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EXISTENCE AND DECAY RATES OF SOLUTION FOR THE MOORE-GIBSON-THOMPSON EQUATION LINEAR

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ABSTRACT. This work is devoted to the study the linear Moore-Gibson-Thompson equation with a viscoelastic memory. Under some appropriate assumptions on the relaxation function g and with certain initial data, the existence and uniqueness of a local solution are obtained via Faedo–Galerkin's method. The global existence of solutions is also established. Furthermore, we prove the decay rates for the energy of Moore-Gibson-Thompson equation linear with a viscoelastic memory of relaxation kernels.

Keywords. Moore-Gibson-Thompson equation; Viscoelastic memory; Galerkin method; global existence; intrinsic decay rates
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1. Introduction

In this paper, we consider the following Moore-Gibson-Thompson equation linear with a viscoelastic memory term

$$v_{ttt}+\alpha v_{tt}-c^2\Delta v-b\Delta v_t+\int_0^t g\left(t-s\right)\Delta v\left(s\right)ds=0, \quad (x,t)\in\Omega\times(0,T)\,, \tag{1.1}$$

with initial data

$$v\left(x,0\right)=\varphi\left(x\right),\ v_t\left(x,0\right)=\omega\left(x\right),\ v_{tt}\left(x,0\right)=\Upsilon\left(x\right), \quad x\in\Omega, \tag{1.2}$$

and boundary conditions

$$v\left(x,t\right)=0, \quad (x,t)\in\partial\Omega\times(0,T)\,, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $T<\infty$, and $g\left(.\right):\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is the relaxation memory kernel, has recently caught the attention of several authors. Here the memory kernel g directly relates to whether or how the energy decays and satisfies some assumptions that will be mentioned later on, and $\varphi\left(x\right),\omega\left(x\right)$ and $\Upsilon\left(x\right)$ are given functions satisfying conditions specified later. Parameters α and b are assumed to be positive. In the context of the acoustic waves, the variable v denotes a scalar acoustic velocity potential $u^{\rightarrow}=-\nabla v$ with $u^{\rightarrow}$ denoting the acoustic particle velocity, c^2 denotes the speed of sound, the

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coefficient $\beta \approxeq \eta + c^2$ where η is the diffusibility of the sound and the coefficient $\beta > 0$ describes natural damping effects associated with an acoustic environment, see Lebon and Cloot [16] . The convolution term $\int_0^t g(t-s)\Delta v(s)\,ds$ represents the memory effects of an acoustic environment.

From the physical view point, (1.1) is a wave-type equation arising in the context of acoustic wave propagation with the so-called second sound, being u the acoustic pressure, where the paradox of the infinite speed of propagation is eliminated by replacing the Fourier law by the Maxwell-Cattaneo one.
This nonlinear equation

$$\tau v_{ttt} + v_{tt} - c^2\Delta v - b\Delta v_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(v_t)^2 + |\nabla v|^2\right), \tag{1.4}$$

is known as the Jordan Moore Gibson Thompson equation (JMGT). For a brief overview on nonlinear acoustics and the derivation of the JMGT equation see the recent papers [8], [12] and [11] and references therein.
In what follows we shall first review the linearized version of equation (1.4), which is known as the Moore Gibson Thompson equation (MGT), in its more general form:

$$\tau v_{ttt} + v_{tt} + c^2 A v + b A v_t = 0, \tag{1.5}$$

where A is a positive self-adjoint operator. Recently many authors have been interested to the equation (1.5) mainly in bounded domains.
The MGT equation with a memory term of the form:

$$\tau v_{ttt} + \alpha v_{tt} - \beta\Delta v - \gamma\Delta v_t + \int_0^t g(t-s)\Delta v(s)\,ds = 0, \quad (x,t)\in\Omega\times(0,T), \tag{1.6}$$

where τ; α; β and γ are positive constants.
The motivation of our work is due to some results regarding the following research papers. In [15], Lasiecka, I and Wang, X., studied, the following Moore–Gibson–Thompson equation with memory

$$\tau v_{ttt} + \alpha v_{tt} + c^2 A v + b A v_t - \int_0^t g(t-s) A w(s)\,ds = 0, \quad (x,t)\in\Omega\times(0,T). \tag{1.7}$$

They showed how a memory term creates damping mechanism and how the memory causes energy decay of problem (1.7), under suitable conditions on the relaxation function g. Typical memory types encountered can be classified in [15] as: Type 1 memory $w(s) = v(s)$; Type 2 memory $w(s) = v_s(s)$; and Type 3 memory $w(s) = \lambda v(s) + v_s(s)$, with an appropriate constant $\lambda > 0$.

[21] is a generalization of the work in [15] as it allows the memory kernel to be more general and shows that the energy decays of problem (1.4) (for $w(s) = v(s)$, Type1 memory in [15]) the same way as the memory kernel does, exponentially ($h'(t) \le -ah(t)$, $a > 0$) or not (Example. $h'(t) \le -h(t)|h(t)|^{\beta-1}$, $\beta \in [1,2]$). For more explanation, we refer the readers to [21].
In [9], Wenjun et al studied new general decay results for a Moore–Gibson–Thompson equation with memory which is a generalization of the the work one in [14, 15] in that it allows the memory kernel to be more general. For a class of relaxation functions satisfying $h'(t) \le -\xi(t) H(h(t))$ for H to be increasing

and convex function near the origin and $\xi(t)$ to be a nonincreasing function, and proved the optimal explicit and general energy decay result.
In [4], the author draifia treated (1.4) and $A = -\Delta$ (Laplacian operator), and $w(s) = v(s)$ (Type1 memory in [15]). They showed the global solution of problem (1.4) and established general decay of the energy under suitable conditions on the relaxation function g (with not necessarily decreasing kernel). For more explanation, we refer the readers to [4].
In [13], Irena Lasiecka studied, Global solvability of Moore–Gibson–Thompson equation with memory arising in nonlinear acoustics. The main result of the paper [13] states that with appropriate calibration of the memory kernel, solutions exist globally for sufficiently small and regular initial data. With exponentially decaying relaxation kernel said solutions exhibit exponential decay rates, for the following Moore–Gibson–Thompson equation with memory arising in nonlinear acoustics

$$\tau v_{ttt}+(\alpha-2kv)\,v_{tt}+c^2Av+bAv_t-\int_0^t h(t-s)\,Aw(s)\,ds = 2kv_t^2,\quad (x,t)\in\Omega\times(0,T),$$

where $k>0$, $w = u_t + \frac{c^2}{b}u$ and $\alpha = \frac{\tau c^2}{b}$.
In [15], Lasiecka, I and Wang, X., studied, the following Moore–Gibson–Thompson equation with memory

$$\tau v_{ttt}+\alpha v_{tt}+c^2Av+bAv_t-\int_0^t h(t-s)\,Aw(s)\,ds = 0,\quad (x,t)\in\Omega\times(0,T).\quad (1.8)$$

For more results, we refer to the previous papers [1, 24].
Our plan in this paper is as follows: In Section 2, we give some lemmas and assumptions, which will be used later. Section 3 is devoted to the study of local existence and uniqueness of the weak solution of $(1.1)-(1.3)$ via Galerkin's method. In Section 4, we obtain global existence of the solution of $(1.1)-(1.3)$. Finally, in Section 5, we prove the decay rates for the energy of $(1.1)-(1.3)$.

2. Assumptions and main results

In this section we prepare some notation and assumptions which will be needed in the proof of our results.
Let us first introduce the following notations:

$$(g\circ\nabla v)(t) := \int_\Omega\int_0^t g(t-s)\,|\nabla v(x,t)-\nabla v(x,s)|^2\,dsdx,$$

$$(g\diamond v)(t) := \int_0^t g(t-s)\,(v(t)-v(s))\,ds,$$

$$(g\Box\nabla v)(t) := \int_0^t g(t-s)\,|\nabla v(x,t)-\nabla v(x,s)|^2\,ds.$$

Set

$$V(D_T) := \left\{v(x,t) : v\in H_0^1(D_T),\ v_t\in H^1(D_T),\ \text{and } v_{tt}\in L^2(D_T)\right\},$$

where $D_T := \Omega\times(0,T)$, and

$$W(D_T) := \{u(x,t) : u\in V(D_T) \text{ and } u(x,T)=0\}.$$

Definition 2.1. A function $v \in V(D_T)$ is called a generalized solution of problems $(1.1) - (1.3)$ if it satisfies, for each $u \in W(D_T)$,

$$\begin{aligned}
& -\left(v_{tt}(t), u_t(t)\right)_{L^2(D_T)} - \alpha\left(v_t(t), u_t(t)\right)_{L^2(D_T)} \\
& +c^2\left(\nabla v(t), \nabla u(t)\right)_{L^2(D_T)} + b\left(\nabla v_t(t), \nabla u(t)\right)_{L^2(D_T)} \\
& -\left(\int_0^t h(t-s)\nabla v(s)\,ds, \nabla u(t)\right)_{L^2(D_T)} \\
= \; & \alpha\left(\omega(x), u(0)\right)_{L^2(\Omega)} + \left(\Upsilon(x), u(0)\right)_{L^2(\Omega)}. \qquad (2.1)
\end{aligned}$$

Lemma 2.2. [24] *For any function $v \in C^1\left((0,T); H_0^1(\Omega)\right)$, we have*

$$\begin{aligned}
& \left(\int_0^t g(t-s)\,\Delta v(s)\,ds, v_t(t)\right)_\Omega \\
= \; & \frac{1}{2}\frac{d}{dt}\left\{(g \circ \nabla v)(t) - \left(\int_0^t g(s)\,ds\right)\|\nabla v(t)\|^2_{L^2(\Omega)}\right\} \\
& -\frac{1}{2}(g' \circ \nabla v)(t) + \frac{1}{2}g(t)\|\nabla v(t)\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
& \left(\int_0^t g(t-s)\,\Delta v(s)\,ds, v_{tt}(t)\right)_\Omega \\
= \; & \frac{1}{2}\frac{d}{dt}\left\{(-g' \circ \nabla v)(t) + g(t)\|\nabla v(t)\|^2_{L^2(\Omega)} - 2\int_0^t g(t-s)\left(\nabla v(s), \nabla v_t(t)\right)_{L^2(\Omega)} ds\right\} \\
& +\frac{1}{2}(g'' \circ \nabla v)(t) - \frac{1}{2}g'(t)\|\nabla v(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Now, let us state the general hypotheses

$(\mathbf{H_1})$ $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a bounded C^2 study the Moore-Gibson-Thompson equation linear-function satisfying $g(0) > 0$ and

$$\bar{g} := \int_0^\infty g(s)\,ds < c^2.$$

$(\mathbf{H_2})$ $g'(t) \leq -H(g(t))$, for all $t \geq 0$, where $H \in C^1(\mathbb{R}_+)$ with $H(0) = 0$ is a given strictly increasing and convex function. Moreover

$$H \in C^2(0,\infty) \text{ and } \lim\inf_{x \to 0^+}\left\{x^2 H''(x) - xH'(x) + H(x)\right\} \geq 0.$$

$(\mathbf{H_3})$ $g''(t) \geq 0$.

