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A NOVEL FOUR-PARAMETER LIFETIME MODEL: PROPERTIES AND APPLICATIONS

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Abstract. In this work, we introduce a new continuous lifetime distribution, called the Power Three-parameters Lindley distribution (PTPL). We explore the maximum likelihood estimation-based inference for the PTPL distribution, and introduce three random data generation algorithms. A Monte Carlo simulation study is conducted to evaluate the performance of the proposed models. Empirical applications to two real-world engineering datasets demonstrate its practical utility, confirm the model's flexibility, and consistently outperform existing lifetime models across various scenarios.

Keywords. Lindley distribution, Lindley distribution, Hazard rate function, Maximum likelihood estimation
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1. Introduction

The Lindley distribution was originally proposed by Lindley [11] within the framework of fiducial and Bayesian statistics. Although this lifetime distribution has received relatively little attention in the statistical literature compared to the widely popular exponential distribution, it has significant applications, particularly in stress-strength reliability modeling, as discussed by [9] and [8], who showed that in many applications, it provides a more suitable model and outperforms the exponential distribution. For instance, a numerical analysis demonstrates that it fits both waiting and survival time data more accurately than the exponential model.

Several researchers have introduced new classes of distributions by extending the Lindley distribution, where the central strategy involves embedding existing distributions into more flexible frameworks. For instance, [17] developed the discrete Poisson-Lindley distribution. Additionally, several extensions of the Lindley distribution have been proposed in the literature, including the generalized Lindley distribution by [23], the power Lindley distribution by [7], the quasi Lindley distribution by [18], and the gamma Lindley distribution by [14]. Recent studies have expanded the Lindley family with new variants include the XLindley [5],

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the composite Pareto-Lindley [24], and the Lindley-based generalized Pareto [10] distributions.

The recently introduced three-parameter Lindley distribution by [20] has garnered growing research interest, particularly due to its applications (see [4] and [22]). This distribution arises from a specific mixture of exponential and gamma distributions. The cumulative distribution function (CDF) and the probability distribution function (PDF) of the three-parameter Lindley distribution with shape parameters λ, α and β are defined respectively as follows.

$$F(t;\lambda,\alpha,\beta) = 1 - (\frac{\beta\lambda+\alpha+\alpha\lambda t}{\beta\lambda+\alpha})\exp\{-\lambda t\}, \quad t,\lambda,\alpha,\beta\lambda+\alpha>0 \tag{1.1}$$

and

$$f(t;\lambda,\alpha,\beta) = \frac{\lambda^2}{\beta\lambda+\alpha}(\beta+\alpha t)\exp\{-\lambda t\} = pf_1(t)+(1-p)f_2(t), \tag{1.2}$$

where $f_1(t) \sim Exp(\lambda)$, $f_2(t) \sim Gamma(2,\lambda)$, and $p = \beta\lambda/(\beta\lambda+\alpha)$.

In the present work, we propose a new lifetime distribution, called the power three-parameter Lindley distribution (PTPL), by applying transformation techniques to the existing three-parameter Lindley distribution. The proposed model contains several well-known distributions as special cases, including Lindley, power Lindley, XLindley, quasi Lindley, Weibull, and exponential distributions, among others. We define and study various properties of the PTPL distribution, including survival and hazard rate functions, moments and related measures, cumulants, quantile function, mean deviation, and entropy. We discuss the maximum likelihood estimation method for our model, propose three algorithms for data generation, and then study the Monte Carlo simulation. Empirical applications to two real-world engineering datasets demonstrate its practical utility and flexibility compared to other existing lifetime models.

The newly suggested PTPL distribution is particularly important, as evidenced by the following points.

- The statistical functions of the proposed PTPL distribution are straightforward and available in closed-form expressions.
- The flexibility of the four-parameter structure makes this distribution well-suited for modeling, especially in survival analysis, actuarial science, engineering, and related areas.
- The proposed distribution provides greater flexibility than current models, especially in capturing various shapes of the hazard rate function, including increasing, decreasing, bathtub-shaped, and upside-down bathtub patterns.‘

The remainder of the paper is structured as follows. Section 2 presents the proposed distribution. Section 3 explores key properties of the PTPL distribution. In Section 4, we address parameter estimation via maximum likelihood and derive asymptotic confidence intervals. Section 5 presents algorithms for random data generation and examines simulation studies to illustrate the performance of ML estimates. Section 6 demonstrates practical applications of the PTPL distribution using engineering datasets. Finally, Section 7 provides concluding remarks.

2. The power three-parameter Lindley distribution

This section presents the development of the PTPL distribution using the power transformation technique $X = T^{\frac{1}{\eta}}$, where T has PDF in (1.2). The PDF of the PTPL distribution can be obtained as follows.

$$\begin{aligned} f(x) &= \frac{\lambda^2\eta}{\beta\lambda+\alpha}(\beta+\alpha x^\eta)x^{\eta-1}\exp\{-\lambda x^\eta\}, \\ &= pg_1(x) + (1-p)g_2(x),\ \ x, \lambda, \eta, \alpha, \beta\lambda+\alpha > 0, \end{aligned} \tag{2.1}$$

where $g_1(x) = \eta\lambda x^{\eta-1}\exp\{-\lambda x^\eta\}$ follows the Weibull distribution with shape parameter η and scale parameter λ, and $g_2(x) = \eta\lambda^2 x^{2\eta-1}\exp\{-\lambda x^\eta\}$ follows the generalized gamma distribution with shape parameters 2 and η and scale λ. Consequently, the PDF of the proposed PTPL distribution is characterized as a mixture of Weibull and generalized gamma distributions with a mixing proportion $p = \beta\lambda/(\beta\lambda+\alpha)$.

The CDF of the PTPL distribution is defined by

$$F(x) = 1 - (\frac{\beta\lambda+\alpha+\alpha\lambda x^\eta}{\beta\lambda+\alpha})\exp\{-\lambda x^\eta\},\quad x, \lambda, \eta, \alpha, \beta\lambda+\alpha > 0. \tag{2.2}$$

We use $X \sim PTPL(\lambda, \eta, \beta, \alpha)$ to denote the random variable having a power three-parameter Lindley distribution with parameters λ,η,β, and α and with PDF and CDF in (2.1) and (2.2), respectively.

The graphical representations of the PTPL distribution's PDF and CDF for various parameter values are illustrated in Figure 1. We see that the PDF can be right-skewed, unimodel, or decreasing.

