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EXISTENCE OF SOLUTIONS FOR NONLINEAR ELLIPTIC SINGULAR INCLUSION PROBLEMS

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Abstract. In this study we investigate the existence of solutions for a particular class of nonlinear elliptic inclusion problems involving singular terms. The analysis is carried out within a weak formulation framework, and particular attention is given to the difficulties arising from the singular nature of the nonlinearities. Using appropriate approximation methods and compactness arguments, we establish the existence of weak solutions assuming broad assumptions on the given data and the nonlinearity.

Keywords. Existence of solution; Inclusion problems; Singular elliptic equations; Weak solution
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1. Introduction

This study focuses on establishing whether solutions exist for the nonlinear elliptic singular inclusion problem described below:

$$(P)\begin{cases} \beta(u)-\operatorname{div}(a(x,\nabla u)) \ni \frac{f}{u^\gamma} & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here Ω represents limited region of $\mathbb{R}^N (N \geq 2)$ and $\gamma > 0$, moreover, function β is a maximal monotone graph satisfying $0 \in \beta(0)$ and the function $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function that fulfils classical Leray-Lions type conditions (see (2.1), (2.2), (2.3)), f here is in $L^\infty(\Omega)$. Furthermore, the main difficulty arises from the singular term $\dfrac{f}{u^\gamma}$, particularly near the boundary of Ω, where its behavior becomes more problematic.

We provide an overview of the relevant literature on singular elliptic problems with $\beta \equiv 0$. The simplest model is represented by

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \frac{f}{u^\gamma} & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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such that $f \in L^1(\Omega)$ denotes a nonnegative function. In [13], L. M. De Cave proved regularity and existence results for (1.2). In [14], the authors replaced f with μ, a Radon measure that is both bounded and nonnegative on Ω. Moreover they studied the existence and uniqueness of solutions to (1.2).

Inclusion problems were explored in [1], [3], [8], [17], and [18], without addressing singularities. A recent work in [2] also considered non-singular problems but focused on non-coercive elliptic inclusion problems of the following form:

$$\begin{cases} \beta(u) - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq \theta < 1$ and the function f on the right-hand side is taken in $L^1(\Omega)$. In that work, the authors establish that entropy solutions exist for the considered differential inclusion, despite the lack of coercivity, along with some regularity results.

An important impetus behind the study of problem (P) arises from the crucial role played by singularities within the study of nonlinear PDEs, which have gained increasing importance in recent years. This interest is driven not only by the mathematical pursuit of extending the scope of existing theories, but also by practical applications in fields such as chemical reactions, phase transitions, boundary layer phenomena as well as within the framework of pseudoplastic fluid models, consult for example, [15] and the references therein.

The significance of our work lies in integrating inclusion problems with singular problems, while also incorporating the maximal monotone mapping, we recommend [7] for readers seeking further details about the maximal monotone mapping.

The appearance of the singular term $\frac{1}{u^\gamma}$ appearing on the right side, together with the maximal monotone mapping β and the homogeneous Dirichlet boundary condition, collectively introduce some fundamental challenges in solving problems of type (1.1). Initially, a proper notion of solution must be established to control the singular effects located on the right side near the boundary of Ω. Next, to establish the existence of a solution, it is necessary to approximate problem (1.1) in an appropriate way to remove the singularity (see (3.1)), then, we need to prove that the approximation of β is bounded. Once the equation is approximated, one must demonstrate a priori estimates, including uniform positivity, along with the asymptotic behaviour of the approximating solutions sequence specifically, their strong convergence in Sobolev spaces to successfully pass to the limit.

This study is organized into the following sections: Section 2, provides the necessary notations, basic hypotheses, and the main result. In Section 3, we concentrate on introducing and solving the approximating problem. The proof of the main result is presented in Section 4.

