}_2(\gamma,\xi;\hbar_1,\hbar_2) &\leq \frac{2\left|\mu_3\right|(1-\mu_2)}{\mu_1}\left\{\hbar_2\sum_{\varpi=3}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-3)!}\right.\\ &\quad+(\hbar_2(2-\hbar_1)+1)\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-2)!}\\ &\quad\left.+(1-\hbar_1)\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}}{(\xi+1)^{\varpi-1}(\varpi-1)!}\right\}\\ &= \frac{2\left|\mu_3\right|(1-\mu_2)}{\mu_1}\left\{\frac{\hbar_2\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}+\frac{(\hbar_2(2-\hbar_1)+1)\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}\right.\\ &\quad\left.+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\right\} \qquad (3.1)\end{aligned}$$

But (3.1) is bounded above by $1-\hbar_1$, this complete the proof of Theorem 3.2. □

Corollary 3.3. *For $\mathbf{q}\in\mathcal{B}^{\mu_3}(\mu_1,\mu_2)$, if*

$$\frac{\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\leq\frac{\mu_1(1-\hbar_1)}{2\left|\mu_3\right|(1-\mu_2)},$$

then $\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q}\in\mathcal{K}(\hbar_1)$.

Corollary 3.4. *For $\mathbf{q}\in\mathcal{B}^{\mu_3}(\mu_1,\mu_2)$, if*

$$\left(\frac{\gamma}{\xi+1}+1\right)e^{\frac{\gamma}{\xi+1}}-1\leq\frac{\mu_1}{2\left|\mu_3\right|(1-\mu_2)},$$

then $\mathcal{R}_{\xi,\gamma}(\kappa)\mathbf{q}\in\mathcal{K}$.

4. An Integral Operator $\mathcal{J}_{\xi,\gamma}(\kappa)$

Theorem 4.1. *Let $\xi\geq 0$ and $\gamma>0$. If integral operator $\mathcal{J}_{\xi,\gamma}$ is given by*

$$\mathcal{J}_{\xi,\gamma}(\kappa)=\int_0^{\kappa}\frac{\mathcal{R}_{\xi,\gamma}(A)}{A}dA,\ |\kappa|<1, \qquad (4.1)$$

then $\mathcal{J}_{\xi,\gamma}(\kappa)\in\mathcal{V}(\hbar_1,\hbar_2)$ if

$$\frac{\hbar_2\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}+\frac{(\hbar_2(2-\hbar_1)+1)\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\leq 1-\hbar_1.$$

Proof. Since

$$\mathcal{J}_{\xi,\gamma}(\kappa)=\kappa+\sum_{\varpi=2}^{\infty}\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\varpi)}\frac{\kappa^{\varpi}}{\varpi},$$

then by Theorem 2.2 it suffices to show that

$$\begin{aligned}\mathcal{P}_3(\gamma,\xi;\hbar_1,\hbar_2) &= \sum_{\varpi=2}^{\infty}\varpi(\varpi-\hbar_1)(\hbar_2(\varpi-1)+1)\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\varpi\Gamma((1+\xi)\varpi)}\\ &\equiv \sum_{\varpi=2}^{\infty}\left[\hbar_2\varpi^2+(1-\hbar_2(\hbar_1+1))\varpi+\hbar_1(\hbar_2-1)\right]\frac{\gamma^{\varpi-1}\Gamma(1+\xi)}{\Gamma((1+\xi)\varpi)}\\ &\leq 1-\hbar_1.\end{aligned}$$

As a proceeding in Theorem 3.2, we get

$$\begin{aligned}&\mathcal{P}_3(\gamma,\xi;\hbar_1,\hbar_2)\\ &\leq \frac{\hbar_2\gamma^2}{(\xi+1)^2}e^{\frac{\gamma}{\xi+1}}+\frac{(\hbar_2(2-\hbar_1)+1)\,\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right).\end{aligned} \tag{4.2}$$

But (4.2) is bounded above by $1-\hbar_1$, this complete the proof of Theorem 4.1. □

Corollary 4.2. *Let $\xi \geq 0$ and $\gamma > 0$. If integral operator $\mathcal{J}_{\xi,\gamma}$ is given by (4.1), then $\mathcal{J}_{\xi,\gamma} \in \mathcal{K}(\hbar_1)$ if and only if*

$$\frac{\gamma}{\xi+1}e^{\frac{\gamma}{\xi+1}}+(1-\hbar_1)\left(e^{\frac{\gamma}{\xi+1}}-1\right)\leq 1-\hbar_1.$$

Corollary 4.3. *Let $\xi \geq 0$ and $\gamma > 0$. If integral operator $\mathcal{J}_{\xi,\gamma}$ is given by (4.1), then $\mathcal{J}_{\xi,\gamma} \in \mathcal{K}$ if and only if*

$$\left(\frac{\gamma}{\xi+1}+1\right)e^{\frac{\gamma}{\xi+1}}\leq 2.$$

5. Conclusions

Using of the Rabotnov function $\mathcal{R}_{\xi,\gamma}(\kappa)$, we examines some inclusion properties, sufficient condition for this function to be in the subclass $\mathcal{V}(\hbar_1,\hbar_2)$ of analytic functions. Furthermore, some corollaries will be implied by our findings. After this work, the function $\mathcal{R}_{\xi,\gamma}(\kappa)$ is applicable to the derivation of novel inclusion relations and conditions for analytic functions in different subclasses in unit disk.

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