The PTPL distribution reduces to several classical distributions under specific parameter constraints, which are summarized in Table 1.

Table 1. The special cases of the PTPL lifetime model.

λ	α	η	β	Name of the distribution	Authors
-	-	1	-	The three-parameter Lindley	Shanker et al. [20]
-	1	-	-	A three-parameter Generalized Lindley	Ekhosuehi and Opone [6]
-	1	1	-	Two-parameter Lindley	Shanker and Mishra [19]
-	λ	1	-	Quasi Lindley	Shanker and Mishra [18]
-	1	-	1	Power Lindley	Ghitany et al. [7]
-	1	-	$\lambda+2$	Power XLindley	Meriem et al. [12]
-	1	1	$\lambda+2$	XLindley	Chouia and Zeghdoudi [5]
-	1	1	1	Lindley	Lindley [11]
-	-	-	0	Generalized gamma	
-	0	-	-	Weibull	
-	-	1	0	gamma	
-	0	1	-	Exponential	

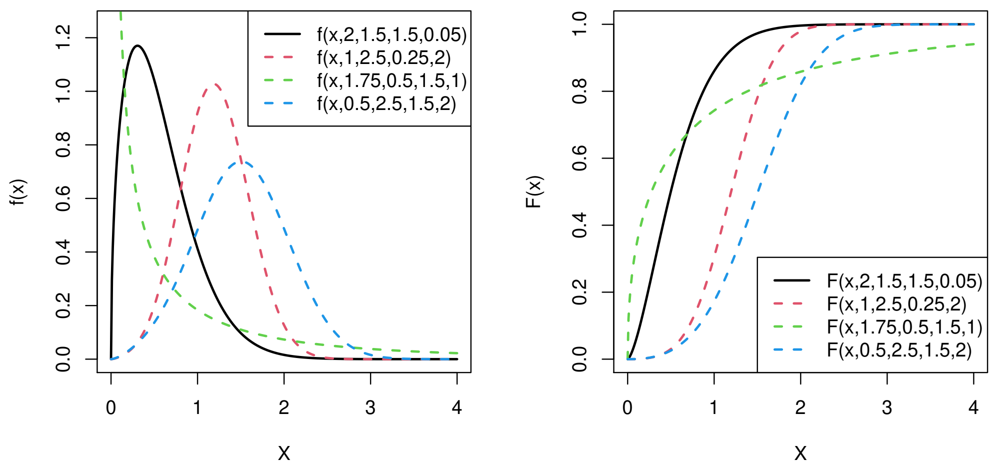

FIGURE 1. PDF (left) and CDF (right) plots of the PTPL model.

3. Statistical Properties

In this section, the fundamental statistical properties of the PTPL distribution are examined, including the derivation of survival and hazard functions, moment-based metrics, cumulants, quantile functions, mean deviations, and entropy measures.

3.1. **Survival and hazard rate functions.** The PTPL distribution is characterized by its survival function $S(x)$ and the hazard rate function $h(x)$, which take the form:

$$S(x) = 1 - F(x) = (\frac{\beta\lambda + \alpha + \alpha\lambda x^{\eta}}{\beta\lambda + \alpha}) \exp\{-\lambda x^{\eta}\},$$

and

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\lambda^2 \eta(\beta + \alpha x^{\eta}) x^{\eta - 1}}{\beta\lambda + \alpha + \alpha\lambda x^{\eta}},$$

where $x > 0$, $\lambda, \eta, \alpha, > 0$, and $\beta\lambda + \alpha > 0$.
As illustrated in Figure 2, the hazard rate of the PTPL distribution exhibits considerable flexibility, producing increasing, decreasing, bathtub and upside-down bathtub shapes under different combinations of parameters. This flexibility suggests the strong potential of the PTPL distribution for modeling diverse datasets in the real world.

3.2. **Moments and related measures.** Let X be a continuous random variable following the PTPL distribution, having the probability density function $f(x)$

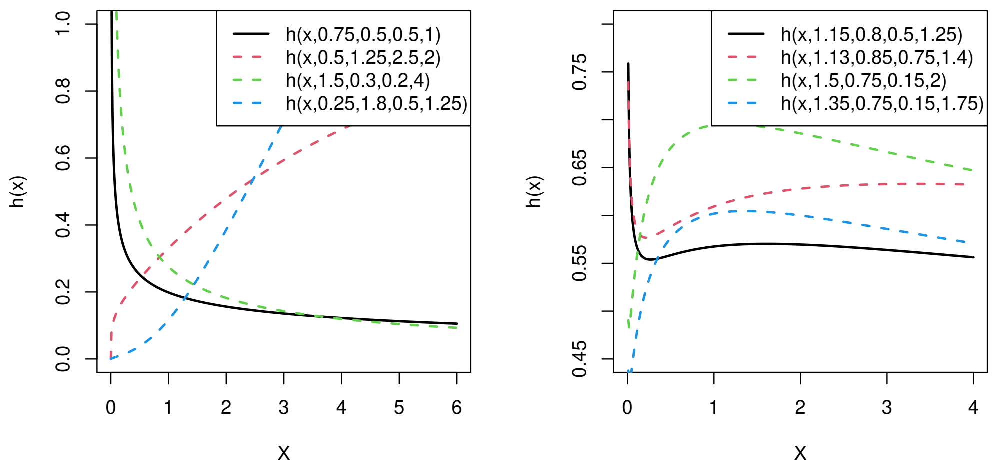

FIGURE 2. Hazard rate function plots of PTPL distribution.