2. Basic setup and principal result

We present several functions that will appear frequently throughout this paper, for $k > 0$, we utilize the well-known truncation function $T_k(s) : \mathbb{R} \to \mathbb{R}$, given by

$$T_k(s) = \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } |s| < k, \\ -k & \text{if } s \leq -k. \end{cases}$$

Let us define the function $G_k(s) : \mathbb{R} \to \mathbb{R}$ by

$$G_k(s) = \begin{cases} s-k & \text{if } s > k, \\ 0 & \text{if } s \leq k. \end{cases}$$

Let us outline the specific assumptions related to the problems we will examine. Assume that the domain Ω is bounded in $\mathbb{R}^N$ $(N \geq 2)$, with its boundary $\partial\Omega$ being sufficiently smooth and $\gamma > 0$. Suppose that $a(x,\ \xi)$: $\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfies the following assumptions:
Hypothesis (H_1)

$$a(x,\xi)\cdot\xi \geq \alpha|\xi|^p \text{ a.e. } x \text{ in } \Omega, \forall\xi \in \mathbb{R}^N, \tag{2.1}$$

$$(a(x,\xi) - a(x,\xi'))\cdot(\xi - \xi') > 0 \ \text{ a.e. } x \text{ in } \Omega,\ \forall\xi,\ \xi' \in \mathbb{R}^N, \tag{2.2}$$

$$|a(x,\xi)| \leq c(\nu(x) + |\xi|^{p-1}) \text{ a.e. } x \text{ in } \Omega, \forall\xi \in \mathbb{R}^N, \tag{2.3}$$

such that $\xi \neq \xi$, $\nu \in L^{p'}(\Omega)$, $\nu \geq 0$, $\alpha, c > 0$ and $1 < p < +\infty$.
Hypothesis (H_2)
We consider $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ to be a set-valued mapping that is both maximal and monotone, and satisfies $0 \in \beta(0)$.

Definition 2.1. *Solution of (P) is characterized by a couple of functions $(u,b) \in W_0^{1,p}(\Omega) \times L^1(\Omega)$ such that the conditions below are fulfilled:*
1) $u = 0$ on $\partial\Omega$ in weak sense,
2) $\forall\omega \subset\subset \Omega$

$$\exists c_\omega > 0 : u \geq c_\omega \ \textit{in } \omega, \tag{2.4}$$

3) $b \in \beta(u)$ a.e. in Ω,
4) $\displaystyle\int_\Omega b\varphi dx + \int_\Omega a(x,\nabla u)\cdot\nabla\varphi dx = \int_\Omega \frac{f\varphi}{u^\gamma}dx,\ \forall\varphi \in C_0^1(\Omega).$

The principal outcome of this paper is the proof of Theorem outlined next:

Theorem 2.1. *Assume that (H_1), (H_2) are satisfied. Let $f \in L^\infty(\Omega)$ be a positive function. Then, the problem (P) admits at least one weak solution (u,b) in the sense of definition 2.1.*

3. A priori estimates

For $n \in \mathbb{N}^*$, let $\beta_n : \mathbb{R} \to \mathbb{R}$ denote the Yosida approximation of $\beta(.)$, and consider the sequence of approximate equations as follow

$$\begin{cases} \beta_n(T_n(u_n)) - \operatorname{div}(a(x,\nabla u_n)) = \frac{f_n}{(|u_n|+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f_n(x) = \min(f(x), n)$, we recall that

$$\forall u \in W_0^{1,p}(\Omega),\ \langle \beta_n(u), u\rangle \geq 0,\ |\beta_n(u)| \leq |\beta_0(u)| \text{ and } \lim_{n\to\infty} \beta_n(u) = \beta(u),$$

where β_0 is the element of minimum norm in $\beta(0)$.

Equation (3.1) can be expressed in its weak form as

$$\int_\Omega \beta_n(T_n(u_n))\varphi dx + \int_\Omega a(x, \nabla u_n)\cdot\nabla\varphi dx = \int_\Omega \frac{f_n\varphi}{(|u_n| + \frac{1}{n})^\gamma}dx,\ \forall\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{3.2}$$

We consider the operator $A_n : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, defined by

$$\langle A_n u, \varphi\rangle = \int_\Omega \beta_n(T_n(u))\varphi dx + \int_\Omega a(x, \nabla u)\cdot\nabla\varphi dx$$

Lemma 3.1. *Under hypothesis (H_1), (H_2) the operator A_n is bounded, coercive and pseudo-monotone.*

Proof. The operator A_n is properly defined and monotone due to (2.2), (2.3). Moreover, A_n is pseudo-monotone, indeed, $\beta_n \circ T_n$ is continuous and bounded, moreover, the condition (2.2) involved (see [12], p.157). Using (2.1) and as a result of $\int_\Omega \beta_n(T_n(u))udx$ is positive, we get

$$\langle A_n u, u\rangle \geq \alpha \int_\Omega |\nabla u|^p dx,\ \forall u \in W_0^{1,p}(\Omega).$$

