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A MARKOVIAN SWITCHING DIFFUSION FOR AN SIRS EPIDEMIC MODEL ON COMPLEX NETWORK

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Abstract. The purpose of this research is to deepen our comprehension of a scale-free network stochastic SIRS epidemic model. Our investigation centers on the global dynamics of a scale-free network that undergoes regime switching. We begin by demonstrating that, for every positive starting value, the system has a unique global positive solution. Then, under the right circumstances, we show that the model has some interesting asymptotic properties. These properties include the extinction of the disease and its persistence in the mean. Through rigorous analysis, we explore the circumstances under which these outcomes occur and provide scientific evidence to support our findings.

Keywords. Complex network, Stochastic epidemic model, Markovian switching, Extinction dynamics, Stochastic persistence
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1. Introduction

For the purpose of modeling epidemics, compartmental models are commonly employed. The proposed models involve the division of the population into multiple classes, including susceptible, infected, and removed individuals. The mathematical studies of these models frequently assume that the population is homogeneous; see for instance [14, 20]. This indicates that each infective individual is a source of the disease, which is equally likely to lead to the subsequent transmission of the disease to other members of the community with whom that individual is in contact, and vice versa. Nevertheless, it has been recorded that in actuality, there are certain individuals, referred to as super-spreaders, who have the ability to spread the virus to a large number of other individuals within the population [16]. For the purpose of incorporating the heterogeneity of contact, a different approach was chosen to analyze the spread of diseases by utilizing network theory [16, 18, 22, 23, 26, 30, 31]. Nodes essentially represent individuals, while edges represent the connections between them. Degree is the number of contacts. The same compartment $S_j(t)$, $I_j(t)$, or $R_j(t)$ is assigned to all individuals with the same degree j at time t. Various types of network topologies, including lattice,

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small world, random, and scale-free, have been well characterized in the literature by graph theorists. These graphs find use in a variety of fields, including biology (for modeling cell interactions) and neuroscience (for describing the intricate brain network) [1, 9, 12].

But in order to get close to the actual data, researchers needed an algorithm that models real-world networks. The simulations demonstrated that the Barabasi-Albert model provides a more accurate representation of real-world network data [2] compared to other models that are presently in use. In subsequent work, Bollobas et al. [5] rigorously demonstrated their concept. To go further into this, Bocaletti et al. [3] provided a comprehensive overview of several complex network architectures and the uses of these structures. Following the previous introduction, numerous authors in the field of epidemiology have utilized complex networks to model the transmission of infectious diseases.

There have been numerous studies published in recent times that are of great interest and concern themselves with the dynamics of epidemic models in networks [16, 18, 26, 27, 28]. In their study, Liu et al. [18] utilized the Latin Hypercube Sampling algorithm to investigate the transmission dynamics of the Zika virus inside a complex network. Using a complex network with a heterogeneous degree distribution, the authors of [28] conducted a study on the global behavior of a two-stage contact process. In reality, they fixed the analysis that Masuda and Konno [19] gave and showed that, in the right conditions, the global stability of disease-free and endemic equilibrium states can be shown. Wang and Cao [27] used the next-generation matrix method to study the transmission of aquatic diseases using a network model. The authors calculated the expression for the basic reproduction number $\mathcal{R}$. The authors demonstrated that the disease-free equilibrium state is globally stable when $\mathcal{R} < 1$. They also investigated the global stability of the endemic equilibrium state by examining two distinct scenarios. The first case occurs when the immunity rate is expected to be permanent. The second occurs when indirect environment-to-human transmission is excluded and just the direct human-to-human transmission rate is considered.

One of the pioneering works was suggested by Li et al. [16]. They formulated and analyzed the following network-based SIRS epidemic model with birth and death rates:

$$\begin{cases} \dfrac{d(S_j)}{\mathrm{d}t} = \mu - \mu S_j - \beta_j S_j \Delta_I + \lambda R_j, \\ \dfrac{d(I_j)}{\mathrm{d}t} = -(\mu+\gamma) I_j + \beta_j S_j \Delta_I, \\ \dfrac{d(R_j)}{\mathrm{d}t} = -(\mu+\lambda) R_j + \gamma I_j. \end{cases} \tag{1.1}$$

The system (1.1) is designed to assure that, for $j = 1, 2, \ldots, n$, the equation $S_j + I_j + R_j = 1$ is satisfied. Given the current situation, the model incorporates essential parameters such as: μ denotes the death and birth rates, $\gamma > 0$ represents the recovery rate of the infected nodes, $\beta_j > 0$ characterizes the degree-dependent infection rate, $\lambda > 0$ denotes the average loss of immunity rate, and

n is the maximum degree of connectivity present in the network, and $\Delta_I(t)$ represents the proportion of infective occupied edges over the entire network, as determined by

$$\Delta_I(t) = \frac{1}{< \jmath >} \sum_{j=1}^{n} \varphi_j \mathcal{P}_j I_j(t). \tag{1.2}$$

Within this context, $\mathcal{P}_j$ stands for the probability that an infected node has (j) neighbors and thus $\sum_1^n \mathcal{P}_j = 1$; $< \jmath >= \sum_1^n j\mathcal{P}_j$ represents the average network connectivity. Furthermore, φ_j denotes the infectivity of a node with degree j, where different types are discussed in [18, 30, 31]. According to [16], the authors found that the dynamics of the network-based SIRS model (1.1) is completely determined by the threshold

$$\mathcal{R}_0 = \frac{1}{(\mu+\gamma) < \jmath >} \sum_{j=1}^{n} \beta_j \varphi_j \mathcal{P}_j.$$

In fact, if $\mathcal{R}_0 \leq 1$, the disease dies out and the disease-free equilibrium is globally attractive. When $\mathcal{R}_0$ exceeds 1, the disease-free equilibrium becomes unstable and in the meantime there exists a unique endemic equilibrium state which is globally asymptotically stable.

Actually, stochastic models are much better than deterministic ones because randomness and doubt are so common in real life. Results from stochastic models are more informative than those from deterministic ones. In contrast to deterministic models, which only provide a single prediction, stochastic models allow us to build a distribution of expected outcomes by running them repeatedly. The analysis of crucial parameters, such as the mean and variance of the number of infected individuals at a given period, is made possible by this distribution of expected results. More information than deterministic models can be obtained by statistical inference, particularly through maximum likelihood estimation of unknown parameters in stochastic differential equations. Therefore, a great deal of research has been done to examine how stochasticity affects epidemic models (see, e.g., [6, 7, 8, 11, 18, 24]). Similar to [21, 22, 23], in our study, we use $\beta_j = j\beta$, where $\beta > 0$ represents the transmission rate. We incorporate stochasticity into the SIRS model (1.1) via the technique of parameter perturbation. Therefore, we replace the infection coefficient β by $\beta + \sigma\frac{d\mathcal{W}}{dt}$, where $\mathcal{W}$ is a Brownian motion defined on the complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions and σ is the intensity of the noise. So, considering a recruitment rate $A > 0$ and a graded cure rate $\rho > 0$, the stochastic model with white noise of the deterministic system (1.1) is

$$\begin{cases} \dfrac{d(S_j)}{\mathrm{d}t} &= A - \mu S_j - j\beta S_j \Delta_I + \lambda R_j - j\sigma S_j \Delta_I \dfrac{d\mathcal{W}}{\mathrm{d}t}, \\ \dfrac{d(I_j)}{\mathrm{d}t} &= -(\mu+\rho+\gamma) I_j + j\beta S_j \Delta_I + j\sigma S_j \Delta_I \dfrac{d\mathcal{W}}{\mathrm{d}t}, \\ \dfrac{d(R_j)}{\mathrm{d}t} &= -(\mu+\lambda) R_j + \gamma I_j, \quad j = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

with an initial condition that verifies $S_j(0) > 0, I_j(0) > 0, R_j(0) > 0, \quad j = 1, 2, \ldots, n$. For φ_j, the infectivity of a node with degree j, we set $\varphi_j = j$ to simplify our model [18]. Consequently, Δ_I takes the form

$$\Delta_I = \sum_{j=1}^{n} \frac{j}{< \jmath >} \mathcal{P}_j I_j, \qquad < \jmath > = \sum_{j=1}^{n} j \mathcal{P}_j. \tag{1.4}$$