specified in equation (2.1). Its r^{th} raw moments are defined by

$$\begin{aligned}\mu_r' = E(x^r) &= \int_0^\infty x^r f(x)dx \\ &= \int_0^\infty x^r(pg_1(x) + (1-p)g_2(x))dx \\ &= pE_1(x^r) + (1-p)E_2(x^r),\end{aligned}$$

where $E_1(x^r)$ and $E_2(x^r)$ are the rth raw moments of the Weibull and generalized gamma distributions, respectively. So

$$\begin{aligned}\mu_r' &= \frac{\beta\lambda}{\beta\lambda+\alpha}\frac{\Gamma(\frac{r}{\eta}+1)}{\lambda^{\frac{r}{\eta}}} + \frac{\alpha}{\beta\lambda+\alpha}\frac{\Gamma(\frac{r}{\eta}+2)}{\lambda^{\frac{r}{\eta}}} \\ &= \frac{r\Gamma(\frac{r}{\eta})(\eta(\beta\lambda+\alpha)+\alpha r)}{\eta^2\lambda^{\frac{r}{\eta}}(\beta\lambda+\alpha)},\end{aligned} \tag{3.1}$$

where $\Gamma(\theta)$ denotes the gamma function.
Note that when $\alpha = \eta = \beta = 1$, that is, in the case of the Lindley distribution, (3.1) reduces to the following simplified form:

$$\mu_r' = \frac{r!(\lambda+1+r)}{\lambda^r(\lambda+1)}.$$

Substituting $r = 1, 2, 3, 4$ in equation (3.1), the PTPL distribution's first 4 raw moments can be calculated as shown below:

$$\mu_1' = \frac{\Gamma(\frac{1}{\eta})(\eta(\beta\lambda+\alpha)+\alpha)}{\eta^2\lambda^{\frac{1}{\eta}}(\beta\lambda+\alpha)}, \quad \mu_2' = \frac{2\Gamma(\frac{2}{\eta})(\eta(\beta\lambda+\alpha)+2\alpha)}{\eta^2\lambda^{\frac{2}{\eta}}(\beta\lambda+\alpha)},$$

$$\mu_3' = \frac{3\Gamma(\frac{3}{\eta})(\eta(\beta\lambda+\alpha)+3\alpha)}{\eta^2\lambda^{\frac{3}{\eta}}(\beta\lambda+\alpha)}, \quad \text{and} \quad \mu_4' = \frac{4\Gamma(\frac{4}{\eta})(\eta(\beta\lambda+\alpha)+4\alpha)}{\eta^2\lambda^{\frac{4}{\eta}}(\beta\lambda+\alpha)}.$$

The central moments take the following form:

$$\mu_r = E((X-\mu_1')^r) = \sum_{k=1}^{r}\binom{r}{k}\mu_k'(-\mu_1')^{r-k}.$$

In particular, the variance of X can be expressed as

$$Var(X) = \mu_2 = \frac{2\eta^2(\beta\lambda+\alpha)\Gamma(\frac{2}{\eta})(\eta(\beta\lambda+\alpha)+2\alpha) - \Gamma^2(\frac{1}{\eta})(\eta(\beta\lambda+\alpha)+\alpha)^2}{\eta^4\lambda^{\frac{2}{\eta}}(\beta\lambda+\alpha)^2}.$$

The coefficient of variation (C.V.), the coefficient of skewness, and the coefficient of kurtosis for random X from PTPL distribution, respectively, obtained according the following relations.

(1) C.V. $= \sqrt{\dfrac{Var(X)}{E(X)}}$

(2) Skewness $= \dfrac{\mu_3}{\mu_2^{3/2}} = \dfrac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(Var(X))^{3/2}}$

(3) Kurtosis $= \dfrac{\mu_4}{\mu_2^2} = \dfrac{\mu_4' - 4\mu_3'\mu_1' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4}{(Var(X))^2}$.

Table 2 presents key statistical measures of the PTPL distribution, computed for various parameter combinations. The distribution demonstrates both right-skewed (positive skewness) and left-skewed (negative skewness) characteristics. Furthermore, the kurtosis values indicate leptokurtic behavior (Kurtosis > 3) in some cases and platykurtic behavior (Kurtosis < 3) in others. This versatility enables the distribution to model diverse data patterns.

3.3. **Cumulants.** The PTPL distribution possesses the characteristic function

$$\Phi_X(t) = E(\exp(itx)).$$

Using the Taylor expansion of the exponential function

$$\exp(itx) = \sum_{k=0}^{\infty}\frac{(it)^k x^k}{k!},$$

we find

$$\Phi_X(t) = E(\sum_{k=0}^{\infty}\frac{(it)^k x^k}{k!}) = \sum_{k=0}^{\infty}\frac{(it)^k E(x^k)}{k!},$$

where $E(x^k)$ is defined in (3.1), we get

$$\Phi_X(t) = \frac{1}{\eta^2(\beta\lambda+\alpha)}\sum_{k=0}^{\infty}\frac{(it)^k}{k!}\frac{k\Gamma(\frac{k}{\eta})(\eta(\beta\lambda+\alpha)+\alpha k)}{\lambda^{\frac{k}{\eta}}},$$

TABLE 2. The PTPL distribution's statistical properties for different parameter values.

λ	η	β	α	E(x)	V(x)	C.V.	Skewness	Kurtosis
0.5	0.5	0.5	0.5	18.67	1059.56	7.53	4.77	45.20
			2.5	22.54	1272.07	7.51	4.41	39.07
		2	0.5	13.33	718.22	7.34	5.57	60.74
			2.5	19.43	1103.67	7.54	4.69	43.76
	1	0.5	0.5	3.33	7.56	1.50	1.51	6.34
			2.5	3.82	7.97	1.44	1.42	6.02
		2	0.5	2.67	6.22	1.53	1.76	7.45
			2.5	3.43	7.67	1.49	1.49	6.25
1.5	2	0.75	1.75	0.94	0.18	0.44	0.37	2.92
			2.5	0.97	0.18	0.43	0.35	2.93
		1.5	1.75	0.88	0.18	0.45	0.45	2.95
			2.5	0.91	0.18	0.45	0.41	2.93
	3	0.75	1.75	0.94	0.08	0.31	-0.05	2.77
			2.5	0.96	0.08	0.29	-0.07	2.82
		1.25	1.75	0.91	0.09	0.32	-0.01	2.71
			2.5	0.93	0.08	0.31	-0.04	2.75

where $i = \sqrt{-1}$ is the imaginary unit. Also, the moment generating function $M_X(t)$ of the PTPL distribution is given by

$$\begin{aligned} M_X(t) = E(\exp(tx)) &= \sum_{k=0}^{\infty} \frac{t^k E(x^k)}{k!} \\ &= \frac{1}{\eta^2(\beta\lambda+\alpha)} \sum_{k=0}^{\infty} \frac{(t)^k}{k!} \frac{k\Gamma(\frac{k}{\eta})(\eta(\beta\lambda+\alpha)+\alpha k)}{\lambda^{\frac{k}{\eta}}}. \end{aligned}$$