$(\mathbf{H_4})$ Let $y(t)$ be the solution of the ODE

$$y'(t) + H(y(t)) = 0, \quad y(0) = g(0) > 0.$$

$(\mathbf{H_5})$ We assume that there exists $\alpha_0 \in [0,1)$ such that $y^{1-\alpha_0} \in L_1(1,\infty)$.

3. Existence and Uniqueness of solution

In this section, we will prove the existence and uniqueness of the weak solution of problem $(1.1) - (1.3)$. We use the Faedo–Galerkin approximations, to prove the existence of the unique solution of problem $(1.1) - (1.3)$.

3.1. **Existence.** We now give the main result on the weak solution of problem $(1.1)-(1.3)$ and prove it by using the Galerkin method.

Theorem 3.1. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_3})$ *hold, the initial data* $\varphi(x)\in H_0^1(\Omega)$, $\omega(x)\in H^1(\Omega)$ *and* $\Upsilon(x)\in L^2(\Omega)$, *then there is at least one generalized solution in* $V(D_T)$ *to problem* $(1.1)-(1.3)$.

Proof. We divide the proof of **Theorem 1**. in two steps: the construction of approximations and then thanks to certain energy estimates, we pass to the limit.
Step 1.: Faedo–Galerkin approximation.
We construct approximations of the solution v by the Faedo–Galerkin method as follows; Let $\{\psi_l(x)\}_{l\geq 1}$ be a fundamental system in $H_0^1(\Omega)$ and assume for convenience the it has been orthonormalized in $L^2(\Omega)$, that is $(\psi_l(x),\psi_k(x))_\Omega=\delta_{l,k}$. We seek an approximate solution $v^m(x)$ in the form

$$v^m(x,t)=\sum_{l=1}^{l=m} f_l(t)\psi_l(x), \tag{3.1}$$

with

$$f_l(t)=(\psi_l(x),v^m(x,t))_{L^2(\Omega)}, \quad l=1,...,m.$$

They can be found, for $k=1,...,m$, via the relations

$$\begin{aligned}
&(v_{ttt}^m(x,t),\psi_k(x))_{L^2(\Omega)}+\alpha(v_{tt}^m(x,t),\psi_k(x))_{L^2(\Omega)}\\
&+c^2(\nabla v^m(x,t),\nabla\psi_k(x))_{L^2(\Omega)}+b(\nabla v_t^m(x,t),\nabla\psi_k(x))_{L^2(\Omega)}\\
&-\left(\int_0^t g(t-s)\nabla v^m(x,s)\,ds,\nabla\psi_k(x)\right)_{L^2(\Omega)}\\
=\;&0.
\end{aligned}\tag{3.2}$$

The above system is a system of ordinary differential equations in $f_l(t)$, $l=1,...,m$, and initial conditions

$$f_l(0)=(\psi_l,\varphi(x))_{L^2(\Omega)},\ f_l'(0)=(\psi_l,\omega(x))_{L^2(\Omega)},\ f_l''(0)=(\psi_l,\Upsilon(x))_{L^2(\Omega)}.$$

Substitution of (3.1) into (3.2) gives for all $l=1,...,m$

$$\begin{aligned}
&\sum_{l=1}^{l=m}\Big\{f_l'''(t)(\psi_l(x),\psi_k(x))_{L^2(\Omega)}+\alpha f_l''(t)(\psi_l(x),\psi_k(x))_{L^2(\Omega)}\\
&+c^2 f_l(t)(\nabla\psi_l(x),\nabla\psi_k(x))_{L^2(\Omega)}+bf_l'(t)(\nabla\psi_l(x),\nabla\psi_k(x))_{L^2(\Omega)}\\
&-\left(\int_0^t g(t-s)f_l(s)\,ds\right)(\nabla\psi_l(x),\nabla\psi_k(x))_{L^2(\Omega)}\Big\}\\
=\;&0.
\end{aligned}$$

If in this last equality, we make the notations

$$(\psi_l(x),\psi_k(x))_{L^2(\Omega)}:=\delta_{lk}=\begin{cases}1, & l=k,\\ 0, & l\neq k,\end{cases}$$

and

$$\left(\nabla\psi_l\left(x\right),\nabla\psi_k\left(x\right)\right)_{L^2(\Omega)}:=\gamma_{lk},$$

and

$$\int_0^t g\left(t-s\right)f_l\left(s\right)ds:=\zeta\left(f_l\right)\left(t\right),$$

then we have the following system

$$\begin{aligned}&\sum_{l=1}^{l=m}\left\{\left(f_l'''\left(t\right)+\alpha f_l''\left(t\right)\right)\delta_{lk}+\left(c^2f_l\left(t\right)+bf_l'\left(t\right)-\zeta\left(f_l\right)\left(t\right)\right)\gamma_{lk}\right\}\\ =\;&0,\end{aligned}\tag{3.3}$$

and initial conditions

$$f_l(0)=\left(\psi_l\left(x\right),\varphi\left(x\right)\right)_{L^2(\Omega)},\; f_l'(0)=\left(\psi_l\left(x\right),\varphi\left(x\right)\right)_{L^2(\Omega)},\; f_l''\left(0\right)=\left(\psi_l\left(x\right),\Upsilon\left(x\right)\right)_{L^2(\Omega)}.\tag{3.4}$$

We obtain a system of differential equations of four orders with respect to the variable t with constant coefficients and the initial conditions (3.4); consequently, we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. Thus for every m there exists a function $v^m\left(x\right)$ satisfying (3.2). Let us obtain bounds for v^m which do not depend on m.

Step 2.: Energy estimates.

We set

$$S^m\left(t\right):=\left\|v^m\left(t\right)\right\|^2_{H_0^1(\Omega)}+\left\|v_t^m\left(t\right)\right\|^2_{H^1(\Omega)}+\left\|v_{tt}^m\left(t\right)\right\|^2_{L^2(\Omega)}.\tag{3.5}$$

We multiply each equation of (3.2) by the appropriate $f_k''\left(t\right)$, add them up from 1 to m and them integrate with respect to t from 0 to τ, with $\tau\leq T$, we obtain

$$\begin{aligned}&\left(v_{ttt}^m\left(t\right),v_{tt}^m\left(t\right)\right)_{L^2(D_\tau)}+\alpha\left(v_{tt}^m\left(t\right),v_{tt}^m\left(t\right)\right)_{L^2(D_\tau)}\\ &+c^2\left(\nabla v^m\left(t\right),\nabla v_{tt}^m\left(t\right)\right)_{L^2(D_\tau)}+b\left(\nabla v_t^m\left(t\right),\nabla v_{tt}^m\left(t\right)\right)_{L^2(D_\tau)}\\ &-\left(\int_0^t g\left(t-s\right)\nabla v^m\left(s\right)ds,\nabla v_{tt}^m\left(t\right)\right)_{L^2(D_\tau)}\\ =\;&0.\end{aligned}\tag{3.6}$$

Evaluation of each term of (3.6). Therefore, using integration by parts and **Lemma 1**. we get,

$$\begin{aligned}
& \frac{g(\tau)}{2}\|\nabla v^m(\tau)\|^2_{L^2(\Omega)} + c^2\left(\nabla v^m(\tau), \nabla v^m_\tau(\tau)\right)_{L^2(\Omega)} \\
& +\frac{b}{2}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v^m_{\tau\tau}(\tau)\|^2_{L^2(\Omega)} \\
& -\int_0^\tau g(\tau-s)\left(\nabla v^m(s), \nabla v^m_\tau(\tau)\right)_{L^2(\Omega)} ds + \frac{1}{2}\left(-g'\circ\nabla v^m\right)(\tau) \\
= \; & \frac{g(0)}{2}\|\nabla v^m(0)\|^2_{L^2(\Omega)} + c^2\left(\nabla v^m(0), \nabla v^m_t(0)\right)_{L^2(\Omega)} \\
& +\frac{b}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v^m_{tt}(0)\|^2_{L^2(\Omega)} \\
& +\frac{1}{2}\int_0^\tau g'(t)\|\nabla v^m(t)\|^2_{L^2(\Omega)} dt + c^2\int_0^\tau \|\nabla v^m_t(t)\|^2_{L^2(\Omega)} dt \\
& -\alpha\int_0^\tau \|v^m_{tt}(t)\|^2_{L^2(\Omega)} dt - \frac{1}{2}\int_0^\tau \left(g''\circ\nabla v^m\right)(t)\, dt. \qquad (3.7)
\end{aligned}$$

Young's inequality with $\delta = \frac{3c^2}{b}$ allows to get

$$\begin{aligned}
& -\frac{3c^4}{2b}\|\nabla v^m(\tau)\|^2_{L^2(\Omega)} - \frac{b}{6}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)} \\
\leq \; & c^2\left(\nabla v^m(\tau), \nabla v^m_\tau(\tau)\right)_{L^2(\Omega)}. \qquad (3.8)
\end{aligned}$$

Young's inequality with $\delta = \frac{3c^2}{b}$, and $-\int_0^\tau g(s)\, ds \geq -\bar{g}$, we obtain

$$\begin{aligned}
& -\frac{6(\bar{g})^2}{2b}\|\nabla v^m(\tau)\|^2_{L^2(\Omega)} - \frac{3\bar{g}}{b}\left(g\circ\nabla v^m\right)(\tau) - \frac{b}{6}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)} \\
\leq \; & -\int_0^\tau g(\tau-s)\left(\nabla v^m(s), \nabla v^m_\tau(\tau)\right)_{L^2(\Omega)} ds. \qquad (3.9)
\end{aligned}$$

Young's inequality with $\delta = 1$ allows to get

$$c^2\left(\nabla v^m(0), \nabla v^m_t(0)\right)_{L^2(\Omega)} \leq \frac{c^2}{2}\|\nabla v^m(0)\|^2_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)}. \qquad (3.10)$$

Combining inequalities (3.8) – (3.10) into (3.7), we have

$$\begin{aligned}
& -\frac{3\left(c^4 + 2(\bar{g})^2\right)}{2b}\|\nabla v^m(\tau)\|^2_{L^2(\Omega)} + \frac{b}{6}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)} \\
& +\frac{1}{2}\|v^m_{\tau\tau}(\tau)\|^2_{L^2(\Omega)} - \frac{3\bar{g}}{b}\left(g\circ\nabla v^m\right)(\tau) + \frac{1}{2}\left(-g'\circ\nabla v^m\right)(\tau) \\
\leq \; & \frac{\left(g(0)+c^2\right)}{2}\|\nabla v^m(0)\|^2_{L^2(\Omega)} + \frac{\left(c^2+b\right)}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)} \\
& +\frac{1}{2}\|v^m_{tt}(0)\|^2_{L^2(\Omega)} + c^2\int_0^\tau \|\nabla v^m_t(t)\|^2_{L^2(\Omega)} dt. \qquad (3.11)
\end{aligned}$$