Then A_n is coercive on $W_0^{1,p}(\Omega)$. □

Proposition 3.1. *The problem (3.1) has a weak solution u_n, such that $u_n \in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$.*

Proof. The proof relies on a typical application of Schauder's fixed point theorem. Let $n \in \mathbb{N}^*$ and fix $v \in L^p(\Omega)$. From the classical results the following non-singular problem

$$\begin{cases} \beta_n(T_n(w)) - \operatorname{div}(a(x, \nabla w)) = \frac{f_n}{(|v|+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

has a unique solution w belongs to $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$. We establish a well-defined mapping for which

$$S : L^p(\Omega) \to L^p(\Omega),$$

where $S(v) = w$. Once more, due to the regularity of the data $\frac{f_n}{(|v|+\frac{1}{n})^\gamma}$, it is possible to use w as a test function to derive

$$\int_\Omega \beta_n(T_n(w))wdx + \int_\Omega a(x, \nabla w)\cdot\nabla w dx = \int_\Omega \frac{f_n w}{(|v| + \frac{1}{n})^\gamma}dx.$$

Given that the leftmost term is nonnegative and thanks to (2.1), it follows

$$\alpha\int_\Omega |\nabla w|^p dx \leq n^{\gamma+1}\int_\Omega |w|dx,$$

applying Poincaré's inequality, we get

$$\int_\Omega |\nabla w|^p dx \leq \frac{Cn^{\gamma+1}}{\alpha}\int_\Omega |\nabla w|dx,$$

through the application of Hölder's inequality to the terms on the right, we have

$$\int_\Omega |\nabla w|^p dx \le \frac{Cn^{\gamma+1}}{\alpha}|\Omega|^{\frac{1}{p'}}\left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}.$$

It follows that

$$\int_\Omega |\nabla w|^p dx \le \left(\frac{Cn^{\gamma+1}}{\alpha}\right)^{p'}|\Omega|.$$

Employing the Poincaré inequality for the left-hand side, we obtain

$$||w||_{L^p(\Omega)} \le \left(\frac{Cn^{\gamma+1}}{\alpha}\right)^{\frac{p'}{p}}|\Omega|^{\frac{1}{p}}.$$

Then

$$||w||_{L^p(\Omega)} \le C' \tag{3.4}$$

where $C' = \left(\frac{Cn^{\gamma+1}}{\alpha}\right)^{\frac{p'}{p}}|\Omega|^{\frac{1}{p}}$ represents a positive constant which remains unaffected by w and v. As a result, we conclude that the ball B with radius C' is unchanged for the map S in $L^p(\Omega)$.

Next, we demonstrate that the map S is continuous in B. We choose a sequence v_k that strongly converges to v in $L^p(\Omega)$, as a result, the Dominated Convergence Theorem implies that

$$\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma} \to \frac{f_n}{(|v|+\frac{1}{n})^\gamma} \text{ in } L^p(\Omega).$$

It remains to demonstrate that $S(v_k)$ converges to $S(v)$ in $L^p(\Omega)$. Using compactness, we know that the sequence $w_k = S(v_k)$ converges in $L^p(\Omega)$ to some function w. It suffices to prove that $w = S(v)$. We recall that $\frac{f_n}{(|v_k|+\frac{1}{n})^\gamma}$ is bounded, and $w_k \in L^\infty(\Omega)$, moreover, there exists a positive constant λ, which is independent of v_k and w_k (though it may depend on n), such that $||w_k||_{L^\infty(\Omega)} \le \lambda$. The sequence w_k is bounded in $W_0^{1,p}(\Omega)$. Consequently, since the weak solution is unique, we are allowed to affirm that $w_k = S(v_k) \to w = S(v)$ in $L^p(\Omega)$. Hence S is continuous in $L^p(\Omega)$. To conclude, we must check that the set $S(B)$ is relatively compact. Consider a bounded sequence v_k and let $w_k = S(v_k)$, we showed that

$$\int_\Omega |\nabla w|^p dx = \int_\Omega |\nabla S(v)|^p dx \le C',$$

then, for any v in $L^p(\Omega)$, considering $v = v_k$, we get

$$\int_\Omega |\nabla w_k|^p dx = \int_\Omega |\nabla S(v_k)|^p dx \le C',$$

then, $S(v)$ is relatively compact in $L^p(\Omega)$, due to Rellich-Kondrachov Theorem. By using the Schauder fixed-point theorem, it follows that a fixed point of the map S exists. We can take u_n in B such that $S(u_n) = u_n$, and thus $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ will solve problem (3.1). □

Lemma 3.2. *For all $n \in \mathbb{N}^*$, the sequence u_n increases with n and for any $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ (regardless of n) ensuring that*

$$u_n(x) \ge c_\omega \text{ in } \omega. \tag{3.5}$$

Specifically, the sequence u_n admits a pointwise limit u, and u satisfies (2.4).