2. Preliminaries

Color noise has the potential to disrupt epidemic models by causing the system to switch between different environmental regimes. Color noise, often known as telegraph noise, is one form of environmental noise [10, 15, 25]. One way to think about telegraph noise is as a transition between multiple environmental regimes that are distinct from one another due to things like dietary preferences, weather patterns, or social and cultural norms. Depending on the former, the disease can spread more quickly or more slowly. Inspired by the previous developments, in the present work we investigate a stochastic SIRS epidemic model in a scale-free network employing the stochastic differential equation for Markovian switching provided below:

$$\begin{cases} \dfrac{dS_j(t)}{\mathrm{d}t} &= \Big[A_{\alpha(t)} - \mu_{\alpha(t)} S_j(t) - j\beta_{\alpha(t)} S_j(t) \times \Delta_I(t) + \lambda_{\alpha(t)} R_j\Big] \\ & \quad -j\sigma_{\alpha(t)} S_j(t) \times \Delta_I(t) \dfrac{d\mathcal{W}(t)}{\mathrm{d}t}, \\ \dfrac{dI_j(t)}{\mathrm{d}t} &= \Big[-\big(\mu_{\alpha(t)} + \rho_{\alpha(t)} + \gamma_{\alpha(t)}\big) I_j(t) + j\beta_{\alpha(t)} S_j(t) \times \Delta_I(t)\Big] \\ & \quad +j\sigma_{\alpha(t)} S_j(t) \times \Delta_I(t) \dfrac{d\mathcal{W}(t)}{\mathrm{d}t}, \\ \dfrac{dR_j(t)}{\mathrm{d}t} &= \Big[-\big(\mu_{\alpha(t)} + \lambda_{\alpha(t)}\big) R_j(t) + \gamma_{\alpha(t)} I_j(t)\Big], \quad j = 1, 2, \ldots, n. \end{cases} \tag{2.1}$$

It is common for the amount of time spent waiting for the next switch to be exponentially scattered, and the process of switching between environments is frequently memoryless. Typically, this kind of switching is described by a continuous-time Markov chain $\{\alpha(t)\}_{t \geq 0}$ with a state space $\mathbb{G}_m = \{1, 2, \ldots, m\}$ and the generator $\Xi = \big(\xi_{\ell k}\big)_{m \times m}$ is specified, for $\tau > 0$, by

$$\mathbb{P}\Big(\alpha(t+\tau) = k | \alpha(t) = \ell\Big) = \begin{cases} \tau \xi_{\ell k} + o(\tau), & \text{if } \ \ell \neq k, \\ 1 + \tau \xi_{\ell\ell} + o(\tau), & \text{if } \ \ell = k. \end{cases} \tag{2.2}$$

Here, $\xi_{\ell k}$ is the rate of change from ℓ to k while $\xi_{\ell\ell} = -\sum_{\ell \neq k} \xi_{\ell k}$. Based on our assumptions, the Markov chain $\{\alpha(t)\}_{t \geq 0}$ is assumed to be irreducible in this study, meaning that the system can transition between regimes. According to our hypothesis, the Markov chain $\{\alpha(t)\}_{t \geq 0}$ has a unique stationary distribution $\Upsilon = \big(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\big)$ which can be determined by solving the linear equation

$\Upsilon\Xi = 0$ subject to

$$\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha = 1 \quad\text{and}\quad \Upsilon_\alpha > 0, \quad \forall\alpha \in \mathbb{G}_m.$$

We assume that A_α, μ_α, γ_α, ρ_α, λ_α, β_α and σ_α are all positive constants for each $\alpha \in \mathbb{G}_m$. Furthermore, we include the following notations:

- For $j \in \{1, 2, \dots, n\} \triangleq \mathbb{A}_n$ and $u \geq 0$,
$$U_j(u) = \big(S_j(u), I_j(u), R_j(u)\big),$$
and
$$\big(S_1(u), I_1(u), R_1(u), \dots, S_n(u), I_n(u), R_n(u)\big) \triangleq \big(U_1(u), \dots, U_n(u)\big).$$
- For $k = 1, 2, 3$,
$$< j^k > = \sum_{j\in\mathbb{A}_n} j^k \mathcal{P}_j.$$
- For every parameter in the model (2.1), such as μ_α, we assign
$$\hat{\mu} = \max_{\alpha\in\mathbb{G}_m} \{\mu_\alpha\}, \text{ and } \check{\mu} = \min_{\alpha\in\mathbb{G}_m} \{\mu_\alpha\}.$$

3. Well-posedness of the model system

Before delving into the asymptotic properties of solutions of (2.1), it is necessary to demonstrate that the solution of system (2.1) does not explode to infinity in a finite time. Furthermore, we prove here that (2.1) has a unique, non-negative, and bounded solution if the start positive initial value is used.

Describe the set

$$\mathcal{H} = \left\{(S_1, I_1, R_1, \dots, S_n, I_n, R_n) \in (0,\infty)^{3n} \middle| \tfrac{\check{A}}{\hat{\rho}+\hat{\mu}} < S_j + I_j + R_j < \tfrac{\hat{A}}{\check{\mu}}, \forall\, j \in \mathbb{A}_n\right\}.$$

Theorem 3.1. *For any specified starting value $\big(U_1(0), \dots, U_n(0)\big) \in \mathcal{H}$, there is a unique solution $\big(U_1(u), \dots, U_n(u)\big) \in \mathcal{H}$ of model (2.1) defined on $u \geq 0$ and the solution will remain in $\mathcal{H}$ with probability 1.*

Proof. In system (2.1), the coefficients are locally Lipschitz continuous. Thus, for the starting value $\big(U_1(0), \dots, U_n(0)\big) \in \mathcal{H}$, there is a unique maximal local solution $\big(U_1(u), \dots, U_n(u)\big)$ on $u \in [0, \varrho_e)$, where ϱ_e is the explosion time. Define the stopping times

$$\begin{aligned}\varrho^+ &= \inf\Big\{u \in (0, \varrho_e);\quad S_1(u) \leq 0 \text{ or } I_1(u) \leq 0 \text{ or } R_1(u) \leq 0 \ \dots \text{ or } \dots \\ &\qquad S_n(u) \leq 0 \text{ or } I_n(u) \leq 0 \text{ or } R_n(u) \leq 0\Big\},\end{aligned}$$

and

$$\begin{aligned}\varrho^{++} &= \inf\left\{u \in (0, \varrho^+);\ \ \mathcal{N}_1(u) < \frac{\check{A}}{\hat{\rho}+\hat{\mu}} \text{ or } \mathcal{N}_2(u) < \frac{\check{A}}{\hat{\rho}+\hat{\mu}} \text{ or } \dots \right. \\ &\qquad \left. \text{or } \mathcal{N}_n(u) < \frac{\check{A}}{\hat{\rho}+\hat{\mu}}\right\},\end{aligned}$$

where, for all $j \in \mathbb{A}_n$, $\mathcal{N}_j(u) = S_j(u) + I_j(u) + R_j(u)$. From system (2.1), $\mathcal{N}_j$ verifies the equation

$$d\mathcal{N}_j = \big(A_\alpha - \mu_\alpha \times \mathcal{N}_j - \rho_\alpha \times I_j\big)\mathrm{d}t. \tag{3.1}$$