3.4. **The Quantile function.** Recent work by [22] established the following quantile function for the three-parameter Lindley distribution:

$$Q_t(u) = -\frac{\beta}{\alpha} - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1}[-\frac{(\beta\lambda+\alpha)(1-u)\exp(-(\beta\lambda+\alpha)/\alpha)}{\alpha}], \quad u \in [0,1]$$

where $W_{-1}(.)$ denotes the negative branch of the Lambert function W.
Using the transformation $X = T^{1/\eta}$, we directly obtain the quantile function of the PTPL distribution as:

$$\begin{aligned} Q_x(u) &= [Q_t(u)]^{\frac{1}{\eta}} \\ &= [-\frac{\beta}{\alpha} - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1}[-\frac{(\beta\lambda+\alpha)(1-u)\exp(-(\beta\lambda+\alpha)/\alpha)}{\alpha}]]^{\frac{1}{\eta}}, \end{aligned} \tag{3.2}$$

where $u \in [0,1]$. By substituting $u = \frac{1}{2}$ into (3.2), we get the median M of X from the PTPL distribution

$$M = Q_x(\frac{1}{2}) = [-\frac{\beta}{\alpha} - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1}[-\frac{(\beta\lambda+\alpha)\exp(-(\beta\lambda+\alpha)/\alpha)}{2\alpha}]]^{\frac{1}{\eta}}.$$

3.5. **Mean deviations.** Let X be a random variable following the PTPL distribution with PDF $f(x)$, CDF $F(x)$, mean $m = E(x)$, and median M. The mean deviation for the mean $D(m)$ and the mean deviation about the median $D(M)$ are calculated, respectively, as follows.

$$\begin{aligned}
D(m) &= E(|X-m|)\\
&= \int_0^m -(x-m)f(x)dx + \int_m^\infty (x-m)f(x)dx\\
&= -\int_0^m xf(x)dx + m\int_0^m f(x)dx + \int_m^\infty xf(x)dx - m\int_m^\infty f(x)dx\\
&= -\int_0^m xf(x)dx + mF(m) - \int_0^m xf(x)dx + m - m(1-F(m))\\
&= 2mF(m) - 2\int_0^m xf(x)dx,
\end{aligned}$$

and

$$\begin{aligned}
D(M) &= E(|X-M|)\\
&= \int_0^M -(x-M)f(x)dx + \int_M^\infty (x-M)f(x)dx\\
&= -\int_0^M xf(x)dx + M\int_0^M f(x)dx + \int_M^\infty xf(x)dx - M\int_M^\infty f(x)dx\\
&= -\int_0^M xf(x)dx + MF(M) - \int_0^M xf(x)dx + m - M(1-F(M))\\
&= -2\int_0^M xf(x)dx + m + 2MF(M) - M.
\end{aligned}$$

By definition of the median $F(M) = \frac{1}{2}$, we get

$$D(M) = m - 2\int_0^M xf(x)dx.$$

Consider the integral

$$I(b) = \int_0^b xf(x)dx,$$

where $b = m$ or $b = M$.

$$\begin{aligned}
I(b) &= \int_0^b x\frac{\lambda^2\eta}{\beta\lambda+\alpha}(\beta+\alpha x^\eta)x^{\eta-1}\exp\{-\lambda x^\eta\}dx\\
&= \frac{\lambda^2\eta}{\beta\lambda+\alpha}\left(\beta\int_0^b x^\eta\exp\{-\lambda x^\eta\}dx + \alpha\int_0^b x^{2\eta}\exp\{-\lambda x^\eta\}dx\right). \qquad (3.3)
\end{aligned}$$

Using the transformations $y=\lambda x^{\eta}$, $x=(\frac{y}{\lambda})^{\frac{1}{\eta}}$, $dx=\frac{1}{\eta\lambda}(\frac{y}{\lambda})^{\frac{1}{\eta}-1}dy$, we get

$$\begin{aligned}\int_0^b x^{\eta}\exp\{-\lambda x^{\eta}\}dx &= \int_0^{\lambda b^{\eta}} \frac{y}{\lambda}\times\frac{1}{\eta\lambda}\times(\frac{y}{\lambda})^{\frac{1}{\eta}-1}\exp\{-y\}dy\\ &= \frac{1}{\eta\lambda^{(\frac{1}{\eta}+1)}}\int_0^{\lambda b^{\eta}} y^{\frac{1+\eta}{\eta}-1}\exp\{-y\}dy\\ &= \frac{1}{\eta\lambda^{(\frac{1}{\eta}+1)}}\gamma(\frac{\eta+1}{\eta},\lambda b^{\eta}). \end{aligned}\tag{3.4}$$

Similarly,

$$\int_0^b x^{2\eta}\exp\{-\lambda x^{\eta}\}dx = \frac{1}{\eta\lambda^{(\frac{1}{\eta}+2)}}\gamma(\frac{2\eta+1}{\eta},\lambda b^{\eta}),\tag{3.5}$$

where $\gamma(\theta,x)$ denotes the lower incomplete gamma function, which defined by $\gamma(\theta,x)=\int_0^x y^{\theta-1}\exp(-y)dy$, by putting (3.4) and (3.5) in the integral (3.3), we find

$$I(b)=\frac{1}{\beta\lambda+\alpha}\left(\frac{\beta}{\lambda^{(\frac{1}{\eta}-1)}}\gamma(\frac{\eta+1}{\eta},\lambda b^{\eta})+\frac{\alpha}{\lambda^{\frac{1}{\eta}}}\gamma(\frac{2\eta+1}{\eta},\lambda b^{\eta})\right).$$

Finally, we have

$$\begin{aligned} D(m) &= 2mF(m)-\frac{2}{\beta\lambda+\alpha}\left(\frac{\beta}{\lambda^{(\frac{1}{\eta}-1)}}\gamma(\frac{\eta+1}{\eta},\lambda m^{\eta})+\frac{\alpha}{\lambda^{\frac{1}{\eta}}}\gamma(\frac{2\eta+1}{\eta},\lambda m^{\eta})\right),\\ D(M) &= m-\frac{2}{\beta\lambda+\alpha}\left(\frac{\beta}{\lambda^{(\frac{1}{\eta}-1)}}\gamma(\frac{\eta+1}{\eta},\lambda M^{\eta})+\frac{\alpha}{\lambda^{\frac{1}{\eta}}}\gamma(\frac{2\eta+1}{\eta},\lambda M^{\eta})\right). \end{aligned}$$