Proof. We will adhere closely to the proof presented in [[6], Lemma 2.1, 2.2].

Since $\frac{f_n}{(|u_n|+\frac{1}{n})^\gamma} \geq 0$ and due to the fact that the operator A_n satisfies the maximum principle, we can infer that $u_n(x) \geq 0$.

Thanks to the monotonicity of the operator A_n and by applying that $0 \leq f_n \leq f_{n+1}$, it follows that the sequence is increasing with respect to n.

Given that for any fixed n, $u_n \in L^\infty(\Omega)$. In particular, for $n = 1$, we obtain that

$$\beta_1(T_1(u_1)) - \operatorname{div}(a(x, \nabla u_1)) = \frac{f_1}{(|u_1| + 1)^\gamma} \geq \frac{f_1}{(||u_1||_\infty + 1)^\gamma} \geq 0.$$

As $\frac{f_1}{(||u_1||_\infty+1)^\gamma} \geq 0$, is not identically zero, implies, via the application of the strong maximum principle that for all $\omega \subset\subset \Omega$, there is a positive constant $c_\omega > 0$ satisfying

$$u_1(x) \geq c_\omega \text{ in } \omega.$$

Therefore, (3.5) holds since $u_n \geq u_1$ for any n, and if u is the pointwise limit of the sequence u_n, which exists due to its monotonicity, then u also satisfies (2.4). □

Lemma 3.3. *Take $f \in L^\infty(\Omega)$ and consider u_n the solution of (3.1). It follows that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. As a result, there is a subsequence, still labeled u_n, and a measurable function u satisfying*

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega.$$

Furthermore, there exist a function $b \in L^\infty(\Omega)$ for which

$$\beta_n(T_n(u_n)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega). \tag{3.6}$$

Proof. Let $k \geq 1$. We use $G_k(u_n)$ in (3.2) as a test function and $\gamma > 0$, we derive

$$\int_\Omega \beta_n(T_n(u_n))G_k(u_n)dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla(G_k(u_n))dx = \int_\Omega \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\gamma}dx.$$

Since the initial expression on the left-hand side is nonnegative and under assumption (2.1), it follows

$$\alpha \int_\Omega |\nabla(G_k(u_n))|^p dx \leq \int_\Omega \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\gamma}dx.$$

We introduce the set $A_{n,k}$ corresponding to each k, defined by:

$$A_{n,k} = \{x \in \Omega \text{ such that } u_n \geq k\}.$$

Observing that $(u_n + \frac{1}{n})^\gamma \geq k^\gamma \geq 1$ on the set $A_{n,k}$ such that $G_k(u_n) \neq 0$ and by the definition of the function $G_k(u_n)$, we obtain

$$\alpha \int_\Omega |\nabla(G_k(u_n))|^p dx \leq \int_\Omega f G_k(u_n)dx,$$

our function f here is in $L^\infty(\Omega)$, so beyond this step, we can proceed following the same method as in the proof of [[4] Theorem 1.1], in order to establish that the sequence u_n is bounded in $L^\infty(\Omega)$. Using the property of weak* lower semicontinuity of the norm, and given that u is the pointwise limit of u_n, it is possible to pass to the limit in the estimates on L^∞ norm of u_n, thus, we obtain that $u \in L^\infty(\Omega)$, and

$$||u||_\infty \leq C(\alpha, N, \Omega,)||f||_\infty = c_\infty. \tag{3.7}$$

Taking u_n as a test function in (3.2) and using (2.1), (3.7), we arrive at

$$\int_\Omega |\nabla u_n|^p dx \leq C||f||_\infty. \tag{3.8}$$

Hence, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω.