Suppose that $\mathbb{P}\left(\varrho^{++} < \infty\right) > 0$, and let $\omega \in (\varrho^{++} < \infty)$. Since for all $u \leq \varrho^{++}$ and for all $j \in \mathbb{A}_n$, $0 < S_j(u), I_j(u), R_j(u)$, then

$$I_j(u) < \mathcal{N}_j(u), \quad u \leq \varrho^{++}, \quad j \in \mathbb{A}_n. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} d\mathcal{N}_j(u) &\geq \big(A_{\alpha(u)} - (\mu_{\alpha(u)} + \rho_{\alpha(u)})\mathcal{N}_j(u)\big)\mathrm{d}u \\ &\geq \big(\check{A} - (\hat{\rho} + \hat{\mu})\mathcal{N}_j(u)\big)\mathrm{d}u, \quad u \leq \varrho^{++}, \quad j \in \mathbb{A}_n. \end{aligned}$$

Which gives by integration between 0 and u that

$$\mathcal{N}_j(u) - \frac{\check{A}}{\hat{\rho} + \hat{\mu}} \geq \left(\mathcal{N}_j(0) - \frac{\check{A}}{\hat{\rho} + \hat{\mu}}\right) e^{-(\hat{\rho}+\hat{\mu})u}, \quad u \leq \varrho^{++}, \quad j \in \mathbb{A}_n.$$

For $\omega \in (\varrho^{++} < \infty)$ and $u = \varrho^{++}$, there exists $j_0 = j_0(\omega)$ such that $\mathcal{N}_{j_0}(\varrho^{++}) = \frac{\check{A}}{\hat{\rho}+\hat{\mu}}$, we have

$$0 \geq \left(\mathcal{N}_{j_0}(0) - \frac{\check{A}}{\hat{\rho} + \hat{\mu}}\right) e^{-(\hat{\rho}+\hat{\mu})\varrho^{++}} > 0.$$

There is a contradiction here. That is, $\varrho^{++} = \infty$ *a.s.*. Hence, $\varrho^{+} = \infty$ *a.s.*, and so $\varrho_e = \infty$ *a.s.*. In other terms, the solution $\big(U_1(u), \dots, U_n(u)\big)$ starting from $\big(U_1(0), \dots, U_n(0)\big) \in \mathcal{H}$ is global, positive and verifies for all $u \geq 0$,

$$\mathcal{N}_j(u) = S_j(u) + I_j(u) + R_j(u) > \frac{\check{A}}{\hat{\rho} + \hat{\mu}}, \quad j \in \mathbb{A}_n.$$

Beside, using the positivity of I_j, we derive from (3.1) that

$$d\mathcal{N}_j(u) \leq \big(\hat{A} - \check{\mu} \times \mathcal{N}_j(u)\big)\mathrm{d}u, \quad j \in \mathbb{A}_n.$$

Integrating the above inequality and using

$$\mathcal{N}_j(0) \leq \frac{\hat{A}}{\check{\mu}}, \quad j \in \mathbb{A}_n,$$

yields for each $u \geq 0$,

$$\mathcal{N}_j(u) \leq \frac{\hat{A}}{\check{\mu}} + \left(\mathcal{N}_j(0) - \frac{\hat{A}}{\check{\mu}}\right) e^{-\check{\mu}u} < \frac{\hat{A}}{\check{\mu}}, \quad j \in \mathbb{A}_n.$$

In such a situation, the domain $\mathcal{H}$ is positively invariant by system (2.1). The proof is complete. $\square$

4. Extinction of Disease

In the present section, we examine the circumstances that lead to the infectious disease of system (2.1) disappearing when the Markov switching is used.

Theorem 4.1. *For any initial values* $\big(U_1(0),\dots,U_n(0),\alpha(0)\big)\in\mathcal{H}\times\mathbb{G}_m$, *if* $\frac{\hat{A}}{\breve{\mu}}\leq\min\limits_{\alpha\in\mathbb{G}_m}\left\{\frac{\beta_\alpha}{\sigma_\alpha^2}\right\}\frac{<j>}{<j^2>}$ *and* $\tilde{\mathcal{R}}_1\triangleq\frac{\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\beta_\alpha\frac{<j^2>\hat{A}}{<j>\breve{\mu}}}{\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(\frac{1}{2}\left(\sigma_\alpha\frac{<j^2>\hat{A}}{<j>\breve{\mu}}\right)^2+\mu_\alpha+\rho_\alpha+\gamma_\alpha\right)}<1$, *then* Δ_I *verifies the following property*

$$\limsup_{t\to\infty}\left[\frac{\log(\Delta_I(t))}{t}\right]\leq\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(\tfrac{1}{2}\left(\sigma_\alpha\tfrac{<j^2>\hat{A}}{<j>\breve{\mu}}\right)^2+\mu_\alpha+\rho_\alpha+\gamma_\alpha\right)(\tilde{\mathcal{R}}_1-1).\tag{4.1}$$

Moreover, the disease will die out with probability one, i.e. for all $j\in\mathbb{A}_n$,

$$\lim_{t\to\infty}I_j(t)=0\ \ a.s..$$

Proof. Using Itô's formula on the function $\log(\Delta_I)$ yields

$$\begin{aligned}
d\big[\log(\Delta_I)\big] &= \Bigg[-(\mu_\alpha+\rho_\alpha+\gamma_\alpha)+\frac{\beta_\alpha}{<j>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\\
&\quad-\frac{\sigma_\alpha^2}{2}\left(\frac{1}{<j>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\right)^2\Bigg]\mathrm{d}t+\frac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_jd\mathcal{W}\\
&= \Bigg[-(\mu_\alpha+\rho_\alpha+\gamma_\alpha)+\frac{\beta_\alpha<j^2>}{<j>}\sum_{j\in\mathbb{A}_n}\frac{j^2}{<j^2>}\mathcal{P}_jS_j\\
&\quad-\frac{1}{2}\left(\frac{\sigma_\alpha<j^2>}{<j>}\right)^2\left(\sum_{j\in\mathbb{A}_n}\frac{j^2}{<j^2>}\mathcal{P}_jS_j\right)^2\Bigg]\mathrm{d}t\\
&\quad+\frac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_jd\mathcal{W}\\
&= \Phi_\alpha(\sum_{j\in\mathbb{A}_n}\frac{j^2}{<j^2>}\mathcal{P}_jS_j)\mathrm{d}t+\frac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_jd\mathcal{W},
\end{aligned}\tag{4.2}$$

where Φ_α is defined by

$$\Phi_\alpha(x)\ \triangleq\ -\frac{1}{2}\left(\sigma_\alpha\frac{<j^2>}{<j>}\right)^2x^2+\beta_\alpha\frac{<j^2>}{<j>}x-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big).\tag{4.3}$$

In view of $\frac{\beta_\alpha}{\sigma_\alpha^2}\frac{<j>}{<j^2>}\geq\frac{\hat{A}}{\breve{\mu}}$ for all $\alpha\in\mathbb{G}_m$, the function $x\longmapsto\Phi_\alpha(x)$ is increasing in the interval $[0,\frac{\hat{A}}{\breve{\mu}}]$. From $S_j\in\big(0,\frac{\hat{A}}{\breve{\mu}}\big)$, it is very easy to see that $\sum_{j\in\mathbb{A}_n}\frac{j^2}{<j^2>}\mathcal{P}_jS_j\in\big(0,\frac{\hat{A}}{\breve{\mu}}\big)$. Hence, we derive

$$\Phi_\alpha(\sum_{j\in\mathbb{A}_n}\frac{j^2}{<j^2>}\mathcal{P}_jS_j)\leq\Phi_\alpha(\frac{\hat{A}}{\breve{\mu}}).\tag{4.4}$$