We multiply each equation of (3.2) by the appropriate $f_k'(t)$ add them up from 1 to m and then integrate on $(0,\tau)$, 1 to m , we obtain

$$\begin{aligned}
& \left(v_{ttt}^m(t), v_t^m(t)\right)_{L^2(D_\tau)} + \alpha\left(v_{tt}^m(t), v_t^m(t)\right)_{L^2(D_\tau)} \\
& + c^2\left(\nabla v^m(t), \nabla v_t^m(t)\right)_{L^2(D_\tau)} + b\left(\nabla v_t^m(t), \nabla v_t^m(t)\right)_{L^2(D_\tau)} \\
& - \left(\int_0^t g(t-s)\nabla v^m(s)\,ds, \nabla v_t^m(t)\right)_{L^2(D_\tau)} \\
= \; & 0. \qquad (3.12)
\end{aligned}$$

Evaluation of each term of (3.12). Therefore, using integration by parts and **Lemma 1** we get,

$$\begin{aligned}
& \frac{1}{2}\left(c^2 - \int_0^\tau g(s)\,ds\right)\|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 \\
& + \left(v_{\tau\tau}^m(\tau), v_\tau^m(\tau)\right)_{L^2(\Omega)} + \frac{1}{2}\left(g\circ\nabla v^m\right)(\tau) \\
= \; & \frac{c^2}{2}\|\nabla v^m(0)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|v_t^m(0)\|_{L^2(\Omega)}^2 \\
& + \left(v_{tt}^m(0), v_t^m(0)\right)_{L^2(\Omega)} - b\int_0^\tau \|\nabla v_t^m(t)\|_{L^2(\Omega)}^2\,dt \\
& + \frac{1}{2}\int_0^\tau \left(g'\circ\nabla v^m\right)(t)\,dt - \frac{1}{2}\int_0^\tau g(t)\|\nabla v^m(t)\|_{L^2(\Omega)}^2\,dt. \qquad (3.13)
\end{aligned}$$

Young's inequality with $\delta = \dfrac{b}{24\bar{g}}$ allows to get

$$-\frac{b}{48\bar{g}}\|v_{\tau\tau}^m(\tau)\|_{L^2(\Omega)}^2 - \frac{12\bar{g}}{b}\|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 \le \left(v_{\tau\tau}^m(\tau), v_\tau^m(\tau)\right)_{L^2(\Omega)}. \qquad (3.14)$$

Young's inequality with $\delta = 1$ allows to get

$$\left(v_{tt}^m(0), v_t^m(0)\right)_{L^2(\Omega)} \le \frac{1}{2}\|v_{tt}^m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v_t^m(0)\|_{L^2(\Omega)}^2. \qquad (3.15)$$

Combining inequalities (3.14) and (3.15) into (3.13), we have

$$\begin{aligned}
& \frac{(c^2-\bar{g})}{2}\|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 + \left\{\frac{\alpha}{2} - \frac{12\bar{g}}{b}\right\}\|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 \\
& - \frac{b}{48\bar{g}}\|v_{\tau\tau}^m(\tau)\|_{L^2(\Omega)}^2 + \frac{1}{2}\left(g\circ\nabla v^m\right)(\tau) \\
\le \; & \frac{c^2}{2}\|\nabla v^m(0)\|_{L^2(\Omega)}^2 + \frac{\{\alpha+1\}}{2}\|v_t^m(0)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v_{tt}^m(0)\|_{L^2(\Omega)}^2. \qquad (3.16)
\end{aligned}$$

Multiplying (3.16) by $\dfrac{12\bar{g}}{b}$ and combining suitably with (3.11) and make use of the following inequalitie

$$\begin{aligned}
& \|v^m(\tau)\|_{L^2(\Omega)}^2 + \beta_1\|\nabla v^m(\tau)\|_{L^2(\Omega)}^2 + \beta_2\|v_\tau^m(\tau)\|_{L^2(\Omega)}^2 \\
\le \; & (1+\beta_1+\beta_2)\int_0^\tau S^m(t)\,dt + \|v^m(0)\|_{L^2(\Omega)}^2 + \beta_1\|\nabla v^m(0)\|_{L^2(\Omega)}^2 + \beta_2\|v_t^m(0)\|_{L^2(\Omega)}^2,
\end{aligned}$$

where

$$\begin{aligned}\beta_1 &: = \frac{3\left(c^4+2\left(\bar{g}\right)^2\right)}{2b} > 0,\\ \beta_2 &: = \left(\frac{12\bar{g}}{b}\right)^2 > 0,\end{aligned}$$

we get

$$\begin{aligned}&\|v^m(\tau)\|^2_{L^2(\Omega)} + \frac{6\bar{h}\left(c^2-\bar{g}\right)}{b}\|\nabla v^m(\tau)\|^2_{L^2(\Omega)}\\ &+\frac{6\bar{g}\alpha}{b}\|v^m_\tau(\tau)\|^2_{L^2(\Omega)} + \frac{b}{6}\|\nabla v^m_\tau(\tau)\|^2_{L^2(\Omega)}\\ &+\frac{1}{4}\|v^m_{\tau\tau}(\tau)\|^2_{L^2(\Omega)} + \frac{3\bar{g}}{b}\left(g\circ\nabla v^m\right)(\tau) + \frac{1}{2}\left(-g'\circ\nabla v^m\right)(\tau)\\ \leq\ &\|v^m(0)\|^2_{L^2(\Omega)} + C_1\|\nabla v^m(0)\|^2_{L^2(\Omega)}\\ &+C_2\|v^m_t(0)\|^2_{L^2(\Omega)} + \frac{\left(c^2+b\right)}{2}\|\nabla v^m_t(0)\|^2_{L^2(\Omega)}\\ &+\frac{\left(12\bar{g}+b\right)}{b}\|v^m_{tt}(0)\|^2_{L^2(\Omega)} + C_3\int_0^\tau S^m(t)\,dt,\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}C_1 &: = \frac{\left(12c^2\bar{g}+b\left(g(0)+c^2\right)+3\left(c^4+2\left(\bar{g}\right)^2\right)\right)}{2b} > 0,\\ C_2 &: = \frac{\left(6\bar{g}b\left(\alpha+1\right)+\left(12\bar{g}\right)^2\right)}{b^2} > 0,\\ C_3 &: = \left(\frac{3\left(c^4+2\left(\bar{g}\right)^2\right)}{2b} + \left(\frac{12\bar{g}}{b}\right)^2 + c^2 + 1\right) > 0.\end{aligned}$$

Set

$$\begin{aligned}S^m(\tau) \ \leq\ &\omega\left\{\|v^m(0)\|^2_{H^1_0(\Omega)} + \|v^m_t(0)\|^2_{H^1(\Omega)} + \|v^m_{tt}(0)\|^2_{L^2(\Omega)}\right\}\\ &+\omega\int_0^\tau S^m(t)\,dt,\end{aligned}\tag{3.18}$$

where

$$\omega := \frac{\max\left\{C_1;\ C_2;\ \dfrac{\left(c^2+b\right)}{2};\ \dfrac{\left(12\bar{g}+b\right)}{b};\ C_3\right\}}{\min\left\{\dfrac{6\bar{g}\left(c^2-\bar{g}\right)}{b};\ \dfrac{6\bar{g}\alpha}{b};\ \dfrac{b}{6};\ \dfrac{1}{4}\right\}} > 0.$$

Applying the **Gronwall lemma** to (3.18) and then integrate from 0 to τ to obtain

$$\int_0^\tau S^m(t)\,dt \leq \omega e^{\omega T}\left\{\|v^m(0)\|^2_{H^1_0(\Omega)} + \|v^m_t(0)\|^2_{H^1(\Omega)} + \|v^m_{tt}(0)\|^2_{L^2(\Omega)}\right\}.$$

We deduce from (3.5) that

$$\|v^m(t)\|^2_{H^1_0(D_T)} + \|v^m_t(t)\|^2_{H^1(D_T)} + \|v^m_{tt}(t)\|^2_{L^2(D_T)} \leq A.$$

Therefore the sequence $\{v^m\}_{m\geq 1} \subset V(D_T)$ is bounded, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(D_T)$ to a limit function $v(x,t)$. We have to show that $v(x,t)$ is a generalized solution of $(1.1)-(1.3)$. Since $\lim_{m\longrightarrow\infty} \|v^m(x,t) \longrightarrow v(x,t)\|_{L^2(D_T)} = 0$ and $\lim_{m\longrightarrow\infty} \|v^m(x,0) \longrightarrow \varphi(x)\|_{L^2(\Omega)} = 0$, then $v(x,0) = \varphi(x)$. Now to prove that (2.1) holds. We multiply each of the relations (3.2) by a function $\Psi_k(t) \in H^1(0,T)$, $\Psi_k(T) = 0$, then add up the obtained equalities ranging from $k=1$ to $k=m$, and integrate over t on $(0,T)$. If we let $\phi^m(x,t) := \sum_{k=1}^{k=m} \Psi_k(t)\psi_k(x)$, then we have

$$\begin{aligned} & -\left(v_{tt}^m(t), \phi_t^m(t)\right)_{L^2(D_T)} - \alpha\left(v_t^m(t), \phi_t^m(t)\right)_{L^2(D_T)} \\ & + c^2\left(\nabla v^m(t), \nabla\phi^m(t)\right)_{L^2(D_T)} + b\left(\nabla v_t^m(t), \nabla\phi^m(t)\right)_{L^2(D_T)} \\ & - \left(\int_0^t g(t-s)\nabla v^m(s)\,ds, \nabla\phi^m(t)\right)_{L^2(D_T)} \\ = \; & \alpha\left(v_t^m(0), \phi^m(0)\right)_{L^2(\Omega)} + \left(v_{tt}^m(0), \phi^m(0)\right)_{L^2(\Omega)}. \end{aligned}$$

Since

$$\begin{aligned} & -\left(\int_0^t g(t-s)\left\{\nabla v^m(s) - \nabla v(s)\right\} ds, \nabla\phi^m(t)\right)_{L^2(D_T)} \\ \leq \; & \frac{\sup_{0\leq t\leq T}\{g(t)\}\,T}{\sqrt{2}} \left\|\nabla\phi^m(t)\right\|_{L^2(D_T)} \left\|\nabla v^m(t) - \nabla v(t)\right\|_{L^2(D_T)}, \end{aligned}$$

and

$$\lim_{m\longrightarrow\infty} \|v^m(t) - v(t)\|_{H_0^1(D_T)} = 0,$$

therefore we have

$$\lim_{m\longrightarrow\infty}\left[\left(\int_0^t g(t-s)\nabla v^m(s)\,ds, \nabla\phi^m(t)\right)_{L^2(D_T)} - \left(\int_0^t g(t-s)\nabla v(s)\,ds, \nabla\phi(t)\right)_{L^2(D_T)}\right] = 0.$$

Thus, the limit function v satisfies (2.1) for every $\phi^m(x,t) := \sum_{k=1}^{k=m} \Psi_k(t)\psi_k(x)$. We denote by $\mathbb{Q}_m$ the totality of all functions of the form $\phi^m(x,t) := \sum_{k=1}^{k=m} \Psi_k(t)\psi_k(x)$, with $\Psi_k(t) \in H^1(0,T)$, $\Psi_k(T) = 0$. But $\cup_{m=1}^{\infty}\mathbb{Q}_m \subset W(D_T)$ is dense, then relation (2.1) holds for all $u \in W(D_T)$. Thus we have shown that the limit function $v(x,t)$ is a generalized solution of problem $(1.1)-(1.3)$ in $V(D_T)$. □

3.2. **Uniqueness.** We now give the main result on the uniqueness of generalized solution of the problem $(1.1)-(1.3)$.