Now, we move on to demonstrate (3.6), for $\delta > 0$, we take

$$\varphi_{\delta,n} = \frac{1}{\delta}\left((T_{k+\delta}(\beta_n(T_n(u_n))) - T_k(\beta_n(T_n(u_n)))\right)$$

as a test function in (3.2), we derive

$$\int_\Omega \beta_n(T_n(u_n))\varphi_{\delta,n} dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi_{\delta,n} dx = \int_\Omega \frac{f_n \varphi_{\delta,n}}{(u_n + \frac{1}{n})^\gamma} dx.$$

As $|\varphi_{\delta,n}| \leq 1$ and β_n is monotone increasing with $\beta_n(0) = 0$, we have

$$\frac{1}{\delta}\int_{\{k \leq |\beta_n(T_n(u_n))| \leq k+\delta\}} a(x, \nabla u_n) \cdot \nabla T_n(u_n) \beta_n'(T_n(u_n)) dx \geq 0.$$

Hence

$$\frac{1}{\delta}\int_{\{|\beta_n(T_n(u_n))| \geq k\}} \beta_n\left(T_n(u_n)\right)\left(T_{k+\delta}(\beta_n(T_n(u_n))) - T_k(\beta_n(T_n(u_n)))\right) dx$$
$$\leq \int_{\{|\beta_n(T_n(u_n))| \geq k\}} \frac{|f|}{|u_n + \frac{1}{n}|^\gamma} dx.$$

Since $\varphi_{\delta,n}$ and u_n has identical sign and by the monotonicity of our operator β_n, we obtain

$k|\{|\beta_n(T_n(u_n))| \geq k+\delta\}| \leq \displaystyle\int_{\{|\beta_n(T_n(u_n))| \geq k+\delta\}} |\beta_n(T_n(u_n))| dx$

$\leq \displaystyle\frac{1}{\delta}\int_{\{|\beta_n(T_n(u_n))| \geq k\}} \beta_n\left(T_n(u_n)\right)\left(T_{k+\delta}(\beta_n(T_n(u_n))) - T_k(\beta_n(T_n(u_n)))\right)$

$\leq \displaystyle\int_{\{|\beta_n(T_n(u_n))| \geq k\}} \frac{|f|}{|u_n + \frac{1}{n}|^\gamma} dx \leq \frac{||f||_\infty}{c_\omega}\{|\beta_n(T_n(u_n))| \geq k\}|$, where $c_\omega > 0$ is a constant (independent of n).

By choosing $k > \frac{||f||_\infty}{c_\omega}$ and letting $\delta \to 0$, we find $|\{|\beta_n(T_n(u_n))| \geq k\}| = 0$ for any $k > \frac{||f||_\infty}{c_\omega}$. Consequently

$$||\beta_n(T_n(u_n))||_\infty \leq \frac{||f||_\infty}{c_\omega}. \tag{3.9}$$

From (3.9), we find $b \in L^\infty(\Omega)$ such that

$$\beta_n(T_n(u_n)) \overset{*}{\rightharpoonup} b \text{ in } L^\infty(\Omega).$$

The proof is thus finished. $\square$

4. Proof of result

We are now able to prove Theorem 2.1, we will use all the results from the previous section

Proof. **First step:** We show that

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx = \int \frac{f\varphi}{u^\gamma}dx \tag{4.1}$$

For any nonnegative test function $\varphi \in C_0^1(\Omega)$, we have

$$\begin{aligned}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx &= \int_\Omega \beta_n(T_n(u_n))\varphi dx + \int_\Omega a(x,\nabla u_n)\nabla\varphi dx\\ &\le \int_\Omega \beta_n(T_n(u_n))\varphi dx + c\int_\Omega \nu(x)\nabla\varphi dx + c\int_\Omega |\nabla u_n|^{p-1}\nabla\varphi dx\end{aligned}$$

due to (2.3), then using Young inequality, we get

$$\begin{aligned}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx \le \int_\Omega \beta_n(T_n(u_n))\varphi dx + \frac{c}{p'}\int_\Omega (\nu(x))^{p'}dx + \frac{2c}{p}\int_\Omega |\nabla\varphi|^p dx\\ + \frac{c(p-1)}{p}\int_\Omega |\nabla u_n|^p dx.\end{aligned}$$

Finally

$$\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx \le \int_\Omega \beta_n(T_n(u_n))\varphi dx + C'[||\varphi||_{C_0^1(\Omega)} + ||u_n||_{W_0^{1,p}(\Omega)}] \tag{4.2}$$

By applying Fatou's lemma in (4.2), yields

$$\int_\Omega \frac{f\varphi}{u^\gamma} \le b||\varphi||_\infty + C$$

where C unaffected by n. Then, for any nonnegative $\varphi \in C_0^1(\Omega)$, we have $\frac{f\varphi}{u^\gamma} \in L^1(\Omega)$. As a result, if $\frac{1}{u^\gamma}$, blows up at 0, we affirm that

$$\{u = 0\} \subset \{f = 0\}, \tag{4.3}$$

up to a set of zero Lebesgue measure.