Substituting (4.4) into (4.2), integrating (4.2) from 0 to t and dividing by t, one can obtain that

$$\begin{aligned}\frac{\log(\Delta_I(t))}{t} &\leq \frac{1}{t}\int_0^t \Phi_{\alpha(u)}(\frac{\hat{A}}{\breve{\mu}})\mathrm{d}u + \frac{1}{t}\int_0^t \frac{\sigma_{\alpha(u)}}{<\jmath>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j S_j(u)d\mathcal{W}(u)\\ &\quad + \frac{\log(\Delta_I(0))}{t}\\ &\leq \frac{1}{t}\int_0^t \Phi_{\alpha(u)}(\frac{\hat{A}}{\breve{\mu}})\mathrm{d}u + \frac{\mathcal{Z}_t}{t} + \frac{\log(\Delta_I(0))}{t}, \end{aligned} \tag{4.5}$$

where

$$\mathcal{Z}_t = \int_0^t \frac{\sigma_{\alpha(u)}}{<\jmath>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j S_j(u)d\mathcal{W}(u).$$

$\mathcal{Z}_t$ is a real-valued local martingale having the following Meyer's angle bracket process

$$\begin{aligned}<\mathcal{Z}_t, \mathcal{Z}_t> &= \int_0^t \left(\frac{\sigma_{\alpha(u)}}{<\jmath>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j S_j(u)\right)^2 \mathrm{d}u\\ &\leq \frac{<\jmath^2>^2\,\hat{\sigma}^2\hat{A}^2}{<\jmath>^2\,\breve{\mu}^2}t < \infty.\end{aligned}$$

So, by applying the large number theorem for martingales [17], we find that

$$\limsup_{t\to\infty}\frac{\mathcal{Z}_t}{t} = 0 \;\; a.s. \tag{4.6}$$

Therefore, according to the ergodic property of Markov chain $\alpha(t)$ (see, Theorem 6.2 of [4]), one can obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\int_0^t \Phi_{\alpha(u)}(\frac{\hat{A}}{\breve{\mu}})\mathrm{d}u\\ =& \sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\Phi_{\alpha(u)}(\frac{\hat{A}}{\breve{\mu}})\\ =& \sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(-\frac{1}{2}\left(\sigma_\alpha\frac{<\jmath^2>\hat{A}}{<\jmath>\breve{\mu}}\right)^2 + \beta_\alpha\frac{<\jmath^2>\hat{A}}{<\jmath>\breve{\mu}} - (\mu_\alpha+\rho_\alpha+\gamma_\alpha)\right)\\ =& \sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(\frac{1}{2}\left(\sigma_\alpha\frac{<\jmath^2>\hat{A}}{<\jmath>\breve{\mu}}\right)^2 + \mu_\alpha+\rho_\alpha+\gamma_\alpha\right)(\tilde{\mathcal{R}}_1 - 1).\end{aligned} \tag{4.7}$$

Taking the superior limit on both sides of (4.5) and combining with (4.6) and (4.7), we have

$$\limsup_{t\to\infty}\left[\frac{\log(\Delta_I(t))}{t}\right] \leq \sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(\frac{1}{2}\left(\sigma_\alpha\frac{<\jmath^2>\hat{A}}{<\jmath>\breve{\mu}}\right)^2 + \mu_\alpha+\rho_\alpha+\gamma_\alpha\right)(\tilde{\mathcal{R}}_1 - 1).$$

Therefore, by the condition $\tilde{\mathcal{R}}_1 < 1$, we have

$$\lim_{t\longrightarrow\infty}\sum_{j\in\mathbb{A}_n}\frac{j}{<\jmath>}\mathcal{P}_jI_j(t)=\lim_{t\longrightarrow\infty}\Delta_I(t)=0.$$

From $I_j \in (0, \frac{\hat{A}}{\tilde{\mu}})$ for all $j \in \mathbb{A}_n$, we get

$$\lim_{t\longrightarrow\infty}I_j(t)=0\ \ a.s.,\ \ j\in\mathbb{A}_n.$$

Therefore, the proof of Theorem 4.1 is completed. □

Theorem 4.2. *For any initial values* $\big(U_1(0),\dots,U_n(0),\alpha(0)\big)\in\mathcal{H}\times\mathbb{G}_m$*, then* Δ_I *verifies the following property*

$$\limsup_{t\to\infty}\left[\frac{\log(\Delta_I(t))}{t}\right]\leq\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)\big(\tilde{\mathcal{R}}_2-1\big)\ \ a.s., \tag{4.8}$$

where

$$\tilde{\mathcal{R}}_2\triangleq\frac{\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\frac{\beta_\alpha^2}{2\sigma_\alpha^2}}{\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)}.$$

Moreover, if $\tilde{\mathcal{R}}_2 < 1$*, the solution of the stochastic differential equation* (2.1) *obeys, for all* $j\in\mathbb{A}_n$*,*

$$\lim_{t\to\infty}I_j(t)=0.$$

Proof. After applying Itô's formula to the function $\log(\Delta_I)$, the result is

$$\begin{aligned}
d\big[\log(\Delta_I)\big] &= \Bigg[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)+\frac{\beta_\alpha}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\\
&\quad-\frac{\sigma_\alpha^2}{2}\left(\frac{1}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\right)^2\Bigg]\mathrm{d}t+\frac{\sigma_\alpha}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_jd\mathcal{W}\\
&= \left[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)+\frac{\beta_\alpha^2}{2\sigma_\alpha^2}-\frac{1}{2}\left(\frac{\beta_\alpha}{\sigma_\alpha}-\frac{\sigma_\alpha}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\right)^2\right]\mathrm{d}t\\
&\quad+\frac{\sigma_\alpha}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_jd\mathcal{W}\\
&\leq \left[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)+\frac{\beta_\alpha^2}{2\sigma_\alpha^2}\right]\mathrm{d}t+\frac{\sigma_\alpha}{<\jmath>}\sum_{j\in\mathbb{A}_n}j^2\mathcal{P}_jS_j\mathrm{d}d\mathcal{W}.
\end{aligned}$$

By integration, we obtain

$$\begin{aligned}
\log(\Delta_I(t)) &\leq \int_0^t\left(\frac{\beta_{\alpha(u)}^2}{2\sigma_{\alpha(u)}^2}-\big(\mu_{\alpha(u)}+\rho_{\alpha(u)}+\gamma_{\alpha(u)}\big)\right)\mathrm{d}u\\
&\quad+\log(\Delta_I(0))+\mathcal{Z}_t,
\end{aligned} \tag{4.9}$$

where

$$\mathcal{Z}_t = \int_0^t \frac{\sigma_{\alpha(u)}}{< \jmath >} \sum_{j \in \mathbb{A}_n} j^2 \mathcal{P}_j S_j(u) d\mathcal{W}(u).$$

From (4.6), we have

$$\limsup_{t\to\infty} \frac{\mathcal{Z}_t}{t} = 0 \ a.s., \tag{4.10}$$

and according to the ergodic property of Markov chain $\alpha(t)$, we obtain

$$\begin{aligned}
& \limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(\frac{\beta^2_{\alpha(u)}}{2\sigma^2_{\alpha(u)}} - (\mu_{\alpha(u)} + \rho_{\alpha(u)} + \gamma_{\alpha(u)}) \right) du \\
= & \sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha \left(\frac{\beta^2_\alpha}{2\sigma^2_\alpha} - \big(\mu_\alpha + \rho_\alpha + \gamma_\alpha\big) \right) \\
= & \sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha \frac{\beta^2_\alpha}{2\sigma^2_\alpha} - \sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha \big(\mu_\alpha + \rho_\alpha + \gamma_\alpha\big) \\
= & \left(\sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha \big(\mu_\alpha + \rho_\alpha + \gamma_\alpha\big) \right) \times \left(\tilde{\mathcal{R}}_2 - 1 \right).
\end{aligned} \tag{4.11}$$

From (4.9), (4.10) and (4.11) we get the required confirmation. □

5. Disease persistence

The property of persistence is critically significant in epidemic models since it indicates that the disease persists regardless of the initial settings. For the purpose of studying the model (2.1) and its persistence, we need the following lemma (refer to, for example, Lemma 17 in [29] for additional details).