3.6. **Entropy.** As a measure of uncertainty dispersion, introduced by Rényi [16], the entropy of a random variable X is given by

$$I_R(s)=\frac{1}{1-s}\ln\left(\int_0^{\infty} f(x)^s dx\right),$$

where $s>0$, $s\neq 1$. For recent developments in generalized entropy measures, see [21]. In particular, for the PTPL distribution, we have

$$I_R(s)=\frac{1}{1-s}\ln\left(\left(\frac{\lambda^2\eta}{\beta\lambda+\alpha}\right)^s\int_0^{\infty}(\beta+\alpha x^{\eta})^s x^{s(\eta-1)}\exp(-\lambda s x^{\eta})dx\right).$$

Using the transformations $y=\lambda s x^{\eta}$, $x=(\frac{y}{\lambda s})^{\frac{1}{\eta}}$, $dx=\frac{1}{\eta\lambda s}(\frac{y}{\lambda s})^{\frac{1}{\eta}-1}dy$, we get

$$I_R(s)=\frac{1}{1-s}\ln\left(\left(\frac{\lambda^2\eta}{\beta\lambda+\alpha}\right)^s\int_0^{\infty}\frac{1}{\eta\lambda s}(\beta+\alpha\frac{y}{\lambda s})^s(\frac{y}{\lambda s})^{\frac{s(\eta-1)+1}{\eta}-1}\exp{(-y)}dy\right).$$

Now, based on the binomial expansion of $(x+y)^r=\sum_{k=0}^r\binom{r}{k}x^{r-k}y^k$, we get

$$(\beta+\alpha\frac{y}{\lambda s})^s=\sum_{k=0}^{s}\binom{s}{k}\beta^{s-k}\alpha^k(\frac{y}{\lambda s})^k.$$

Then, by substituting into the preceding integral we get

$$
\begin{aligned}
I_R(s) &= \frac{1}{1-s}\ln\left(\left(\frac{\lambda^2\eta}{\beta\lambda+\alpha}\right)^s \sum_{k=0}^{s}\binom{s}{k}\beta^{s-k}\frac{\alpha^k}{\eta(\lambda s)^{k+\frac{s(\eta-1)+1}{\eta}}}\int_0^{\infty}(y)^{k+\frac{s(\eta-1)+1}{\eta}-1}\exp\left(-y\right)dy\right)\\
&= \frac{1}{1-s}\ln\left(\left(\frac{\lambda^2\eta}{\beta\lambda+\alpha}\right)^s \sum_{k=0}^{s}\binom{s}{k}\beta^{s-k}\frac{\alpha^k}{\eta(\lambda s)^{k+\frac{s(\eta-1)+1}{\eta}}}\Gamma(k+\frac{s(\eta-1)+1}{\eta})\right).
\end{aligned}
$$

4. Estimation of parameters

4.1. **Maximum Likelihood Estimation.** Let $X_1, X_2, ..., X_n$ be a random sample of size n from the PTPL distribution with the parameter vector $\Theta = (\lambda, \eta, \beta, \alpha)^T$. The log-likelihood function for Θ can be written as

$$
\begin{aligned}
l(\Theta) &= \log\prod_{i=1}^{n} f(x_i;\Theta) = \sum_{i=1}^{n}\log f(x_i;\Theta)\\
&= n[2\log(\lambda)+\log(\eta)-\log(\beta\lambda+\alpha)] + \sum_{i=1}^{n}\log(\beta+\alpha x_i^{\eta}) + (\eta-1)\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}x_i^{\eta}.
\end{aligned}
$$

Below are the score functions (partial derivatives) calculated for parameters λ, η, β, and α:

$$\frac{\partial l(\Theta)}{\partial\lambda} = \frac{2n}{\lambda} - \frac{n\beta}{\beta\lambda+\alpha} - \sum_{i=1}^{n}x_i^{\eta},$$

$$\frac{\partial l(\Theta)}{\partial\eta} = \frac{n}{\eta} + \sum_{i=1}^{n}\frac{\alpha\log(x_i)x_i^{\eta}}{\beta+\alpha x_i^{\eta}} + \sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}\log(x_i)x_i^{\eta},$$

$$\frac{\partial l(\Theta)}{\partial\beta} = -\frac{n\lambda}{\beta\lambda+\alpha} + \sum_{i=1}^{n}\frac{1}{\beta+\alpha x_i^{\eta}},$$

and

$$\frac{\partial l(\Theta)}{\partial\alpha} = -\frac{n}{\beta\lambda+\alpha} + \sum_{i=1}^{n}\frac{x_i^{\eta}}{\beta+\alpha x_i^{\eta}}.$$

Due to the inherent complexity of solving numerically these four coupled non-linear equations, iterative methods such as Newton-Raphson could theoretically be employed. However, we successfully resolve them through numerical implementation using the R-4.3.2 programming language.

4.2. **Asymptotic confidence Intervals.** The asymptotic confidence interval can be derived for unknown parameters λ, η, β and α with the assumption that the MLEs $(\lambda, \eta, \beta, \alpha)$ of these parameters are approximately multivariate normal with mean $(\lambda, \eta, \beta, \alpha)$ and inverse Fisher information observed variance-covariance matrix (I_n^{-1}). For our model, the Fisher information matrix takes the form of a 4×4 matrix defined by:

$$I_n^{-1} = \begin{pmatrix} -\frac{\partial^2 l(\Theta)}{\partial \lambda^2} & -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \eta} & -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \beta} & -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \alpha} \\ -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \eta} & -\frac{\partial^2 l(\Theta)}{\partial \eta^2} & -\frac{\partial^2 l(\Theta)}{\partial \eta \partial \beta} & -\frac{\partial^2 l(\Theta)}{\partial \eta \partial \alpha} \\ -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \beta} & -\frac{\partial^2 l(\Theta)}{\partial \eta \partial \beta} & -\frac{\partial^2 l(\Theta)}{\partial \beta^2} & -\frac{\partial^2 l(\Theta)}{\partial \beta \partial \alpha} \\ -\frac{\partial^2 l(\Theta)}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l(\Theta)}{\partial \eta \partial \alpha} & -\frac{\partial^2 l(\Theta)}{\partial \beta \partial \alpha} & -\frac{\partial^2 l(\Theta)}{\partial \alpha^2} \end{pmatrix}^{-1}$$

The asymptotic confidence intervals $(1-\alpha)100\%$ for the parameters of the PTPL distribution are constructed by

$$\hat{\lambda} \pm z_{\alpha/2}\sqrt{Var(\hat{\lambda})}, \quad \hat{\eta} \pm z_{\alpha/2}\sqrt{Var(\hat{\eta})}, \quad \hat{\beta} \pm z_{\alpha/2}\sqrt{Var(\hat{\beta})}, \quad and \quad \hat{\alpha} \pm z_{\alpha/2}\sqrt{Var(\hat{\alpha})},$$

where $z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of the standard $N(0,1)$ distribution.