From this point onward, we suppose that $\frac{1}{u^\gamma}$ blows up at 0. We fix $\delta > 0$, then

$$\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx = \int_{\{u_n\le\delta\}} \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx + \int_{\{u_n>\delta\}} \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx. \tag{4.4}$$

Moreover, we introduce $V_\delta(u_n)$ defined in $W_0^{1,p}(\Omega)$, thanks to Lemma 1.1 in [17], where

$$V_\delta(t) = \begin{cases} 1 & t\le\delta\\ \frac{2\delta-t}{\delta} & \delta<t<2\delta\\ 0 & t\ge 2\delta.\end{cases} \tag{4.5}$$

Then, taking as a test function $\varphi V_\delta(u_n)$ in (3.2), we derive

$$\int_\Omega \frac{f_n\varphi V_\delta(u_n)}{(u_n+\frac{1}{n})^\gamma}dx = \int_\Omega \beta_n(T_n(u_n))\varphi V_\delta(u_n)dx + \int_\Omega a(x,\nabla u_n)\cdot\nabla(\varphi V_\delta(u_n))dx, \tag{4.6}$$

in addition, we have

$$\int_{\{u_n\leq\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}dx\leq\int_{\Omega}\frac{f_n\varphi V_\delta(u_n)}{(u_n+\frac{1}{n})^{\gamma}}dx \tag{4.7}$$

using (4.5), it follows that

$$\int_{\{u_n\leq\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}dx\leq\int_{\Omega}\beta_n(T_n(u_n))\varphi V_\delta(u_n)dx+\int_{\Omega}a(x,\nabla u_n)\cdot\nabla\varphi V_\delta(u_n)dx$$
$$-\frac{1}{\delta}\int_{\{\delta<u_n<2\delta\}}a(x,\nabla u_n)\cdot\nabla u_n\varphi dx$$

from (2.1),(2.3), we deduce

$$\begin{aligned}\int_{\{u_n\leq\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}dx&\leq\int_{\Omega}\beta_n(T_n(u_n))\varphi V_\delta(u_n)dx+c\int_{\Omega}(\nu(x)+|\nabla u_n|^{p-1})\nabla\varphi V_\delta(u_n)dx\\&-\frac{\alpha}{\delta}\int_{\{\delta<u_n<2\delta\}}|\nabla u_n|^{p}\varphi\nabla u_n dx\\&\leq\int_{\Omega}\beta_n(T_n(u_n))\varphi V_\delta(u_n)dx+c\int_{\Omega}(\nu(x)+|\nabla u_n|^{p-1})\nabla\varphi V_\delta(u_n)dx.\end{aligned}$$

Since V_δ is bounded, we conclude that

$$(\nu(x)+|\nabla u_n|^{p-1})\nabla\varphi V_\delta(u_n)\rightharpoonup(\nu(x)+|\nabla u|^{p-1})\nabla\varphi V_\delta(u)\text{ in }(L^{p'}(\Omega))^N\text{ as }n\to\infty.$$

Thus, it follows that

$$\lim_{n\to\infty}\int_{\{u_n\leq\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}dx\leq\int_{\Omega}b\varphi V_\delta(u)dx+c\int_{\Omega}\left(\nu(x)+|\nabla u|^{p-1}\right)\nabla\varphi V_\delta(u)dx. \tag{4.8}$$

Given that $V_\delta(u)\underset{\delta\to0}{\to}\chi_{\{u=0\}}$ a.e. in Ω, and as $u\in W_0^{1,p}(\Omega)$, this implies that $(\nu(x)+|\nabla u|^{p-1})\nabla\varphi V_\delta(u)\underset{\delta\to0}{\to}0$ a.e. in Ω. Using the Lebesgue theorem to the terms on the right of (4.8), we find that

$$\lim_{\delta\to0^+}\lim_{n\to+\infty}\int_{\{u_n\leq\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}dx=0. \tag{4.9}$$

Now, for the second term in (4.4), we observe that

$$0\leq\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n>\delta\}}\leq f\sup_{s\in]\delta,\infty)}\left[\frac{1}{s^{\gamma}}\right]\varphi\in L^1(\Omega). \tag{4.10}$$