Lemma 5.1. *Consider $\mathcal{U} \in \mathcal{C}(\mathbb{R}_+ \times \Omega; \mathbb{R}_+)$ and $\mathcal{V} \in \mathcal{C}(\mathbb{R}_+ \times \Omega; \mathbb{R})$. If there exist positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that for each $s \geq 0$,*

$$\log \mathcal{U}(s) \ \geq \ \mathcal{M}_1 s - \mathcal{M}_2 \int_0^s \mathcal{U}(u) du + \mathcal{V}(s) \ \ and \ \ \lim_{s\to\infty} \frac{\mathcal{V}(s)}{s} = 0 \ a.s.,$$

then

$$\liminf_{s\to\infty} \frac{1}{s} \int_0^s \mathcal{U}(u) du \ \geq \ \frac{\mathcal{M}_1}{\mathcal{M}_2} \ a.s..$$

To begin with, let us put

$$\tilde{\mathcal{R}}^* \ \triangleq \ \frac{\frac{\check{\beta} < \jmath^2 >}{\hat{\mu} < \jmath >} \sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha A_\alpha}{\sum_{\alpha \in \mathbb{G}_m} \Upsilon_\alpha \left(\mu_\alpha + \rho_\alpha + \gamma_\alpha + \frac{\sigma^2_\alpha}{2} \left(\frac{\hat{A} < \jmath^2 >}{\check{\mu} < \jmath >} \right)^2 \right)}. \tag{5.1}$$

Theorem 5.2. *For given initial value* $\big(U_1(0),\ldots,U_n(0),\alpha(0)\big)\in\mathcal{H}\times\mathbb{G}_m$, *if* $\tilde{\mathcal{R}}^*>1$, *then* Δ_I *verifies the following property*

$$\liminf_{t\to\infty}\left[\frac{1}{t}\int_0^t \Delta_I(u)\,du\right] \;\geq\; \zeta_{\Delta_I}, \tag{5.2}$$

where

$$\zeta_{\Delta_I} \;\triangleq\; \tfrac{\hat{\mu}\check{\mu}<j>}{\check{\beta}\hat{\beta}\hat{A}<j^3>}\times\left[\sum_{\alpha\in\mathbb{G}_m}\Upsilon_\alpha\left(\mu_\alpha+\rho_\alpha+\gamma_\alpha+\tfrac{\sigma_\alpha^2}{2}\left(\tfrac{\hat{A}<j^2>}{\check{\mu}<j>}\right)^2\right)\right]\times(\tilde{\mathcal{R}}^*-1).$$

Proof. Using Itô's formula on the function $\log(\Delta_I)$ induces

$$\begin{aligned} d\big[\log(\Delta_I)\big] \;=\; & \Bigg[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)+\tfrac{\beta_\alpha}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_jS_j \\ & -\tfrac{\sigma_\alpha^2}{2}\left(\tfrac{1}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_jS_j\right)^2\Bigg]\mathrm{d}t+\tfrac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_jS_jd\mathcal{W}. \end{aligned} \tag{5.3}$$

From $S_j\in(0,\frac{\hat{A}}{\check{\mu}})$, we have

$$0 \;\leq\; \tfrac{1}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_jS_j \;\leq\; \tfrac{1}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j\tfrac{\hat{A}}{\check{\mu}} \;=\; \tfrac{\hat{A}<j^2>}{\check{\mu}<j>}. \tag{5.4}$$

Combining (5.3) and (5.4) yields

$$\begin{aligned} d\big[\log(\Delta_I)\big] \;\geq\; & \Bigg[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)+\check{\beta}\tfrac{<j^2>}{<j>}\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jS_j \\ & -\tfrac{\sigma_\alpha^2}{2}\left(\tfrac{\hat{A}<j^2>}{\check{\mu}<j>}\right)^2\Bigg]\mathrm{d}t+\tfrac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_jS_jd\mathcal{W}. \end{aligned} \tag{5.5}$$

On the other hand, from system (2.1), we get

$$\begin{aligned} d\left(\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jS_j\right) \;=\; & \Bigg[A_\alpha-\mu_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jS_j-\beta_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^3}{<j^2>}\mathcal{P}_jS_j\Delta_I \\ & +\lambda_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jR_j\Bigg]\mathrm{d}t-\sigma_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^3}{<j^2>}\mathcal{P}_jS_j\Delta_Id\mathcal{W}. \end{aligned}$$

Since

$$\begin{aligned} \left[\mu_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jS_j\right]\mathrm{d}t \;\geq\; & \left[A_\alpha-\beta_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^3}{<j^2>}\mathcal{P}_jS_j\Delta_I\right]\mathrm{d}t-d\left(\sum_{j\in\mathbb{A}_n}\tfrac{j^2}{<j^2>}\mathcal{P}_jS_j\right) \\ & -\sigma_\alpha\sum_{j\in\mathbb{A}_n}\tfrac{j^3}{<j^2>}\mathcal{P}_jS_j\Delta_Id\mathcal{W}. \end{aligned} \tag{5.6}$$

Considering that $S_j \in (0, \frac{\hat{A}}{\check{\mu}})$,

$$0 \;\leq\; \beta_\alpha \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j^2>}\mathcal{P}_j S_j \Delta_I \;\leq\; \hat{\beta} \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j^2>}\mathcal{P}_j \tfrac{\hat{A}}{\check{\mu}} \Delta_I \;=\; \tfrac{\hat{\beta}\hat{A}<j^3>}{\check{\mu}<j^2>}\Delta_I. \tag{5.7}$$

Combining (5.6) and (5.7) yields

$$\begin{aligned}
\hat{\mu}\left[\sum_{j\in\mathbb{A}_n} \tfrac{j^2}{<j^2>}\mathcal{P}_j S_j\right] \mathrm{d}t \;&\geq\; \left[A_\alpha - \tfrac{\hat{\beta}\hat{A}<j^3>}{\check{\mu}<j^2>}\Delta_I\right] \mathrm{d}t - d\left(\sum_{j\in\mathbb{A}_n} \tfrac{j^2}{<j^2>}\mathcal{P}_j S_j\right) \\
&\qquad -\sigma_\alpha \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j^2>}\mathcal{P}_j S_j \Delta_I d\mathcal{W}.
\end{aligned}$$

Substituting the previous inequality into (5.5) and rearrange

$$\begin{aligned}
d\big[\log(\Delta_I)\big] \;&\geq\; \Bigg[-\big(\mu_\alpha + \rho_\alpha + \gamma_\alpha\big) + \tfrac{\check{\beta}<j^2>}{\hat{\mu}<j>} A_\alpha - \tfrac{\check{\beta}\hat{\beta}\hat{A}<j^3>}{\hat{\mu}\check{\mu}<j>} \times \Delta_I \\
&\qquad -\tfrac{\sigma_\alpha^2}{2}\left(\tfrac{\hat{A}<j^2>}{\check{\mu}<j>}\right)^2\Bigg]\mathrm{d}t + \tfrac{\sigma_\alpha}{<j>}\sum_{j\in\mathbb{A}_n} j^2 \mathcal{P}_j S_j d\mathcal{W} \\
&\qquad -d\left(\sum_{j\in\mathbb{A}_n} \tfrac{\check{\beta}j^2}{\hat{\mu}<j>}\mathcal{P}_j S_j\right) - \tfrac{\check{\beta}\sigma_\alpha}{\hat{\mu}} \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j>}\mathcal{P}_j S_j \Delta_I d\mathcal{W}.
\end{aligned}$$