Theorem 3.2. *Under* $(\mathbf{H_1})-(\mathbf{H_3})$, *problems* $(1.1)-(1.3)$ *admit, in* $V(D_T)$, *a unique generalized solution.*

Proof. Suppose that $v_1, v_2 \in V(D_T)$ are two different solutions of problem (1.1)−(1.3). Then $v := v_1 - v_2$ solves

$$\begin{cases} v_{ttt} + \alpha v_{tt} - c^2 \Delta v - b\Delta v_t + \int_0^t g(t-s)\Delta v(s)\,ds = 0, \\ v(x,0) = v_t(x,0) = v_{tt}(x,0) = 0, \\ v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{3.19}$$

and (2.1) gives

$$\begin{aligned} & -(v_{tt}(t), u_t(t))_{L^2(D_T)} - \alpha (v_t(t), u_t(t))_{L^2(D_T)} \\ & + c^2 (\nabla v(t), \nabla u(t))_{L^2(D_T)} + b(\nabla v_t(t), \nabla u(t))_{L^2(D_T)} \\ & - \left(\int_0^t g(t-s)\nabla v(s)\,ds, \nabla u(t) \right)_{L^2(D_T)} \\ = \ & 0. \end{aligned} \tag{3.20}$$

Definite the function $u(x,t)$ by

$$u(x,t) := \begin{cases} \int_t^\tau v(x,s)\,ds, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{3.21}$$

It is obvious that $u \in W(D_T)$ and $u_t(x,t) = -v(x,t)$ for all $t \in [0,\tau]$. Integration by parts allows to write

$$\begin{aligned} & \frac{\alpha}{2}\|v(\tau)\|^2_{L^2(\Omega)} + (v_\tau(\tau), v(\tau))_{L^2(\Omega)} + \frac{c^2}{2}\|\nabla u(0)\|^2_{L^2(\Omega)} \\ = \ & \left(\int_0^t g(t-s)\nabla v(s)\,ds, \nabla u(t) \right)_{L^2(D_T)} \\ & + \int_0^\tau \|v_t(t)\|^2_{L^2(\Omega)}\,dt - b\int_0^\tau \|\nabla u_t(t)\|^2_{L^2(\Omega)}\,dt. \end{aligned} \tag{3.22}$$

We set

$$\Psi(t) := \|v(t)\|^2_{H_0^1(\Omega)} + \|v_t(t)\|^2_{H^1(\Omega)} + \|v_{tt}(t)\|^2_{L^2(\Omega)} + \|\nabla\theta(t)\|^2_{L^2(\Omega)}, \tag{3.23}$$

where

$$\theta(x,t) := \int_0^t v(x,s)\,ds. \tag{3.24}$$

Using (3.21) and (3.24), we get

$$u(x,t) = \theta(x,\tau) - \theta(x,t), \ \nabla u(x,0) = \nabla\theta(x,\tau),$$

and

$$\int_0^\tau \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \le 2\tau \|\nabla\theta(\tau)\|^2_{L^2(\Omega)} + 2\int_0^\tau \|\nabla\theta(t)\|^2_{L^2(\Omega)}\,dt. \tag{3.25}$$

Using (3.21) and (3.23), we get

$$\int_0^\tau \|u(t)\|^2_{L^2(\Omega)}\,dt \le \frac{T^2}{2}\int_0^\tau \Psi(t)\,dt. \tag{3.26}$$

Young's inequality with $\delta = 1$ allows to get

$$-\frac{1}{2}\left\|v_\tau\left(\tau\right)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|v\left(\tau\right)\right\|^2_{L^2(\Omega)} \le \left(v_\tau\left(\tau\right), v\left(\tau\right)\right)_{L^2(\Omega)}. \tag{3.27}$$

Young's inequality with $\delta = 1,$ (3.23) and (3.25), we obtain

$$\begin{aligned} & \left(\int_0^t g\left(t-s\right)\nabla v\left(s\right)ds, \nabla u\left(t\right)\right)_{L^2(D_T)} \\ \le \quad & \frac{\left\{\sup_{0\le t\le T}\left\{g^2\left(t\right)\right\}T^2+4\right\}}{4}\int_0^\tau \Psi\left(t\right)dt + \tau\left\|\nabla\theta\left(\tau\right)\right\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.28}$$

Combination of (3.27) and (3.28) into (3.22) results in

$$\begin{aligned} & \frac{\{\alpha-1\}}{2}\left\|v\left(\tau\right)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|v_\tau\left(\tau\right)\right\|^2_{L^2(\Omega)} + \left\{\frac{c^2}{2}-\tau\right\}\left\|\nabla\theta\left(\tau\right)\right\|^2_{L^2(\Omega)} \\ \le \quad & \frac{\left\{\sup_{0\le t\le T}\left\{g^2\left(t\right)\right\}T^2+12+2T^2\right\}}{4}\int_0^\tau \Psi\left(t\right)dt. \end{aligned} \tag{3.29}$$

Now multiply the differential equation in (3.19) by $v_t\left(t\right)$ and integrate over $D_\tau :=$ $\Omega\times(0,\tau)$, and using integration by parts and **Lemma 1** we get

$$\begin{aligned} & \frac{1}{2}\left(c^2-\int_0^\tau g\left(s\right)ds\right)\left\|\nabla v\left(\tau\right)\right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\|v_\tau\left(\tau\right)\right\|^2_{L^2(\Omega)} \\ & +\left(v_{\tau\tau}\left(\tau\right), v_\tau\left(\tau\right)\right)_{L^2(\Omega)} + \frac{1}{2}\left(g\circ\nabla v\right)\left(\tau\right) \\ = \quad & -b\int_0^\tau \left\|\nabla v_t\left(t\right)\right\|^2_{L^2(\Omega)}dt + \int_0^\tau \left\|v_{tt}\left(t\right)\right\|^2_{L^2(\Omega)}dt \\ & +\frac{1}{2}\int_0^\tau \left(g'\circ\nabla v\right)\left(t\right)dt - \frac{1}{2}\int_0^\tau g\left(t\right)\left\|\nabla v\left(t\right)\right\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{3.30}$$

Young's inequality with $\delta = \dfrac{b}{24\bar{g}}$ allows to get

$$-\frac{b}{48\bar{g}}\left\|v_{\tau\tau}\left(\tau\right)\right\|^2_{L^2(\Omega)} - \frac{12\bar{g}}{b}\left\|v_\tau\left(\tau\right)\right\|^2_{L^2(\Omega)} \le \left(v_{\tau\tau}\left(\tau\right), v_\tau\left(\tau\right)\right)_{L^2(\Omega)}. \tag{3.31}$$

Combining the inequalitie (3.31) into (3.30), we have

$$\begin{aligned} & \frac{\left(c^2-\bar{g}\right)}{2}\left\|\nabla v\left(\tau\right)\right\|^2_{L^2(\Omega)} + \left\{\frac{\alpha}{2}-\frac{12\bar{g}}{b}\right\}\left\|v_\tau\left(\tau\right)\right\|^2_{L^2(\Omega)} \\ & -\frac{b}{48\bar{g}}\left\|v_{\tau\tau}\left(\tau\right)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left(g\circ\nabla v\right)\left(\tau\right) \\ \le \quad & \int_0^\tau \Psi\left(t\right)dt. \end{aligned} \tag{3.32}$$

Now multiply the differential equation in (3.19) by $v_{tt}(t)$ and integrate over $D_\tau := \Omega \times (0,\tau)$, and using integration by parts and **Lemma 1** we get

$$\begin{aligned}
& c^2 \left(\nabla v(\tau), \nabla v_\tau(\tau)\right)_{L^2(\Omega)} + \frac{b}{2}\left\|\nabla v_\tau(\tau)\right\|^2_{L^2(\Omega)} \\
& + \frac{g(\tau)}{2}\left\|\nabla v(\tau)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left(-g' \circ \nabla v\right)(\tau) + \frac{1}{2}\left\|v_{\tau\tau}(\tau)\right\|^2_{L^2(\Omega)} \\
& - \int_0^\tau g(\tau - s)\left(\nabla v(s), \nabla v_\tau(\tau)\right)_{L^2(\Omega)} ds \\
= \; & \frac{1}{2}\int_0^\tau g'(t)\left\|\nabla v(t)\right\|^2_{L^2(\Omega)} dt + c^2 \int_0^\tau \left\|\nabla v_t(t)\right\|^2_{L^2(\Omega)} dt \\
& - \alpha \int_0^\tau \left\|v_{tt}(t)\right\|^2_{L^2(\Omega)} dt - \frac{1}{2}\int_0^\tau \left(g'' \circ \nabla v\right)(t)\, dt.
\end{aligned} \tag{3.33}$$

Young's inequality with $\delta = \dfrac{3c^2}{b}$, we obtain

$$-\frac{3c^4}{2b}\left\|\nabla v(\tau)\right\|^2_{L^2(\Omega)} - \frac{b}{6}\left\|\nabla v_\tau(\tau)\right\|^2_{L^2(\Omega)} \leq c^2\left(\nabla v(\tau), \nabla v_\tau(\tau)\right)_{L^2(\Omega)}. \tag{3.34}$$

Young's inequality with $\delta = \dfrac{6\bar{g}}{b}$, and $-\int_0^\tau g(s)\, ds \geq -\bar{g}$, we obtain

$$\begin{aligned}
& -\frac{3(\bar{g})^2}{b}\left\|\nabla v(\tau)\right\|^2_{L^2(\Omega)} - \frac{3\bar{g}}{b}\left(g \circ \nabla v\right)(\tau) - \frac{b}{6}\left\|\nabla v_\tau(\tau)\right\|^2_{L^2(\Omega)} \\
\leq \; & -\int_0^\tau g(\tau - s)\left(\nabla v(s), \nabla v_\tau(\tau)\right)_{L^2(\Omega)} ds.
\end{aligned} \tag{3.35}$$

Combining the two inequalities (3.34) and (3.35) into (3.33), we have

$$\begin{aligned}
& -\left\{\frac{3c^4}{2b} + \frac{3(\bar{g})^2}{b}\right\}\left\|\nabla v(\tau)\right\|^2_{L^2(\Omega)} + \frac{b}{6}\left\|\nabla v_\tau(\tau)\right\|^2_{L^2(\Omega)} \\
& + \frac{1}{2}\left\|v_{\tau\tau}(\tau)\right\|^2_{L^2(\Omega)} - \frac{3\bar{g}}{b}\left(g \circ \nabla v\right)(\tau) \\
\leq \; & \frac{\{2c^2 + 3\}}{2}\int_0^\tau \Psi(t)\, dt.
\end{aligned} \tag{3.36}$$