We note that it is necessary to exclude δ from the set $W=\{\eta;|u=\eta|>0\}$ to avoid cases where the characteristic function $\chi_{\{u>\delta\}}$ can behave discontinuously and cause loss of convergence, the set W is at most countable. Then, one can conclude that $\chi_{\{u_n>\delta\}}$ converges almost everywhere to $\chi_{\{u>\delta\}}$ in Ω. It follows that

$$\frac{f_n\varphi}{(u_n+\frac{1}{n})^{\gamma}}\chi_{\{u_n>\delta\}}\underset{n\to+\infty}{\longrightarrow}\frac{f\varphi}{u^{\gamma}}\chi_{\{u>\delta\}}\text{ in }L^1(\Omega),$$

furthermore, we get $fh(u)\varphi\chi_{\{u>\delta\}}\in L^1(\Omega)$, then

$$\frac{f\varphi}{u^{\gamma}}\chi_{\{u>\delta\}}\underset{\delta\to0}{\longrightarrow}\frac{f\varphi}{u^{\gamma}}\chi_{\{u>0\}}\text{ in }L^1(\Omega),$$

therefore, applying the Lebesgue Theorem once more, we get

$$\lim_{\delta\to 0^+}\lim_{n\to\infty}\int_{\{u_n>\delta\}}\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx=\int_{\{u>0\}}\frac{f\varphi}{u^\gamma}dx. \tag{4.11}$$

Combining (4.9), (4.11), we finally arrive at for any positive $\varphi\in C_0^1(\Omega)$

$$\lim_{n\to\infty}\int_\Omega\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx=\int_\Omega\frac{f\varphi}{u^\gamma}dx. \tag{4.12}$$

For $\varphi=\varphi^+-\varphi^-$, we conclude that (4.12) is valid $\forall\varphi\in C_0^1(\Omega)$.
Second step: Due to (3.8) there exist a subsequence, remaining notated by u_n, and a measurable function u satisfying

$$u_n\rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega.$$

Our goal is to establish that

$$u_n\to u \text{ strongly in } W_0^{1,p}(\Omega). \tag{4.13}$$

We use as a test function $\varphi=u_n-u$ in (3.2), we get

$$\int_\Omega\beta_n(T_n(u_n))(u_n-u)dx+\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx=\int_\Omega\frac{f_n(u_n-u)}{(u_n+\frac{1}{n})^\gamma}dx.$$

Then

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx=\int_\Omega\frac{f_n(u_n-u)}{(u_n+\frac{1}{n})^\gamma}dx-\int_\Omega\beta_n(T_n(u_n))(u_n-u)dx.$$

The right-hand side limit is zero as $n\to\infty$. In another way, we can write

$$\begin{aligned}&\int_\Omega(a(x,\nabla u_n)-a(x,\nabla u))\cdot\nabla(u_n-u)dx\\&=\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx-\int_\Omega a(x,\nabla u)\cdot\nabla(u_n-u)dx,\end{aligned} \tag{4.14}$$

we have that

$$\lim_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx=0.$$

Thus, we derive that

$$\lim_{n\to\infty}\int_\Omega(a(x,\nabla u_n)-a(x,\nabla u))\cdot\nabla(u_n-u)dx=0, \tag{4.15}$$

owing to (2.2), the integrand function on the left-hand side of (4.15) is positive, then

$$(a(x,\nabla u_n)-a(x,\nabla u))\cdot\nabla(u_n-u)\longrightarrow 0 \text{ in } L^1(\Omega).$$

Therefore, for a subsequence still denoted by u_n, one arrives at

$$(a(x,\nabla u_n)-a(x,\nabla u))\cdot\nabla(u_n-u)\longrightarrow 0 \text{ for a.e. } x \text{ in } \Omega,$$

there is a subset $Z\subseteq\Omega$ with zero measure, so that for each x outside Z, it holds that

$$D_n(x)=(a(x,\nabla u_n)-a(x,\nabla u))\cdot\nabla(u_n-u)\longrightarrow 0, \tag{4.16}$$