By integration the previous inequality, we get

$$\begin{aligned}
\log \Delta_I(t) \;&\geq\; \int_0^t \Bigg[-\big(\mu_{\alpha(u)} + \rho_{\alpha(u)} + \gamma_{\alpha(u)}\big) + \tfrac{\check{\beta}<j^2>}{\hat{\mu}<j>} A_{\alpha(u)} - \tfrac{\sigma^2_{\alpha(u)}}{2}\left(\tfrac{\hat{A}<j^2>}{\check{\mu}<j>}\right)^2\Bigg]\mathrm{d}u \\
&\qquad - \tfrac{\check{\beta}\hat{\beta}\hat{A}<j^3>}{\hat{\mu}\check{\mu}<j>} \times \int_0^t \Delta_I(u)\mathrm{d}u + \int_0^t \left(\tfrac{\sigma_{\alpha(u)}}{<j>}\sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j S_j(u)\right) d\mathcal{W}(u) \\
&\qquad - \sum_{j\in\mathbb{A}_n} \tfrac{\check{\beta}j^2}{\hat{\mu}<j>}\mathcal{P}_j (S_j(t) - S_j(0)) + \log \Delta_I(0) \\
&\qquad - \int_0^t \left(\tfrac{\check{\beta}\sigma_{\alpha(u)}}{\hat{\mu}} \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j>}\mathcal{P}_j S_j(u)\Delta_I(u)\right) d\mathcal{W}(u) \\
&\geq\; \int_0^t C_{\alpha(u)}\mathrm{d}u - \tfrac{\check{\beta}\hat{\beta}\hat{A}<j^3>}{\hat{\mu}\check{\mu}<j>} \times \int_0^t \Delta_I(u)\mathrm{d}u + V(t),
\end{aligned} \tag{5.8}$$

where

$$C_{\alpha(u)} \;=\; -\big(\mu_{\alpha(u)} + \rho_{\alpha(u)} + \gamma_{\alpha(u)}\big) + \tfrac{\check{\beta}<j^2>}{\hat{\mu}<j>} A_{\alpha(u)} - \tfrac{\sigma^2_{\alpha(u)}}{2}\left(\tfrac{\hat{A}<j^2>}{\check{\mu}<j>}\right)^2,$$

and

$$\begin{aligned} V(t) &= \int_0^t \left(\tfrac{\sigma_{\alpha(u)}}{<j>} \sum_{j\in\mathbb{A}_n} j^2\mathcal{P}_j S_j(u) \right) d\mathcal{W}(u) - \sum_{j\in\mathbb{A}_n} \tfrac{\breve{\beta} j^2}{\hat{\mu}<j>} \mathcal{P}_j \big(S_j(t) - S_j(0)\big) \\ &\quad + \log \Delta_I(0) - \int_0^t \left(\tfrac{\breve{\beta}\sigma_{\alpha(u)}}{\hat{\mu}} \sum_{j\in\mathbb{A}_n} \tfrac{j^3}{<j>} \mathcal{P}_j S_j(u)\Delta_I(u) \right) d\mathcal{W}(u). \end{aligned}$$

Therefore, by the condition $\tilde{\mathcal{R}}^* > 1$, we have $\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha > 0$. According to the ergodic property of Markov chain $\alpha(t)$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t C_{\alpha(u)} \mathrm{d}u = \sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha.$$

Then for all sufficiently small $\varepsilon > 0$, we obtain for any t sufficiently large

$$\int_0^t C_{\alpha(u)} \mathrm{d}u \geq \left(\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha - \varepsilon \right) t. \tag{5.9}$$

From (5.8) and (5.9), we deduce

$$\log \Delta_I(t) \;\geq\; \left[\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha - \varepsilon \right] t - \tfrac{\breve{\beta}\hat{\beta}\hat{A}<j^3>}{\hat{\mu}\breve{\mu}<j>} \times \int_0^t \Delta_I(u)\mathrm{d}u + V(t). \tag{5.10}$$

Furthermore, given that $S_j, \Delta_I \in (0, \frac{\hat{A}}{\hat{\mu}})$, then from large number theorem for martingales, we have

$$\lim_{t\to\infty} \frac{V(t)}{t} = 0.$$

It then follows from Lemma 5.1 that

$$\liminf_{t\to\infty} \left[\frac{1}{t} \int_0^t \Delta_I(u)\mathrm{d}u \right] \;\geq\; \left(\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha - \varepsilon \right) \times \tfrac{\hat{\mu}\breve{\mu}<j>}{\breve{\beta}\hat{\beta}\hat{A}<j^3>} \;\; a.s..$$

By letting $\varepsilon \longrightarrow 0$, we get

$$\liminf_{t\to\infty} \left[\frac{1}{t} \int_0^t \Delta_I(u)\mathrm{d}u \right] \;\geq\; \left(\sum_{\alpha\in\mathbb{G}_m} \Upsilon_\alpha C_\alpha \right) \times \tfrac{\hat{\mu}\breve{\mu}<j>}{\breve{\beta}\hat{\beta}\hat{A}<j^3>} \;=\; \zeta_{\Delta_I} \;\; a.s..$$

The proof is now complete. □

Theorem 5.3. *For any given initial value* $\big(U_1(0), \ldots, U_n(0), \alpha(0)\big) \in \mathcal{H} \times \mathbb{G}_m$, *if* $\tilde{\mathcal{R}}^* > 1$, *then the solution of system* (2.1) *obeys, for* $j \in \mathbb{A}_n$

(i) $\displaystyle \liminf_{t\to\infty} \left(\frac{1}{t} \int_0^t S_j(u)\,du \right) \;\geq\; \zeta_{S,j} \;\; a.s.,$

(ii) $\displaystyle \liminf_{t\to\infty} \left(\frac{1}{t} \int_0^t I_j(u)\,du \right) \;\geq\; \zeta_{I,j} \;\; a.s.,$

(iii) $\displaystyle \liminf_{t\to\infty} \left(\frac{1}{t} \int_0^t R_j(u)\,du \right) \;\geq\; \zeta_{R,j} \;\; a.s.,$

where the positive constants $\zeta_{S,j}, \zeta_{I,j}$ and $\zeta_{R,j}$ are defined by

$$\zeta_{S,j} \triangleq \check{A}\left[\hat{\mu}+\frac{j\hat{A}\hat{\beta}}{\check{\mu}}\right]^{-1}, \zeta_{I,j} \triangleq \zeta_{\Delta_I}\times j\check{\beta}\left(\frac{\check{A}}{\hat{\rho}+\hat{\mu}}\right)\left[\hat{\mu}+\hat{\rho}+\hat{\gamma}+\frac{j\hat{A}\hat{\beta}}{\check{\mu}}\times\frac{\check{\mu}+\hat{\gamma}+\check{\lambda}}{\check{\lambda}+\check{\mu}}\right]^{-1},$$

and

$$\zeta_{R,j} \triangleq \frac{\check{\gamma}\times\zeta_{I,j}}{\hat{\lambda}+\hat{\mu}}.$$

Then the solutions of stochastic model system (2.1) starting from any point in $\mathcal{H}$ are strongly persistent in mean.