Multiplying (3.36) by $\dfrac{b}{12\bar{g}}$ and combining suitably with (3.29) and (3.32) and using the following inequalitie

$$\frac{1}{2}\left\|v(\tau)\right\|^2_{L^2(\Omega)} + \beta_3\left\|\nabla v(\tau)\right\|^2_{L^2(\Omega)} + \beta_4\left\|v_\tau(\tau)\right\|^2_{L^2(\Omega)} \leq \frac{\{1 + 2\beta_3 + 2\beta_4\}}{2}\int_0^\tau \Psi(t)\, dt,$$

where

$$\begin{cases}
\beta_3 := \dfrac{\left(c^4 + 2(\bar{g})^2\right)}{8\bar{g}} > 0, \\[2ex]
\beta_4 := \dfrac{(b + 24\bar{g})}{2b} > 0,
\end{cases}$$

we get

$$\begin{aligned}
& \frac{\alpha}{2}\left\|v\left(\tau\right)\right\|_{L^2(\Omega)}^2+\frac{\left(c^2-\bar{g}\right)}{2}\left\|\nabla v\left(\tau\right)\right\|_{L^2(\Omega)}^2+\frac{\alpha}{2}\left\|v_\tau\left(\tau\right)\right\|_{L^2(\Omega)}^2 \\
& +\frac{b^2}{72\bar{g}}\left\|\nabla v_\tau\left(\tau\right)\right\|_{L^2(\Omega)}^2+\frac{b}{48\bar{g}}\left\|v_{\tau\tau}\left(\tau\right)\right\|_{L^2(\Omega)}^2 \\
& +\frac{\left(c^2-2\tau\right)}{2}\left\|\nabla\theta\left(\tau\right)\right\|_{L^2(\Omega)}^2+\frac{1}{4}\left(g\circ\nabla v\right)\left(\tau\right) \\
\leq\ & C_5\int_0^\tau\Psi\left(t\right)dt, \qquad (3.37)
\end{aligned}$$

where

$$C_5:=\left\{\frac{\left[\sup_{0\leq t\leq T}\left\{g^2\left(t\right)\right\}+2\right]T^2}{4}+\frac{\left\{2c^2+3\right\}b}{24\bar{g}}+\frac{\left(c^4+2\left(\bar{g}\right)^2\right)}{8\bar{g}}+\frac{\left(b+24\bar{g}\right)}{2b}+\frac{9}{2}\right\}>0.$$

Since τ is arbitrary, we may assume that $c^2-2\tau>0$; thus, inequality (3.37) takes the form

$$\Psi\left(\tau\right)\leq\omega'\int_0^\tau\Psi\left(t\right)dt, \qquad (3.38)$$

where

$$\omega':=\frac{C_5}{\min\left\{\frac{\alpha}{2};\ \frac{\left(c^2-\bar{g}\right)}{2};\ \frac{b^2}{72\bar{g}};\ \frac{b}{48\bar{g}};\ \frac{\left(c^2-2\tau\right)}{2};\ \frac{1}{4}\right\}}>0.$$

Applying the **Gronwall lemma** to (3.38), we deduce that

$$\begin{aligned}
& \left\|v\left(\tau\right)\right\|_{H_0^1(\Omega)}^2+\left\|v_\tau\left(\tau\right)\right\|_{H^1(\Omega)}^2+\left\|v_{\tau\tau}\left(\tau\right)\right\|_{L^2(\Omega)}^2+\left\|\nabla\theta\left(\tau\right)\right\|_{L^2(\Omega)}^2 \\
\leq\ & 0, \quad \forall\tau\in\left[0,\frac{c^2}{2}\right].
\end{aligned}$$

Proceeding in the same way for the intervals $\tau\in\left[\frac{\left(N-1\right)c^2}{2},\frac{Nc^2}{2}\right]$ to cover the whole intervale $\left[0,T\right]$, and thus proving that $v\left(x,\tau\right)=0$, for all τ in $\left[0,T\right]$. □

4. Global Existence

In this section, we state and proved global existence of solution $(1.1)-(1.3)$. We define the functional energy of problem $(1.1)-(1.3)$ as

$$\begin{aligned}
E(t) \quad : \quad &= \frac{1}{2}\left(c^2 - \int_0^t g(s)\,ds\right)\|\nabla v(t)\|^2_{L^2(\Omega)} + \frac{\lambda}{2} g(t)\|\nabla v(t)\|^2_{L^2(\Omega)} \\
&+\lambda c^2 (\nabla v(t), \nabla v_t(t))_{L^2(\Omega)} + \frac{\alpha}{2}\|v_t(t)\|^2_{L^2(\Omega)} + (v_{tt}(t), v_t(t))_{L^2(\Omega)} \\
&+\frac{\lambda b}{2}\|\nabla v_t(t)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|v_{tt}(t)\|^2_{L^2(\Omega)} \\
&-\lambda \int_0^t g(t-s)(\nabla v(s), \nabla v_t(t))_{L^2(\Omega)}\,ds \\
&+\frac{1}{2}\int_\Omega (g\square\nabla v)(t)\,dx + \frac{\lambda}{2}\int_\Omega (-g'\square\nabla v)(t)\,dx, \qquad (4.1)
\end{aligned}$$

where the constant λ satisfies

$$\frac{4}{\alpha} < \lambda < \frac{b}{9c^2}.$$

Lemma 4.1. *Let u be a solution of problem $(1.1)-(1.3)$ then $E(t)$ is a non-increasing function on $[0,t)$ and*

$$\begin{aligned}
\frac{d}{dt}\{E(t)\} \quad &= \quad \frac{\{\lambda g'(t) - g(t)\}}{2}\|\nabla v(t)\|^2_{L^2(\Omega)} + \left\{\lambda c^2 - b\right\}\|\nabla v_t(t)\|^2_{L^2(\Omega)} \\
&\quad + \{1-\lambda\alpha\}\|v_{tt}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (g'\square\nabla v)(t)\,dx - \frac{\lambda}{2}\int_\Omega (g''\square\nabla v)(t)\,dx \\
&\leq \quad 0. \qquad (4.2)
\end{aligned}$$

Proof. Multiplying equation of (1.1) by $v_t(t) + \lambda v_{tt}(t)$, and integrating by parts and **Lemma 1** we obtain (4.2). □

Lemma 4.2. *There exist two positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t), \qquad (4.3)$$

where

$$\begin{aligned}
F(t) \quad : \quad &= \|\nabla v(t)\|^2_{L^2(\Omega)} + \|v_t(t)\|^2_{L^2(\Omega)} + \|\nabla v_t(t)\|^2_{L^2(\Omega)} + \|v_{tt}(t)\|^2_{L^2(\Omega)} \\
&+ \int_\Omega (g\square\nabla v)(t)\,dx + \int_\Omega (-g'\square\nabla v)(t)\,dx,
\end{aligned}$$

and

$$\begin{cases}
\alpha_1 := \min\left\{\dfrac{(c^2-\bar{g})}{6};\ \dfrac{\alpha}{4};\ \dfrac{\lambda b}{6};\ \dfrac{\lambda}{4};\ \dfrac{1}{4}\right\} > 0, \\
\text{and} \\
\alpha_2 := \max\left\{\dfrac{(\lambda c^2 + \lambda \max_{0\leq t\leq T}\{g(t)\} + (c^2-\bar{g}) + \lambda\bar{g})}{2}\right. \\
\left.;\ \dfrac{(\alpha+1)}{2};\ \dfrac{\lambda(c^2+b+2\bar{g})}{2};\ \dfrac{(\lambda+1)}{2}\right\} > 0.
\end{cases}$$

Theorem 4.3. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_3})$ *hold, the initial data* $\varphi(x)\in H_0^1(\Omega)$, $\omega(x)\in H^1(\Omega)$ *and* $\Upsilon(x)\in L^2(\Omega)$, *then the solution* $(1.1)-(1.3)$ *is global and bounded.*

Proof. It suffices to show that $\|\nabla v(t)\|^2_{L^2(\Omega)}+\|v_t(t)\|^2_{L^2(\Omega)}$ is bounded independently of t. By using **Lemma 3**. and **Lemma 2**.

$$\alpha_1\left\{\|\nabla v(t)\|^2_{L^2(\Omega)}+\|v_t(t)\|^2_{L^2(\Omega)}\right\}\le E(t)\le E(0).$$

Therefore,

$$\|\nabla v(t)\|^2_{L^2(\Omega)}+\|v_t(t)\|^2_{L^2(\Omega)}\le \frac{E(0)}{\alpha_1},$$

where α_1 is a positive constant, which depends only on $c^2-\bar{g},\ \alpha,\ \lambda$ and b. His infers that the solution of $(1.1)-(1.3)$ is bounded and global in time. □

5. Decay of solutions

In this section, we prove the decay rates for the energy of the posed problem.

Lemma 5.1. *Let us assume that* $(\mathbf{H_1})-(\mathbf{H_5})$ *are in place. Then, there exists a positive constant* $T_0>0$ *such that*

$$E((n+1)T)+\tilde{H}\left(C_9^{-1}E((n+1)T)\right)\le E(nT),\quad n=1,2,3...$$

for all $T>T_0$ *and all* $n\in N$, *where* $\tilde{H}$ *is given in* (5.32) *and* C_9 *is given in* (5.33).

Proof. For this purpose, multiplying (1.1) by the viscoelastic multiplier $(g\diamond u)(t)$ and integrating over $\Omega\times(nT,(n+1)T)$, we obtain

$$\begin{aligned}
&\int_{nT}^{(n+1)T}(v_{ttt}(t),(g\diamond v)(t))_\Omega\,dt\\
&+\alpha\int_{nT}^{(n+1)T}(v_{tt}(t),(g\diamond v)(t))_\Omega\,dt\\
&-c^2\int_{nT}^{(n+1)T}(\Delta v(t),(g\diamond v)(t))_\Omega\,dt\\
&-b\int_{nT}^{(n+1)T}(\Delta v_t(t),(g\diamond v)(t))_\Omega\,dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\Delta v(s)\,ds,(g\diamond v)(t)\right)_\Omega dt\\
=\ &0. \qquad (5.1)
\end{aligned}$$