$|u(x)|<\infty, |\nabla u(x)|<\infty, |\nu(x)|<\infty$ and $u_n(x)\to u(x)$, as a result of the growth condition (2.1), (2.3) and $||u_n||_\infty\le c_\infty$, we have

$$D_n(x)\ge\alpha|\nabla u_n|^p-c(x)(1+|\nabla u_n|+|\nabla u_n|^{p-1}),$$

where $c(x)$ depending only on x, which demonstrates, due to (4.16) that the sequence $|\nabla u_n|$ is bounded uniformly in $\mathbb{R}^N$. We adopt a similar approach to that in Lemma 5 in [5] with respect to the Carathéodory function a and n, to get (4.13). Returning to equation (3.2), we can now pass to the limit. For this, we consider φ in $C_0^1(\Omega)$, it results that

$$\int_\Omega \beta_n(T_n(u_n))\varphi dx + \int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi dx = \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx. \tag{4.17}$$

Using (4.13), we get $\nabla u_n \to \nabla u$ strongly in $(L^p(\Omega))^N$ and a.e. in Ω. Thus, by Vitali's Theorem, we conclude that

$$a(x, \nabla u_n) \longrightarrow a(x, \nabla u) \text{ strongly in } (L^{p'}(\Omega))^N.$$

Hence, taking the limit in (4.17), and by applying the result from step 1, we find

$$\int_\Omega b\varphi dx + \int_\Omega a(x, \nabla u) \cdot \nabla \varphi dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx. \tag{4.18}$$

Last step: We still need to show that $u(x) \in D(\beta(x))$, and also $b(x) \in \beta(u(x))$ for almost every x in Ω. The fact that β is a maximal monotone graph ensures the presence of a lower semicontinuous, proper, and convex function such that

$$j : \mathbb{R} \to [0,\ \infty] \text{ such that } \beta(r) = \partial j(r)\ \forall r \in \mathbb{R}.$$

Referring to [7], for $n \in \mathbb{N}^*$, $j_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$\forall s \in \mathbb{R}\ j_n(s) = \int_0^s \beta_n(r) dr.$$

Exhibits the properties outlined below
1) For any $n \in \mathbb{N}^*, j_n$ is convex and diferentiable $\forall s \in \mathbb{R}$, such that

$$j_n'(s) = \beta_n(s)\ \forall s \in \mathbb{R} \text{ with every } n \in \mathbb{N}^*$$

2) $j_n(s) \to j(s)$ for all $s \in \mathbb{R}$ as $n \to \infty$.
For any $n \in \mathbb{N}^*$, it follows from 1) that

$$j_n(s) \geq j_n(T_n(u_n)) + (s - T_n(u_n))\beta_n(T_n(u_n)), \tag{4.19}$$

applies to all $s \in \mathbb{R}$ and holds almost everywhere in Ω.
Let $F \subset \Omega$ denote any arbitrary measurable set, and let χ_F represent its characteristic function.
We start by fixing $n_0 > 0$, then multiply (4.19) by $g_l(u_n)\chi_F$, where $g_l : \mathbb{R} \to \mathbb{R}$ defined by

$$g_l(r) = \begin{cases} 0 & \text{if } |r| \geq l+1, \\ l+1-|r| & \text{if } l < |r| < l+1, \\ 1 & \text{if } |r| \leq l. \end{cases}$$

We integrate over Ω and applying 2), to arrive at

$$j(s) \int_F g_l(u_n) dx \geq \int_F (j_{n_0}(T_{l+1}(u_n)) g_l(u_n) + (r - T_{l+1}(u_n)) g_l(u_n) \beta_n(T_n(u_n))) dx \tag{4.20}$$

for all $s \in \mathbb{R}$ and for $n > \min(n_0, \dfrac{1}{l})$. Letting $n \to \infty$, bearing in mind that F arbitrary, and from (4.20), we obtain

$$j(s)g_l(u) \geq j_{n_0}(T_{l+1}(u))g_l(u) + bg_l(u)(s - T_{l+1}(u)) \ \forall s \in \mathbb{R} \text{ and a.e. } x \text{ in } \Omega. \tag{4.21}$$

By first letting $l \to \infty$ and then $n_0 \to \infty$ in (4.21), we eventually get

$$j(s) \geq j(u(x)) + b(x)(s - u(x)) \ \forall s \in \mathbb{R} \text{ and a.e. } x \text{ in } \Omega. \tag{4.22}$$

We finally get $u(x) \in D(\beta(x))$, and also $b(x) \in \beta(u(x))$ for almost every x in Ω. This concludes the proof of Theorem 2.1.

□

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Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability Statement

The manuscript has no associate data.

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