Proof. (i) Let $j\in\mathbb{A}_n$. From system (2.1), we have

$$d(S_j)=\Big[A_\alpha-\mu_\alpha S_j-j\beta_\alpha S_j\Delta_I+\lambda_\alpha R_j\Big]\mathrm{d}t-j\sigma_\alpha S_j\Delta_I d\mathcal{W}.$$

Using $S_j, R_j, \Delta_I \in \big(0,\frac{\hat{A}}{\check{\mu}}\big)$, integrating the equation above and dividing by t, we have

$$\begin{aligned}\frac{1}{t}\left(\hat{\mu}+\frac{j\hat{A}\hat{\beta}}{\check{\mu}}\right)\int_0^t S_j(u)\mathrm{d}u \quad &\geq \quad \check{A}-\frac{S_j(t)-S_j(0)}{t}\\ &-\frac{j}{t}\int_0^t \sigma_{\alpha(u)}S_j(u)\Delta_I(u)d\mathcal{W}(u). \qquad (5.11)\end{aligned}$$

From (5.11), $S_j\in\big(0,\frac{\hat{A}}{\check{\mu}}\big)$ and the large number theorem for martingales, we get the required result (i).

(ii) Let $j\in\mathbb{A}_n$. From the system (2.1), we have

$$\begin{aligned}d(I_j) &= \Big[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)I_j+j\beta_\alpha S_j\Delta_I\Big]\mathrm{d}t+j\sigma_\alpha S_j\Delta_I d\mathcal{W}\\ &= \Big[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)I_j+j\beta_\alpha\big(\mathcal{N}_j-I_j-R_j\big)\Delta_I\Big]\mathrm{d}t\\ &\quad +j\sigma_\alpha S_j\Delta_I d\mathcal{W}. \qquad (5.12)\end{aligned}$$

From $\mathcal{N}_j\in\big(\frac{\check{A}}{\hat{\rho}+\hat{\mu}},\frac{\hat{A}}{\check{\mu}}\big)$ and (5.12), we have

$$\begin{aligned}d(I_j) \quad \geq \quad &\Big[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)I_j+j\beta_\alpha\frac{\check{A}}{\hat{\rho}+\hat{\mu}}\Delta_I-j\beta_\alpha I_j\Delta_I-j\beta_\alpha R_j\Delta_I\Big]\mathrm{d}t\\ &+j\sigma_\alpha S_j\Delta_I d\mathcal{W}.\end{aligned}$$

From $\Delta_I\in\big(0,\frac{\hat{A}}{\check{\mu}}\big)$, we have

$$\begin{aligned}d(I_j) \quad \geq \quad &\left[-\big(\mu_\alpha+\rho_\alpha+\gamma_\alpha\big)I_j+j\beta_\alpha\frac{\check{A}}{\hat{\rho}+\hat{\mu}}\Delta_I-j\beta_\alpha\frac{\hat{A}}{\check{\mu}}I_j-j\beta_\alpha\frac{\hat{A}}{\check{\mu}}R_j\right]\mathrm{d}t\\ &+j\sigma_\alpha S_j\Delta_I d\mathcal{W}.\end{aligned}$$

Taking the above inequality and integrating it, then dividing the two sides by t, yields

$$\begin{aligned}
&\frac{1}{t}\left(\hat{\mu}+\hat{\rho}+\hat{\gamma}+\frac{j\hat{A}\hat{\beta}}{\check{\mu}}\right)\int_0^t I_j(u)\mathrm{d}u\\
\geq\ &\frac{j\check{\beta}}{t}\left(\frac{\check{A}}{\hat{\rho}+\hat{\mu}}\right)\int_0^t \Delta_I(u)\mathrm{d}u-\frac{j\hat{A}\hat{\beta}}{t\check{\mu}}\int_0^t R_j(u)\mathrm{d}u\\
&-\frac{I_j(t)-I_j(0)}{t}+\frac{j}{t}\int_0^t \sigma_{\alpha(u)}S_j(u)\Delta_I(u)d\mathcal{W}(u). \qquad (5.13)
\end{aligned}$$

According to the system (2.1), we are given

$$d(R_j)=\big[-(\mu_\alpha+\lambda_\alpha)R_j+\gamma_\alpha I_j\big]\mathrm{d}t, \qquad (5.14)$$

from which we deduce

$$-\frac{1}{t}\int_0^t R_j(u)\mathrm{d}u \geq \frac{R_j(t)-R_j(0)}{t(\check{\mu}+\check{\lambda})}-\frac{\hat{\gamma}}{t(\check{\mu}+\check{\lambda})}\int_0^t I_j(u)\mathrm{d}u. \qquad (5.15)$$

Combining (5.13) and (5.15) and rearranging to get

$$\begin{aligned}
&\frac{1}{t}\left(\hat{\mu}+\hat{\rho}+\hat{\gamma}+\frac{j\hat{A}\hat{\beta}}{\check{\mu}}\times\frac{\check{\mu}+\check{\lambda}+\hat{\gamma}}{\check{\mu}+\check{\lambda}}\right)\int_0^t I_j(u)\mathrm{d}u\\
\geq\ &\frac{j\check{\beta}}{t}\left(\frac{\check{A}}{\hat{\rho}+\hat{\mu}}\right)\int_0^t \Delta_I(u)\mathrm{d}u-\frac{I_j(t)-I_j(0)}{t}\\
&+\frac{j\hat{A}\hat{\beta}}{\check{\mu}(\check{\mu}+\check{\lambda})}\times\frac{R_j(t)-R_j(0)}{t}+\frac{j}{t}\int_0^t \sigma_{\alpha(u)}S_j(u)\Delta_I(u)d\mathcal{W}(u). \qquad (5.16)
\end{aligned}$$

From $I_j, R_j, \Delta_I \in \big(0, \frac{\hat{A}}{\check{\mu}}\big)$ and the large number theorem for martingales, we have

$$\lim_{t\to\infty}\left(-\frac{I_j(t)-I_j(0)}{t}+\frac{j\hat{A}\hat{\beta}(R_j(t)-R_j(0))}{t\check{\mu}(\check{\mu}+\check{\lambda})}+\frac{j}{t}\int_0^t \sigma_{\alpha(u)}S_j(u)\Delta_I(u)d\mathcal{W}(u)\right)=0. \qquad (5.17)$$

Considering (5.16), (5.17), and (5.2), the assertion (ii) follows immediately.

(iii) Let $j\in\mathbb{A}$. From (5.14), we have

$$\frac{\hat{\mu}+\hat{\lambda}}{t}\int_0^t R_j(u)\mathrm{d}u \ \geq\ -\frac{R_j(t)-R_j(0)}{t}+\frac{\check{\gamma}}{t}\int_0^t I_j(u)\mathrm{d}u. \qquad (5.18)$$

Considering that $R_j\in(0,\frac{\hat{A}}{\check{\mu}})$,

$$\lim_{t\longrightarrow\infty}\frac{R_j(t)-R_j(0)}{t}=0. \qquad (5.19)$$

From (5.18), (5.19), and (ii) we derive (iii). The proof is completed. $\square$

6. Numerical simulations

In the following, in order to verify the correctness of the theoretical results presented in our research, by the Milstein's higher order method [13], we will give the numerical simulations of stochastic regime-switching SRIS system based on the scale-free network.

Here, the degree distribution $\mathcal{P}_j = cj^{-3}$, where c is a constant satisfying $\sum_{j\in\mathbb{A}_n} \mathcal{P}_j = 1$. In model (2.1), let $\{\alpha(t)\}_{t\geq 0}$ is a right-continuous Markov switching taking values in a two-state space $\mathbb{G}_m = \{1, 2\}$ with the generator Ξ of the Markov chain is

$$\Xi = \begin{pmatrix} -\xi_{1,2} & \xi_{1,2} \\ \xi_{2,1} & -\xi_{2,1} \end{pmatrix}.$$

By solving the linear equation $\Upsilon\Xi = 0$, we obtain the unique stationary distribution

$$\Upsilon = \big(\Upsilon_1, \Upsilon_2\big) = \big(\frac{\xi_{2,1}}{\xi_{1,2} + \xi_{2,1}}, \frac{\xi_{1,2}}{\xi_{1,2} + \xi_{2,1}}\big).$$

To begin with, let us put $\xi_{1,2} = 1$ and $\xi_{2,1} = 2$, then the unique stationary distribution is given by $\big(\Upsilon_1, \Upsilon_2\big) = (0.6667, 0.3333)$.