Evaluation of each term of (5.1). Therefore, using integration by parts, we get

$$\begin{aligned}
&\frac{b}{2}\int_{nT}^{(n+1)T} g\left(t\right)\left\|\nabla v\left(t\right)\right\|_{L^2(\Omega)}^2 dt+\frac{1}{2}\int_{nT}^{(n+1)T} g\left(t\right)\left\|v_t\left(t\right)\right\|_{L^2(\Omega)}^2 dt\\
&-\alpha\int_{nT}^{(n+1)T}\left(\int_0^t g\left(s\right)ds\right)\left\|v_t\left(t\right)\right\|_{L^2(\Omega)}^2 dt\\
=\;&\sum_{i=1}^{i=11} J_i, \qquad (5.2)
\end{aligned}$$

where

$$\begin{aligned}
J_1 &:= \frac{b}{2}\left[\left\|\nabla v\left(t\right)\right\|_{L^2(\Omega)}^2\left(\int_0^t g\left(s\right)ds\right)\right]_{nT}^{(n+1)T},\\
J_2 &:= \frac{1}{2}\left[\left\|v_t\left(t\right)\right\|_{L^2(\Omega)}^2\left(\int_0^t g\left(s\right)ds\right)\right]_{nT}^{(n+1)T},\\
J_3 &:= -b\left[\left(\nabla v\left(t\right),\left(\int_0^t g\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds\right)\right)_\Omega\right]\Big|_{nT}^{(n+1)T},\\
J_4 &:= -\alpha\left[\left(v_t\left(t\right),\left(\int_0^t g\left(t-s\right)\left(v\left(t\right)-v\left(s\right)\right)ds\right)\right)_\Omega\right]_{nT}^{(n+1)T},\\
J_5 &:= -\left[\left(v_{tt}\left(t\right),\left(\int_0^t g\left(t-s\right)\left(v\left(t\right)-v\left(s\right)\right)ds\right)\right)_\Omega\right]_{nT}^{(n+1)T},\\
J_6 &:= -c^2\int_{nT}^{(n+1)T}\left(\nabla v\left(t\right),\int_0^t g\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds\right)_\Omega dt,\\
J_7 &:= \int_{nT}^{(n+1)T}\left(\int_0^t g\left(t-s\right)\nabla v\left(t\right)ds,\int_0^t g\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds\right)_\Omega dt,\\
J_8 &:= -\int_{nT}^{(n+1)T}\left(\int_0^t g\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds,\int_0^t g\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds\right)_\Omega dt,\\
J_9 &:= b\int_{nT}^{(n+1)T}\left(\nabla v\left(t\right),\left(\int_0^t g'\left(t-s\right)\left(\nabla v\left(t\right)-\nabla v\left(s\right)\right)ds\right)\right)_\Omega dt,\\
J_{10} &:= \alpha\int_{nT}^{(n+1)T}\left(v_t\left(t\right),\left(\int_0^t g'\left(t-s\right)\left(v\left(t\right)-v\left(s\right)\right)ds\right)\right)_\Omega dt,\\
J_{11} &:= \int_{nT}^{(n+1)T}\left(v_{tt}\left(t\right),\left(\int_0^t g'\left(t-s\right)\left(v\left(t\right)-v\left(s\right)\right)ds\right)\right)_\Omega dt.
\end{aligned}$$

Using (A) and **Lemma 3**. and Poincaré inequality, we get

$$\begin{aligned}
&\left|J_1\right|+\left|J_2\right|+\left|J_3\right|+\left|J_4\right|+\left|J_5\right|\\
\leq\;&\frac{\left\{\left(2b+1\right)\bar{g}+b+\left(\alpha+1\right)\left(1+\bar{g}C\right)\right\}}{2\alpha_1}\left[E\left(\left(n+1\right)T\right)+E\left(nT\right)\right], \qquad (5.3)
\end{aligned}$$

where C is the Poincaré constant. Young's inequality with $\delta = \dfrac{3\rho}{\gamma_1}$, and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} |J_6| &\leq \frac{3\rho c^2}{2\gamma_1} \int_{nT}^{(n+1)T} \|\nabla v(t)\|_{L^2(\Omega)}^2 \, dt \\ &\quad + \frac{\gamma_1 c^2}{2\rho} \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t) \, dx dt, \end{aligned} \tag{5.4}$$

where the constants γ_1 and ρ satisfies

$$\begin{cases} \dfrac{2c^2}{b \min_{0 \leq t \leq T} \{g(t)\}} \leq \gamma_1 \leq \dfrac{3\rho}{\alpha}, \\ \text{and} \\ \rho < \dfrac{1}{3}. \end{cases}$$

Young's inequality with $\delta = \dfrac{6\rho}{\gamma_1 c^2}$, (A) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} |J_7| &\leq \frac{3\rho}{\gamma_1} \int_{nT}^{(n+1)T} \left(\int_0^t g(s) \, ds \right) \|\nabla v(t)\|_{L^2(\Omega)}^2 \, dt \\ &\quad + \frac{\gamma_1 c^4}{12\rho} \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t) \, dx dt. \end{aligned} \tag{5.5}$$

Using Cauchy Schwarz inequality, we get

$$|J_8| \leq c^2 \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t) \, dx dt. \tag{5.6}$$

Young's inequality with $\delta = \dfrac{3\rho c^2}{b\gamma_1}$, (A) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} |J_9| &\leq \frac{3\rho c^2}{2\gamma_1} \int_{nT}^{(n+1)T} \|\nabla v(t)\|_{L^2(\Omega)}^2 \, dt \\ &\quad + \frac{g(0) \gamma_1 b^2}{6\rho c^2} \int_{nT}^{(n+1)T} \int_\Omega (-g' \Box \nabla v)(t) \, dx dt. \end{aligned} \tag{5.7}$$

Young's inequality with $\delta = \dfrac{6\rho c^2}{\alpha\gamma_1}$, (A) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} |J_{10}| &\leq \frac{3\rho c^2}{\gamma_1} \int_{nT}^{(n+1)T} \|v_t(t)\|_{L^2(\Omega)}^2 \, dt \\ &\quad + \frac{g(0) \gamma_1 \alpha^2}{12\rho c^2} \int_{nT}^{(n+1)T} \int_\Omega (-g' \Box \nabla v)(t) \, dx dt. \end{aligned} \tag{5.8}$$

Young's inequality with $\delta = \dfrac{\{\lambda\alpha - 1\}}{\gamma_1}$, (A) and Cauchy Schwarz inequality, we obtain

$$\begin{aligned} |J_{11}| &\leq \frac{\{\lambda\alpha - 1\}}{2\gamma_1} \int_{nT}^{(n+1)T} \|v_{tt}(t)\|^2_{L^2(\Omega)} dt \\ &+ \frac{g(0)\gamma_1}{2\{\lambda\alpha - 1\}} \int_{nT}^{(n+1)T} \int_\Omega (-g' \Box \nabla v)(t)\, dxdt. \end{aligned} \tag{5.9}$$

Combining $(5.3) - (5.9)$ into (5.2), we write

$$\begin{aligned} &\frac{b}{2} \int_{nT}^{(n+1)T} g(t) \|\nabla v(t)\|^2_{L^2(\Omega)} dt - \frac{3\rho c^2}{\gamma_1} \int_{nT}^{(n+1)T} \|v_t(t)\|^2_{L^2(\Omega)} dt \\ &+ \frac{1}{2} \int_{nT}^{(n+1)T} g(t) \|v_t(t)\|^2_{L^2(\Omega)} dt - \alpha \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds \right) \|v_t(t)\|^2_{L^2(\Omega)} dt \\ &- \frac{\{\lambda\alpha - 1\}}{2\gamma_1} \int_{nT}^{(n+1)T} \|v_{tt}(t)\|^2_{L^2(\Omega)} dt \\ \leq\ & \frac{\{(2b+1)\bar{g} + b + (\alpha+1)(1+\bar{g}C)\}}{2\alpha_1} [E((n+1)T) + E(nT)] \\ &+ \frac{3\rho c^2}{\gamma_1} \int_{nT}^{(n+1)T} \|\nabla v(t)\|^2_{L^2(\Omega)} dt \\ &+ \frac{3\rho}{\gamma_1} \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds \right) \|\nabla v(t)\|^2_{L^2(\Omega)} dt \\ &+ \frac{\{\gamma_1 c^2 + 4\rho\} c^2}{4\rho} \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t)\, dxdt \\ &+ \gamma_1 \left\{ \frac{b^2 + \alpha^2}{6\rho c^2} + \frac{1}{2\{\lambda\alpha - 1\}} \right\} g(0) \int_{nT}^{(n+1)T} \int_\Omega (-g' \Box \nabla v)(t)\, dxdt. \end{aligned} \tag{5.10}$$

Now multiply the differential equation in (1.1) by $v(t)$ and integrate over $\Omega \times (nT, (n+1)T)$, we get

$$\begin{aligned} &\int_{nT}^{(n+1)T} (v_{ttt}(t), v(t))_\Omega\, dt + \alpha \int_{nT}^{(n+1)T} (v_{tt}(t), v(t))_\Omega\, dt \\ &- c^2 \int_{nT}^{(n+1)T} (\Delta v(t), v(t))_\Omega\, dt - b \int_{nT}^{(n+1)T} (\Delta v_t(t), v(t))_\Omega\, dt \\ &+ \int_{nT}^{(n+1)T} \left(\int_0^t g(t-s) \Delta v(s)\, ds, v(t) \right)_\Omega dt \\ =\ & 0. \end{aligned} \tag{5.11}$$

Evaluation of each term of (5.11). Therefore, using integration by parts, we get

$$\begin{aligned} & c^2 \int_{nT}^{(n+1)T} \|\nabla v(t)\|^2_{L^2(\Omega)} dt - \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds \right) \|\nabla v(t)\|^2_{L^2(\Omega)} dt \\ & -\alpha \int_{nT}^{(n+1)T} \|v_t(t)\|^2_{L^2(\Omega)} dt \\ = & \sum_{\alpha=1}^{\alpha=5} I_\alpha, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} I_1 &:= -\frac{b}{2} \|\nabla v(t)\|^2_{L^2(\Omega)} \Big|_{nT}^{(n+1)T}, \\ I_2 &:= \frac{1}{2} \|v_t(t)\|^2_{L^2(\Omega)} \Big|_{nT}^{(n+1)T}, \\ I_3 &:= -\alpha \left(v_t(t), v(t)\right)_\Omega \Big|_{nT}^{(n+1)T}, \\ I_4 &:= -\left(v_{tt}(t), v(t)\right)_\Omega \Big|_{nT}^{(n+1)T}, \\ I_5 &:= -\int_{nT}^{(n+1)T} \int_0^t g(t-s) \left(\nabla v(t) - \nabla v(s), \nabla v(t)\right)_\Omega ds dt. \end{aligned}$$

Using Young's inequality with $\delta = \frac{1}{2}$, Poincaré inequality and **Lemma 3.**, we get

$$\begin{aligned} & |I_1| + |I_2| + |I_3| + |I_4| \\ \le & \frac{\{b + 1 + (\alpha+1)(1+C)\}}{2\alpha_1} \left[E((n+1)T) + E(nT)\right]. \end{aligned} \tag{5.13}$$

Young's inequality with $\delta = \frac{1}{2}$, and (2.1), we obtain

$$\begin{aligned} |I_5| \le & \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds \right) \|\nabla v(t)\|^2_{L^2(\Omega)} dt \\ & + \frac{1}{4} \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t)\, dx dt. \end{aligned} \tag{5.14}$$

Combining (5.13) and (5.14) into (5.12), we write

$$\begin{aligned} & c^2 \int_{nT}^{(n+1)T} \|\nabla v(t)\|^2_{L^2(\Omega)} dt - \alpha \int_{nT}^{(n+1)T} \|v_t(t)\|^2_{L^2(\Omega)} dt \\ \le & \frac{\{b + 1 + (\alpha+1)(1+C_p)\}}{2\alpha_1} \left[E((n+1)T) + E(nT)\right] \\ & + 2 \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds \right) \|\nabla v(t)\|^2_{L^2(\Omega)} dt \\ & + \frac{1}{4} \int_{nT}^{(n+1)T} \int_\Omega (g \Box \nabla v)(t)\, dx dt. \end{aligned} \tag{5.15}$$