5. Generation algorithms and simulation study

In this section, we develop three alternative sampling methods for the PTPL distribution. The first approach generates random samples by drawing from the three-parameter Lindley distribution via its exponential-gamma mixture formulation in equation (1.2), and applying the power transformation $x^{1/\eta}$ to the obtained values. A data generation procedure founded on equation (2.1) forms the basis of the second algorithm. Random sample generation in the third algorithm is accomplished via the inverse transform method applied to the PTPL CDF.

Algorithm 1:

(1) Generate U_i from a standard uniform distribution, $i = 1, ..., n$
(2) Generate $E_i \sim Exponential(\lambda)$, $i = 1, ..., n$
(3) Generate $G_i \sim Gamma(2, \lambda)$, $i = 1, ..., n$
(4) if $U_i \leq p = \frac{\beta\lambda}{\beta\lambda+\alpha}$, then set $X_i = E_i^{\frac{1}{\eta}}$; otherwise, set $X_i = G_i^{\frac{1}{\eta}}$; $i = 1, ..., n$

Algorithm 2:

(1) Generate U_i from a standard uniform distribution, $i = 1, ..., n$
(2) Generate $W_i \sim Weibul(\eta, \lambda)$, $i = 1, ..., n$
(3) Generate $G_i \sim Generalized\ Gamma(2, \eta, \lambda)$, $i = 1, ..., n$
(4) if $U_i \leq p = \frac{\beta\lambda}{\beta\lambda+\alpha}$, then set $X_i = W_i$; otherwise, $X_i = G_i$; $i = 1, ..., n$

Algorithm 3:

(1) Generate U_i from a standard uniform distribution, $i = 1, ..., n$
(2) Set for $i = 1, ..., n$:

$$X_i = [-\frac{\beta}{\alpha} - \frac{1}{\lambda} - \frac{1}{\lambda}W_{-1}[-\frac{(\beta\lambda+\alpha)(1-u)\exp(-(\beta\lambda+\alpha)/\alpha)}{\alpha}]]^{\frac{1}{\eta}},$$

where $W_{-1}(.)$ is specified in equation (3.2).

In the following, we conducted an extensive Monte Carlo simulation study to evaluate the finite-sample performance of the ML estimators for the PTPL distribution parameters. The study design involved generating $m = 1000$ samples with sizes 30, 50, 100, and 150 from the PTPL distribution, for three selected parameter value sets, detailed below:

- **Set 1**: $(\lambda = 2, \eta = 1.5, \beta = 0.5, \alpha = 1)$
- **Set 2**: $(\lambda = 0.5, \eta = 2, \beta = 1, \alpha = 3)$
- **Set 3**: $(\lambda = 1, \eta = 0.5, \beta = 1.5, \alpha = 2)$

A simulation study was conducted following the steps outlined below:
Step 1: Choose the initial values of $\lambda_0, \eta_0, \beta_0, \alpha_0$ for the respective components of parameter vector $\Theta = (\lambda, \eta, \beta, \alpha)$ that define the PTPL distribution.
Step 2: Choose the sample size n.
Step 3: Generate m independent samples of size n from PTPL distribution by using **Algorithm 1**.
Step 4: Calculate the estimate $\hat{\Theta}_{ML}$ of Θ for each of the m samples.
Step 5: Compute the average bias and mean square error (MSE) of the simulated estimates:

$$bias(b) = \frac{1}{m}\sum_{i=1}^{m} \hat{b}_i - b_0, \text{ and } MSE(b) = \frac{1}{m}\sum_{i=1}^{m} (\hat{b}_i - b_0)^2,$$

where $b = \lambda, \eta, \beta, \alpha$.

The results of the simulation are shown in Table 3.

Table 3. Average biases and MSEs of the simulated estimates.

Set 1	$bias(\lambda)$	$MSE(\lambda)$	$bias(\eta)$	$MSE(\eta)$	$bias(\beta)$	$MSE(\beta)$	$bias(\alpha)$	$MSE(\alpha)$
n=30	-0.08	0.23	0.08	0.08	0.40	0.64	-0.26	0.64
n=50	-0.08	0.21	0.05	0.06	0.37	0.61	-0.22	0.27
n=100	-0.08	0.18	0.03	0.04	0.33	0.54	-0.19	0.24
n=150	-0.07	0.17	0.02	0.03	0.31	0.53	-0.18	0.23
Set 2								
n=30	-0.05	0.04	0.22	0.26	1.54	11.59	-0.54	1.37
n=50	-0.05	0.03	0.17	0.19	1.38	10.23	-0.48	1.22
n=100	-0.04	0.03	0.14	0.13	1.34	9.31	-0.47	1.14
n=150	-0.04	0.02	0.11	0.10	1.21	8.63	-0.43	1.08
Set 3								
n=30	-0.05	0.11	0.05	0.02	0.22	0.34	-0.11	0.19
n=50	-0.04	0.09	0.03	0.02	0.18	0.31	-0.08	0.20
n=100	-0.02	0.08	0.01	0.01	0.15	0.29	-0.04	0.21
n=150	-0.02	0.07	0.01	0.01	0.16	0.29	-0.06	0.19

The results in Table 3 indicate that as the sample size n grows, the simulated estimates' biases and mean square errors converge to zero.

6. Engineering real data application

The current section showcases the distribution's fitting effectiveness by analyzing two engineering datasets. Our PTPL distribution compared with other related distributions : Lindley (L) [11], XLindley (XL) [5] , gamma Lindley (gaL) [14], three-parameter Lindley (TPL) [20], and Beta generalized Lindley (BGL) [15]. All model parameters were estimated by maximizing the likelihood function using the package named "maxLik" in the R-4.3.2 programming language. The data are listed as follows:

Data set 1: The first set of data concerns 50 component failures, which was initially examined by [13]. The following are the observations made: 0.04, 0.06, 0.06, 0.07, 0.08, 0.09, 0.10, 0.10, 0.11, 0.12, 0.15, 0.18, 0.19, 0.25, 0.26, 0.37, 0.38, 0.54, 0.57, 0.58, 0.59, 0.62, 0.64, 0.96, 1.23, 1.60, 2.01, 2.05, 2.80, 3.06, 3.08, 3.15, 3.62, 3.70, 3.93, 4.07, 4.39, 4.53, 4.89, 6.27, 6.82, 7.89, 7.90, 8.02, 9.34, 10.94, 11.02, 13.88, 14.73, 15.08.