Now multiply the differential equation in (1.1) by $v_t(t)$ and integrate over $\Omega \times (nT,(n+1)T)$, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} (v_{ttt}(t), v_t(t))_\Omega \, dt + \alpha \int_{nT}^{(n+1)T} (v_{tt}(t), v_t(t))_\Omega \, dt \\
&-c^2 \int_{nT}^{(n+1)T} (\Delta v(t), v_t(t))_\Omega \, dt - b \int_{nT}^{(n+1)T} (\Delta v_t(t), v_t(t))_\Omega \, dt \\
&+ \int_{nT}^{(n+1)T} \left(\int_0^t g(t-s) \Delta v(s) \, ds, v_t(t) \right)_\Omega dt \\
= \;& 0. \qquad (5.16)
\end{aligned}$$

Evaluation of each term of (5.15). Therefore, using integration by parts and **Lemma 1.**, we get

$$\begin{aligned}
&\frac{1}{2} \int_{nT}^{(n+1)T} g(t) \|\nabla v(t)\|^2_{L^2(\Omega)} \, dt + b \int_{nT}^{(n+1)T} \|\nabla v_t(t)\|^2_{L^2(\Omega)} \, dt \\
&- \int_{nT}^{(n+1)T} \|v_{tt}(t)\|^2_{L^2(\Omega)} \, dt + \frac{1}{2} \int_{nT}^{(n+1)T} \int_\Omega (-g' \Box \nabla v)(t) \, dx dt \\
= \;& \sum_{\alpha=1}^{\alpha=5} k_\alpha, \qquad (5.17)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &: = -\frac{c^2}{2} \|\nabla v(t)\|^2_{L^2(\Omega)} \Big|_{nT}^{(n+1)T}, \\
k_2 &: = \frac{1}{2} \left(\int_0^t g(s) \, ds \right) \|\nabla v(t)\|^2_{L^2(\Omega)} \Big|_{nT}^{(n+1)T}, \\
k_3 &: = -\frac{\alpha}{2} \|v_t(t)\|^2_{L^2(\Omega)} \Big|_{nT}^{(n+1)T}, \\
k_4 &: = -(v_{tt}(t), v_t(t))_\Omega \big|_{nT}^{(n+1)T}, \\
k_5 &: = -\frac{1}{2} \int_\Omega (g \Box \nabla v)(t) \, dx \Big|_{nT}^{(n+1)T}.
\end{aligned}$$

Using Young's inequality with $\delta = \frac{1}{2}$, and **Lemma 3.**, we get

$$|k_1| + |k_2| + |k_3| + |k_4| + |k_5| \leq w \left[E((n+1)T) + E(nT) \right], \qquad (5.18)$$

where

$$w := \frac{\{c^2 + \bar{g} + \alpha + 3\}}{2\alpha_1} > 0.$$

Using (5.17) into (5.16), we get

$$\begin{aligned}
&\frac{1}{2}\int_{nT}^{(n+1)T} g\left(t\right)\left\|\nabla v\left(t\right)\right\|_{L^2(\Omega)}^2 dt + b\int_{nT}^{(n+1)T}\left\|\nabla v_t\left(t\right)\right\|_{L^2(\Omega)}^2 dt \\
&-\int_{nT}^{(n+1)T}\left\|v_{tt}\left(t\right)\right\|_{L^2(\Omega)}^2 dt + \frac{1}{2}\int_{nT}^{(n+1)T}\int_{\Omega}\left(-g'\Box\nabla v\right)\left(t\right)dxdt \\
\leq\ & w\left[E\left(\left(n+1\right)T\right)+E\left(nT\right)\right]. \qquad (5.19)
\end{aligned}$$

Now multiply the differential equation in (1.1) by $v_{tt}\left(t\right)$ and integrate over $\Omega\times\left(nT,\left(n+1\right)T\right)$, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(v_{ttt}\left(t\right),v_{tt}\left(t\right)\right)_{\Omega}dt + \alpha\int_{nT}^{(n+1)T}\left(v_{tt}\left(t\right),v_{tt}\left(t\right)\right)_{\Omega}dt \\
&-c^2\int_{nT}^{(n+1)T}\left(\Delta v\left(t\right),v_{tt}\left(t\right)\right)_{\Omega}dt - b\int_{nT}^{(n+1)T}\left(\Delta v_t\left(t\right),v_{tt}\left(t\right)\right)_{\Omega}dt \\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g\left(t-s\right)\Delta v\left(s\right)ds, v_{tt}\left(t\right)\right)_{\Omega}dt \\
=\ & 0. \qquad (5.20)
\end{aligned}$$

Evaluation of each term of (5.19). Therefore, using integration by parts and **Lemma 1** we get

$$\begin{aligned}
&-\frac{1}{2}\int_{nT}^{(n+1)T} g'\left(t\right)\left\|\nabla v\left(t\right)\right\|_{L^2(\Omega)}^2 dt - c^2\int_{nT}^{(n+1)T}\left\|\nabla v_t\left(t\right)\right\|_{L^2(\Omega)}^2 dt \\
&+\alpha\int_{nT}^{(n+1)T}\left\|v_{tt}\left(t\right)\right\|_{L^2(\Omega)}^2 dt + \frac{1}{2}\int_{nT}^{(n+1)T}\int_{\Omega}\left(g''\Box\nabla v\right)\left(t\right)dxdt \\
=\ & \sum_{\alpha=6}^{\alpha=11} k_{\alpha}, \qquad (5.21)
\end{aligned}$$

where

$$\begin{aligned}
k_6 &:= -\frac{1}{2}\, g\left(t\right)\left\|\nabla v\left(t\right)\right\|_{L^2(\Omega)}^2\Big|_{nT}^{(n+1)T}, \\
k_7 &:= -\frac{b}{2}\left\|\nabla v_t\left(t\right)\right\|_{L^2(\Omega)}^2\Big|_{nT}^{(n+1)T}, \\
k_8 &:= -c^2\left(\nabla v\left(t\right),\nabla v_t\left(t\right)\right)_{\Omega}\Big|_{nT}^{(n+1)T}, \\
k_9 &:= \int_0^t g\left(t-s\right)\left(\nabla v(s),\nabla v_t\left(t\right)\right)_{L^2(\Omega)}ds\Big|_{nT}^{(n+1)T}, \\
k_{10} &:= -\left\|v_{tt}\left(t\right)\right\|_{L^2(\Omega)}^2\Big|_{nT}^{(n+1)T}, \\
k_{11} &:= \frac{1}{2}\int_{\Omega}\left(g'\Box\nabla v\right)\left(t\right)dx\Big|_{nT}^{(n+1)T}.
\end{aligned}$$

Using Young's inequality with $\delta = \frac{1}{2}$, and **Lemma 3.**, we get

$$|k_6| + |k_7| + |k_8| + |k_9| + |k_{10}| + |k_{11}| \leq w' \left[E\left((n+1)T\right) + E\left(nT\right) \right], \tag{5.22}$$

where

$$w' := \frac{\left(\max_{0\leq t\leq T}\{g(t)\} + b + 2c^2 + 3\bar{g} + 4\right)}{2\alpha_1} > 0.$$

Using (5.22) into (5.21), we get

$$\begin{aligned}
&-\frac{1}{2}\int_{nT}^{(n+1)T} g'(t)\,\|\nabla v(t)\|^2_{L^2(\Omega)}\,dt - c^2\int_{nT}^{(n+1)T}\|\nabla v_t(t)\|^2_{L^2(\Omega)}\,dt \\
&+\alpha\int_{nT}^{(n+1)T}\|v_{tt}(t)\|^2_{L^2(\Omega)}\,dt + \frac{1}{2}\int_{nT}^{(n+1)T}\int_\Omega \left(g''\Box\nabla v\right)(t)\,dxdt \\
\leq\;& w'\left[E\left((n+1)T\right) + E\left(nT\right)\right].
\end{aligned} \tag{5.23}$$

Multiplying (5.10) by γ_1 and (5.15) by γ_2 and (5.23) by λ and combining suitably with (5.19), where

$$\gamma_2 \leq \min\left\{\frac{\gamma_1 \min_{0\leq t\leq T}\{g(t)\}}{2\alpha};\ \frac{1}{2}\right\},$$

we get

$$\begin{aligned}
& c^2\int_{nT}^{(n+1)T}\|\nabla v(t)\|^2_{L^2(\Omega)}\,dt + c^2\int_{nT}^{(n+1)T}\|v_t(t)\|^2_{L^2(\Omega)}\,dt \\
& + \left\{b - \lambda c^2\right\}\int_{nT}^{(n+1)T}\|\nabla v_t(t)\|^2_{L^2(\Omega)}\,dt + \frac{\{\lambda\alpha - 1\}}{2}\int_{nT}^{(n+1)T}\|v_{tt}(t)\|^2_{L^2(\Omega)}\,dt \\
& +\frac{1}{2}\int_{nT}^{(n+1)T}\int_\Omega\left(-g'\Box\nabla v\right)(t)\,dxdt + \frac{\lambda}{2}\int_{nT}^{(n+1)T}\int_\Omega\left(g''\Box\nabla v\right)(t)\,dxdt \\
\leq\;& C_3\left[E\left((n+1)T\right) + E\left(nT\right)\right] + 3\rho c^2\int_{nT}^{(n+1)T}\|v_t(t)\|^2_{L^2(\Omega)}\,dt \\
& + (3\rho + 1)\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|v_t(t)\|^2_{L^2(\Omega)}\,dt \\
& +3\rho c^2\int_{nT}^{(n+1)T}\|\nabla v(t)\|^2_{L^2(\Omega)}\,dt \\
& + (3\rho + 1)\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla v(t)\|^2_{L^2(\Omega)}\,dt \\
& +C_4\int_{nT}^{(n+1)T}\int_\Omega\left(g\Box\nabla v\right)(t)\,dxdt + C_5\int_{nT}^{(n+1)T}\int_\Omega\left(-g'\Box\nabla v\right)(t)\,dxdt,
\end{aligned} \tag{5.24}$$

where

$$\begin{cases} C_3 := \dfrac{\gamma_1 \left\{(2b+1)\,\bar{g} + b + (\alpha+1)\,(1+\bar{g}C)\right\} + \gamma_2 \left\{b+1+(\alpha+1)\,(1+C)\right\} + 2\alpha_1\,(w+\lambda w')}{2\alpha_1} > 0, \\ C_4 := \dfrac{\left\{\gamma_1\left(\gamma_1 c^2 + 4\rho\right)c^2 + \rho\gamma_2\right\}}{4\rho} > 0, \\ C_5 := (\gamma_1)^2 \left\{\dfrac{b^2+\alpha^2}{6\rho c^2} + \dfrac{1}{2\left\{\lambda\alpha - 1\right\}}\right\} g(0) > 0. \end{cases}$$

Adding and subtracting in (5.24) the term

$$-\int_{nT}^{(n+1)T} \left(\int_0^t g(s)\,ds\right) \|v_t(t)\|^2_{L^2(\Omega)}\,dt \quad \text{and} \quad -\int_{nT}^{(n+1)T} \left(\int_0^t g(s)\,ds\right) \|\nabla v(t)\|^2_{L^2(\Omega)}\,dt,$$

and

$$\int_{nT}^{(n+1)T} \int_\Omega (g\Box\nabla v)(t)\,dxdt,$$

in order to recover the energy $E(t)$, we obtain

$$\begin{aligned} & (1-3\rho)\int_{nT}^{(n+1)T} \left(c^2 - \int_0^t g(s)\,ds\right) \|\nabla v(t)\|^2_{L^2(\Omega)}\,dt \\ & + (1-3\rho)\int_{nT}^{(n+1)T} \left(c^2 - \int_0^t g(s)\,ds\right) \|v_t(t)\|^2_{L^2(\Omega)}\,dt \\ & + \left(b - \lambda c^2\right)\int_{nT}^{(n+1)T} \|\nabla v_t(t)\|^2_{L^2(\Omega)}\,dt + \frac{\left\{\lambda\alpha - 1\right\}}{2}\int_{nT}^{(n+1)T} \|v_{tt}(t)\|^2_{L^2(\Omega)}\,dt \\ & + \int_{nT}^{(n+1)T}\int_\Omega (g\Box\nabla v)(t)\,dxdt + \frac{1}{2}\int_{nT}^{(n+1)T}\int_\Omega (-g'\Box\nabla v)(t)\,dxdt \\ & + \frac{\lambda}{2}\int_{nT}^{(n+1)T}\int_\Omega (g''\Box\nabla v)(t)\,dxdt \\ \leq\; & C_3\left[E((n+1)T) + E(nT)\right] + C_4\int_{nT}^{(n+1)T}\int_\Omega (g\Box\nabla v)(t)\,dxdt \\ & + C_4\int_{nT}^{(n+1)T}\int_\Omega k_1(-g'\Box\nabla v)(t)\,dxdt, \end{aligned} \tag{5.25}$$

where $k_1 := \dfrac{C_5}{C_4} > 0$. From (5.25), choosing ρ sufficiently small, $k_1 > 0$ and T large enough and using **Lemma 3.**, we get

$$\begin{aligned} \int_{nT}^{(n+1)T} E(t)\,dt \quad \leq \quad & C_6\left[E((n+1)T) + E(nT)\right] \\ & + C_7\int_{nT}^{(n+1)T}\int_\Omega (g\Box\nabla v)(t)\,dxdt \\ & + C_7\int_{nT}^{(n+1)T}\int_\Omega k_1(-g'\Box\nabla v)(t)\,dxdt, \end{aligned} \tag{5.26}$$

where

$$\begin{cases} C_6 := \dfrac{\alpha_2 C_3}{\min\left\{(1-3\rho)\left(c^2-\bar{g}\right);\ \left(b-\lambda c^2\right);\ \dfrac{\{\lambda\alpha-1\}}{2};\ \dfrac{1}{2};\ \dfrac{\lambda}{2}\right\}} > 0, \\ \text{and} \\ C_7 := \dfrac{\alpha_2 C_4}{\min\left\{(1-3\rho)\left(c^2-\bar{g}\right);\ \left(b-\lambda c^2\right);\ \dfrac{\{\lambda\alpha-1\}}{2};\ \dfrac{1}{2};\ \dfrac{\lambda}{2}\right\}} > 0. \end{cases}$$

In the last step, we need to relate the viscoelastic energy to the viscoelastic damping. In the case when the relaxation function obeys a linear equation, this relation is straightforward and is expressed by a suitable multiplication. However, in the case of general decays, additional arguments are used. Here, we follow [18]. From the ($\mathbf{H_2}$) made on the viscoelastic kernel g and from [18, **Lemma 4**] we obtain

$$(g\Box\nabla v)(t) \leq \hat{H}_\alpha^{-1}\left(-g'\Box\nabla v\right)(t), \quad t \in [nT, (n+1)T], \tag{5.27}$$

where $\hat{H}_\alpha$ is a rescaling of H_α with

$$H_\alpha(s) = \alpha s^{1-\frac{1}{\alpha}} H\left(s^{\frac{1}{\alpha}}\right),$$

and $\alpha \in (0,1)$ is such that

$$\sup_{t>0}\int_0^t g^{1-\alpha}(t-s)\left\|\nabla v(t)-\nabla v(s)\right\|^2 ds < \infty.$$

From ($\mathbf{H_2}$) it is clear that $\alpha \geq \alpha_0$. The main point, however, is that the argument can be reiterated (based on [18, **Lemma 8**] leading to $\alpha = 1$). This allows us to replace H_α, the function in (5.26), by the original function $\hat{H}$ which is a rescaling of $H(s)$. This means that $\hat{H} = cH\left(\dfrac{C}{s}\right)$ for some $c,\ C > 0$. Now, from (5.26) and taking (5.27) into account, we deduce that

$$\begin{aligned} \int_{nT}^{(n+1)T} E(t)\,dt \quad &\leq \quad C_6\left[E((n+1)T) + E(nT)\right] \\ &+ C_7\int_{nT}^{(n+1)T}\int_\Omega \left[\hat{H}^{-1}+k_1\right]\left(-g'\Box\nabla v\right)(t)\,dxdt. \end{aligned} \tag{5.28}$$

Next, we shall employ the following version of **Jensen's inequality** applied to measures and convex functions F. We shall use (5.28) in order to bring the functions H in front of the integrals. Let us denote $\alpha_0 := meas(\Omega)$. We note that the function $\hat{H}^{-1}+k_1$ is concave. Let $F^{-1} = \hat{H}^{-1}+k_1,\ f(x) = \left(-g'\Box\nabla v\right)(t),\ h(x) =$

$T,\ h_0 = T\alpha_0$ and $h_0^{-1} = \alpha_0^{-1}T^{-1}$, thus, we have

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\int_\Omega \left[\hat{H}^{-1}+k_1\right]\left(-g'\Box\nabla v\right)(t)\,dxdt \\
\leq\ & \alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega \left(-g'\Box\nabla v\right)(t)\,dxdt\right]. \quad (5.29)
\end{aligned}$$

On the other hand, from the identity (4.2) for the energy, we can write

$$E\left((n+1)T\right) - E\left(nT\right) = -\int_{nT}^{(n+1)T} D\left(t\right)dt, \quad (5.30)$$

where

$$\begin{aligned}
D\left(t\right)\ :\ &= \frac{\left\{g\left(t\right)-\lambda g'\left(t\right)\right\}}{2}\left\|\nabla v\left(t\right)\right\|^2_{L^2(\Omega)} + \left\{b-\lambda c^2\right\}\left\|\nabla v_t\left(t\right)\right\|^2_{L^2(\Omega)} \\
&+\left\{\lambda\alpha-1\right\}\left\|v_{tt}\left(t\right)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega\left(-g'\Box\nabla v\right)(t)\,dx + \frac{\lambda}{2}\int_\Omega\left(g''\Box\nabla v\right)(t)\,dx.
\end{aligned}$$

En replacement (5.29) into (5.28) and using (5.30), we get

$$\begin{aligned}
\int_{nT}^{(n+1)T} E\left(t\right)dt\ &\leq\ C_6\left\{2E\left((n+1)T\right)+\int_{nT}^{(n+1)T} D\left(t\right)dt\right\} \\
&+C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega\left(-g'\Box\nabla v\right)(t)\,dxdt\right],
\end{aligned}$$

and using

$$\int_{nT}^{(n+1)T}\int_\Omega\left(-g'\Box\nabla u\right)(t)\,dxdt \leq 2\int_{nT}^{(n+1)T} D\left(t\right)dt,$$

thus, we get

$$\int_{nT}^{(n+1)T} E\left(t\right)dt \leq 2C_6 E\left((n+1)T\right) + C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D\left(t\right)dt\right], \quad (5.31)$$

where

$$\begin{cases}
C_8 := \max\left\{2C_7;\ 1\right\} > 0, \\
\tilde{H} := \left[\hat{H}^{-1}+k_2\right]^{-1}, \\
k_2 := \left(C_6 + 2C_7 k_1\right) > 0.
\end{cases} \quad (5.32)$$

By using $TE\left((n+1)T\right) \leq \int_{nT}^{(n+1)T} E\left(t\right)dt$ in (5.31), we get

$$\left(T-2C_6\right)E\left((n+1)T\right) \leq C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D\left(t\right)dt\right].$$

For T large enough, where C_6 is a positive constant, which implies that

$$E\left((n+1)T\right) \leq C_9\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D\left(t\right)dt\right],$$

where

$$C_9 := \frac{C_8}{(T - 2C_6)} > 0, \tag{5.33}$$

which gives that

$$\tilde{H}\left(C_9^{-1} E\left((n+1)\,T\right)\right) \leq \int_{nT}^{(n+1)T} D\left(t\right) dt, \tag{5.34}$$

using (5.30) into (5.34) , we get

$$\tilde{H}\left(C_9^{-1} E\left((n+1)\,T\right)\right) \leq E\left(nT\right) - E\left((n+1)\,T\right),$$

from the above we have

$$E\left((n+1)\,T\right) + \tilde{H}\left(C_9^{-1} E\left((n+1)\,T\right)\right) \leq E\left(nT\right), \quad n = 1, 2, 3....$$

This completes the proof. □

Lemma 5.2. ([18]) *Let p be a positive, increasing function such that $p\left(0\right) = 0$. Let $q\left(x\right) \equiv x - \left(I + p\right)^{-1}\left(x\right)$. Consider a sequence F_n of positive numbers which satisfies*

$$F_{m+1} + p\left(F_{m+1}\right) \leq F_m. \tag{5.35}$$

Then $F_m \leq S\left(m\right)$ where $S\left(t\right)$ is a solution of the differential equation

$$\frac{d}{dt}\left\{S\left(t\right)\right\} + q\left(S\left(t\right)\right) = 0, \quad S\left(0\right) = F_0. \tag{5.36}$$

Moreover, if $p\left(x\right) > 0$ for $x > 0$ then $\lim_{t \longrightarrow \infty} S\left(t\right) = 0$.

Theorem 5.3. ([18])*Let us assume that $(\mathbf{H_1}) - (\mathbf{H_5})$, (2.2). Then there exist positive constants c_1, c_2 and T_0 such that the solution of problem (1.1) − (1.3) satisfies $E\left(t\right) \leq s\left(t\right)$, where $s\left(t\right)$ verifies the ODE*

$$s_t + \hat{H}\left(s\right) = 0, \ s\left(0\right) = E\left(0\right), \ t \geq T_0 > 0,$$

with $\hat{H}\left(s\right) = c_1 H\left(c_2 s\right)$.

Proof. We are in a position to apply the result of **Lemma 5.** with

$$F_m \equiv E\left(mt\right), \quad F_0 \equiv E\left(0\right).$$

This yields

$$E\left(mT\right) \leq S\left(m\right), \quad m = 0, 1, 2, 3....$$

Setting $t = mT + \tau$ and recalling the evolution property gives

$$E\left(t\right) \leq E\left(mT\right) \leq S\left(m\right) \leq S\left(\frac{t - \tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right).$$

This completes the proof. □

6. Conclusion

In this paper, we have determined su¢ cient conditions that can lead the solution to tend towards zero when t tends to infinity for the linear Moore-Gibson-Thompson equation with a viscoelastic memory. Despite the complexity of the technical calculations of the methods, furthermore, we prove the decay rates for the energy of Moore Gibson-Thompson equation linear with a viscoelastic memory of relaxation kernels. Our ultimate objective after this thesis work is to deal with other more complicated problems with fractional dissipations.

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