Data set 2: Breaking stress data (measured in GPa) for 50 mm-long carbon fibers measurements serve as the foundation for our secondary dataset, and is also used by [2]. The following are the observations made: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Table 4 summarizes the datasets statistical properties.

Table 4. Statistical properties of the data sets

	Min	1stQu	Median	Mean	3rdQu	Max
Data set 1	0.04	0.21	1.41	3.34	4.49	15.08
Data set 2	0.39	2.18	2.84	2.76	3.28	4.90

The fitted distributions are compared using -2LL, in which LL indicates the log-likelihood evaluated at the ML parameter estimates, the Bayesian information criterion (BIC), the Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), and Hannan-quinn information criterion (HQIC). The following criteria formulas are as follows.

- $AIC = -2LL + 2k$,
- $BIC = -2LL + k\log(n)$,
- $CAIC = -2LL + \frac{2kn}{n-k-1}$,
- $HQIC = -2LL + 2k\log(\log(n))$.

where n is the sample size and k is the number of estimated parameters.

Tables 5, and 7 provide the estimated values of the distribution parameters, and Tables 6, and 8 provide the numerical values of the information criterion (e.g., -2LL, AIC, BIC, CAIC and HQIC) for each model. While most estimates demonstrate reasonable stability, we observe numerical instabilities manifested through the extremely large β estimate for gaL distribution $(5.04e+06)$ in Table 7. This phenomenon primarily occurs when the parameter is near-nonidentifiable, or with insufficient sample sizes ($n = 66$ in our study). Such computational instabilities may limit the practical utility of this distribution in certain applications.

In Figure 5, we show the total time on test (TTT) plots for both datasets. For Data 1, the TTT plot displays a convex curve, indicating a monotonically decreasing hazard rate. The TTT plot for Data 2 shows a concave curve, suggesting that the hazard rate function is increasing. In contrast, The PTPL distribution's hazard rate function provides statistically significant fits to both datasets. For further details on TTT plots, see [1]. Across all evaluated scenarios, the PTPL distribution consistently demonstrates consistent dominance, as evidenced by systematically lower values across all information criteria when compared to competing distributions. This comprehensive outperformance strongly suggests that, among the considered models, our PTPL distribution provides the best fit to the data.

Figures 3 and 4 provide graphical validation of the PTPL distribution's superior fit through four complementary representations. These visual diagnostics collectively demonstrate excellent agreement between the empirical data and our proposed model. Furthermore, Figure 5 confirms the distribution's capability to capture the underlying hazard rate patterns, with the estimated function closely following the observed failure trends.

Table 5. Parameter ML estimates for data set 1

Distribution	Estimates			
PTPL	$\hat{\lambda}$ =0.65	$\hat{\eta}$ =0.65	$\hat{\beta}$ =2.01	$\hat{\alpha}$ =0.28
BGL	$\hat{\alpha}$ =0.07	$\hat{\lambda}$ =0.01	$\hat{a}$ =14.78	$\hat{b}$ =15.09
TPL	$\hat{\lambda}$ =0.30	$\hat{\beta}$ =2.68	$\hat{\alpha}$ =0.01	
gaL	$\hat{\lambda}$ =0.30	$\hat{\beta}$ = 0.23		
XL	$\hat{\lambda}$ =0.44			
L	$\hat{\lambda}$ =0.49			

Table 6. The numerical values of the information criterion for dataset 1

Distribution	-2LL	AIC	BIC	CAIC	HQIC
PTPL	204.66	212.66	220.31	213.55	215.57
BGL	204.72	212.72	220.37	213.61	215.63
TPL	220.68	226.68	232.42	227.21	228.87
gaL	220.68	224.68	228.51	224.94	226.14
XL	229.56	231.56	233.47	231.64	232.28
L	240.36	242.36	244.27	242.44	243.08

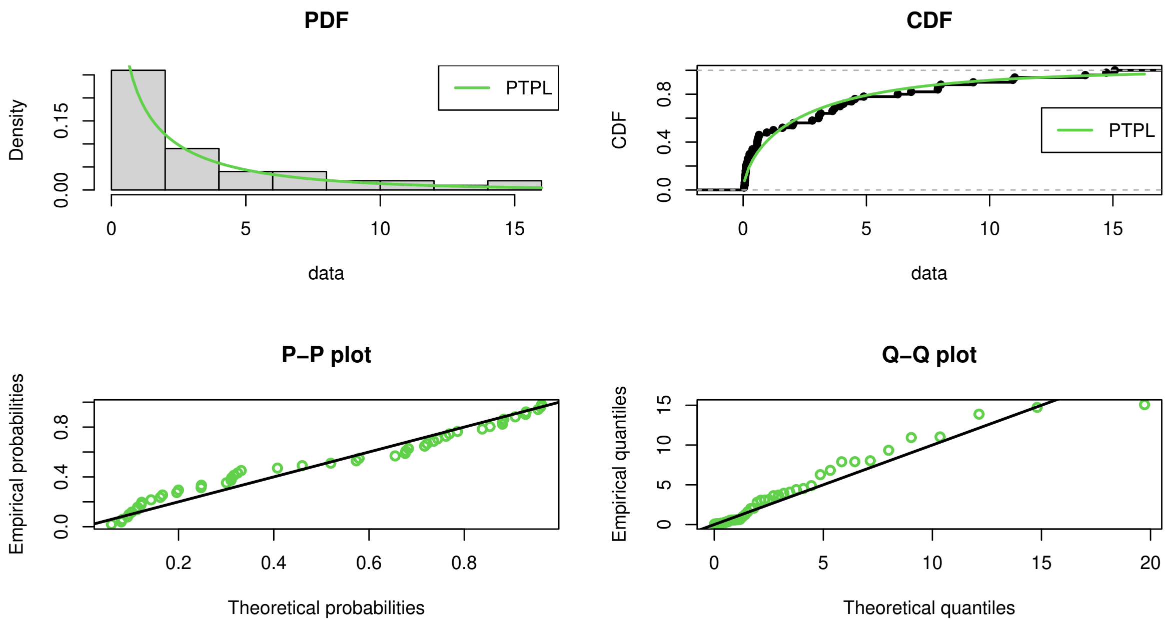

FIGURE 3. The estimated PDF and CDF along with their corresponding P-P and Q-Q diagnostic plots of the PTPL distribution for dataset 1.

TABLE 7. Parameter ML estimates for dataset 2

Distribution	Estimates			
PTPL	$\hat{\lambda} = 0.08$	$\hat{\eta}$ =2.78	$\hat{\beta}$ =1.90	$\hat{\alpha}$ =0.43
BGL	$\hat{\alpha} = 4.33$	$\hat{\lambda}$ =0.29	$\hat{a} = 0.61$	$\hat{b}$ =82.97
TPL	$\hat{\lambda} = 0.91$	$\hat{\beta}$ =-170.96	$\hat{\alpha}$ =458.24	
gaL	$\hat{\lambda}$ =0.72	$\hat{\beta}$ = 5.04e+06		
XL	$\hat{\lambda}$ =0.51			
L	$\hat{\lambda}$ =0.59			

TABLE 8. The numerical values of the information criterion for dataset 2

Distribution	-2LL	AIC	BIC	CAIC	HQIC
PTPL	171.01	179.01	187.77	179.67	182.47
BGL	171.55	179.55	188.31	180.21	183.02
TPL	204.46	210.46	217.03	210.85	213.05
gaL	224.01	228.01	232.38	228.19	229.74
XL	254.93	256.93	259.12	256.99	257.79
L	244.77	246.77	248.96	246.83	247.63

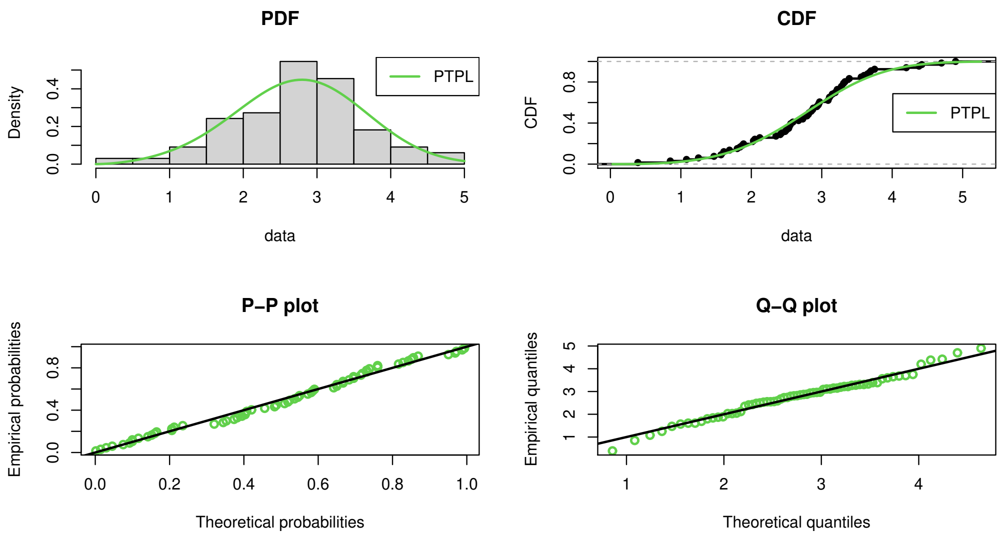

FIGURE 4. The estimated PDF and CDF along with their corresponding P-P and Q-Q diagnostic plots of the PTPL distribution for dataset 2.

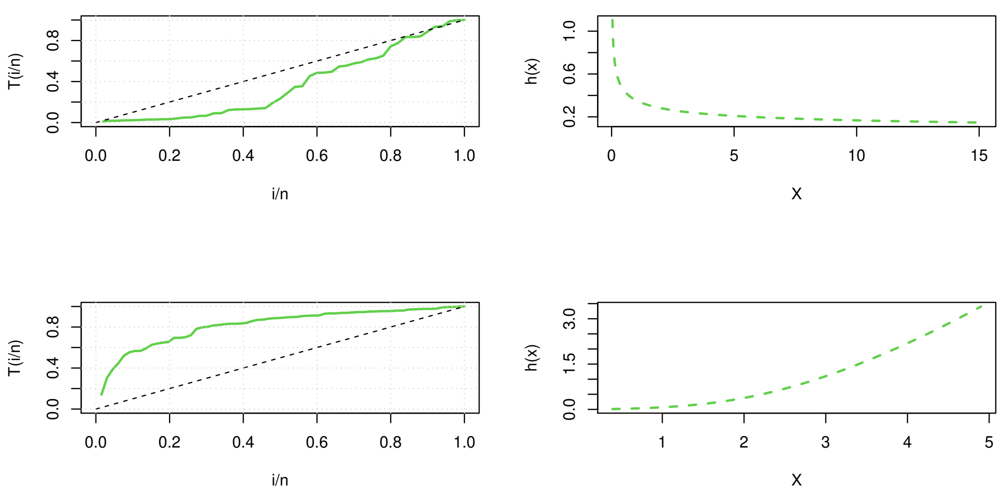

FIGURE 5. Estimated hazard rate function plots of the PTPL distribution and TTT plot for the two data sets.

7. Conclusion

This study has developed a new flexible lifetime distribution that contains twelve special cases of established models, with complete derivation of its density and distribution functions. The investigation further establishes essential statistical properties of the introduced model.

Our methodological contributions include the maximum likelihood estimation with asymptotic confidence intervals inference. Providing the necessary tools for real-world application, we developed three distinct algorithms for random variate generation from the proposed distribution. Through extensive simulation studies, we evaluated the finite-sample properties of the estimators, particularly examining their bias and mean squared error. The distribution's practical utility is demonstrated through superior performance in modeling two real-world engineering datasets, outperforming existing distributions and showing promise for diverse applications.

Finally, in our further research we will study the Gini-index goodness-of-fit statistical test by generalizing the work of [3]. Also, address the challenging theoretical problems of distribution extension under incomplete progressively censored data scenarios. These directions represent natural and important extensions of the current work.

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