Example 6.1. In model (2.1), choose the values of the parameters in the following manner: $A_1 = 1.6$, $\mu_1 = 0.4$, $\beta_1 = 0.031$, $\lambda_1 = 0.2$, $\rho_1 = 0.10$, $\gamma_1 = 0.31$, $\sigma_1 = 0.015$, $A_2 = 1$, $\mu_2 = 0.2$, $\beta_2 = 0.08$, $\lambda_2 = 0.1$, $\rho_2 = 0.12$, $\gamma_2 = 0.21$, $\sigma_2 = 0.051$. Here, we find that there are 20 different values of degree, i.e., $n = 20$. We calculate $\frac{\hat{A}}{\hat{\mu}} = 8 \leq \min_{\alpha\in\mathbb{G}_m}\left\{\frac{\beta_\alpha}{\sigma_\alpha^2}\right\} \times \frac{<j>}{<j^2>} = 14.19$, and $\tilde{\mathcal{R}}_1 = 0.9926 < 1$. According to Theorem 4.1, the requirements for extinction are satisfied, then $\lim_{t\longrightarrow\infty} \Delta_I(t) = 0$, and $\forall j \in \mathbb{A}_n$, $\lim_{t\longrightarrow\infty} I_j(t) = 0$ a.s. The computer simulations in Figure 1 provide unambiguous support for our findings.

Example 6.2. In model (2.1), choose the values of the parameters in the following manner: $A_1 = 2$, $\mu_1 = 0.6$, $\beta_1 = 0.27$, $\lambda_1 = 0.02$, $\rho_1 = 0.22$, $\gamma_1 = 0.1$, $\sigma_1 = 0.2$, $A_2 = 1$, $\mu_2 = 0.4$, $\beta_2 = 0.23$, $\lambda_2 = 0.08$, $\rho_2 = 0.25$, $\gamma_2 = 0.3$, $\sigma_2 = 0.17$. From Theorem 4.2, it is interesting to find that $\tilde{\mathcal{R}}_2$ is not dependent on the values of degree n in network. Here, we simulate three cases $n = 20$, $n = 45$, and $n = 90$. We compute $\tilde{\mathcal{R}}_2 = 0.9813 < 1$. Consequently, the condition of Theorem 4.2 about extinction is met, then $I_1, \ldots, I_n$ go to extinction. Computer simulations of Figure 2 provide strong evidence for these conclusions.

Example 6.3. In model (2.1), choose the values of the parameters in the following manner: $A_1 = 0.6$, $\mu_1 = 0.1$, $\beta_1 = 0.11$, $\lambda_1 = 0.23$, $\rho_1 = 0.1$, $\gamma_1 = 0.31$, $\sigma_1 = 0.02$, $A_2 = 0.5$, $\mu_2 = 0.2$, $\beta_2 = 0.25$, $\lambda_2 = 0.11$, $\rho_2 = 0.11$, $\gamma_2 = 0.22$, $\sigma_2 = 0.04$. Here, we find that there are 20 different values of degree. We compute $\tilde{\mathcal{R}}^* = 1.2257 > 1$. The requirement of Theorem 5.2 holds, consequently Δ_I persists in the mean, i.e.

$$\liminf_{t\to\infty}\left(\frac{1}{t}\int_0^t \Delta_I(u)\mathrm{d}u\right) > 0.$$

Figure 3 shows that for every $j \in \mathbb{A}_n$, S_j, I_j, and R_j are persistent in network, this supports the result of Theorem 5.3.

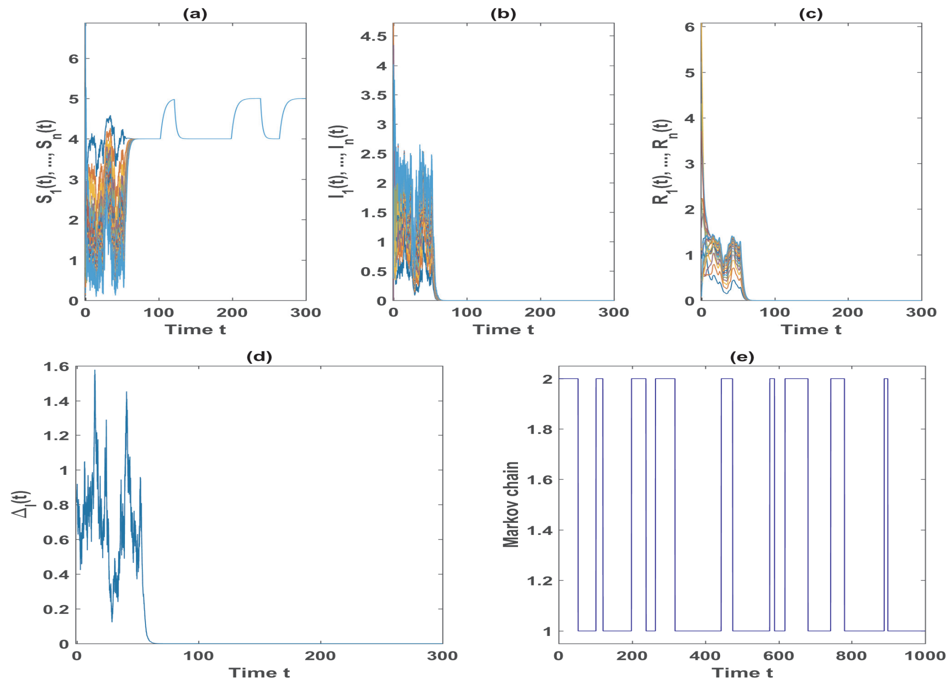

FIGURE 1. The simulations of the solution in system (2.1) with parameters from Example 6.1 for different initial values. (a) Reflects the simulations of susceptible individuals $S_1(t), \ldots, S_n(t)$; (b) represents the simulations of infective individuals $I_1(t), \ldots, I_n(t)$; (c) depicts the simulations of recovered individuals $R_1(t), \ldots, R_n(t)$; (d) shows the proportion of infective occupied edges over the entire network $\Delta_I(t)$; (e) is the Markov chain $\alpha(t)$.

Example 6.4 (The influence of random switching)**.** To demonstrate how the Markov chain $\alpha(t)$ affects disease transmission dynamics, we use the generator Ξ with the same stationary distribution but different elements. We choose 3 different generators Ξ,

$$\Xi_1 = \begin{pmatrix} -0.1 & 0.1 \\ 0.02 & -0.02 \end{pmatrix}, \ \Xi_2 = \begin{pmatrix} -1 & 1 \\ 0.2 & -0.2 \end{pmatrix}, \ \Xi_3 = \begin{pmatrix} -10 & 10 \\ 2 & -2 \end{pmatrix}.$$

Solving the linear equation $\Upsilon\Xi = 0$ reveals that the Markov chain $\alpha(t)$ has a unique stationary distribution $(\Upsilon_1, \Upsilon_2) = (0.1667, 0.8333)$ for all three cases. The present example uses the same additional parameters as Example 6.3. We calculate that $\tilde{\mathcal{R}}^* = 1.12 > 1$. In accordance with Theorem 5.3, we are able to determine that the disease is persistent for the model (2.1). As can be observed in Figure 4, the solutions of stochastic model (2.1) with generators Ξ_1, Ξ_2, and Ξ_3 fluctuate between determining solutions of the two states $\alpha = 1$ and $\alpha = 2$. Furthermore, the amplitude of the oscillations increases with increasing the elements of the generator Ξ. In other words, the disease becomes more unstable when the switching tempo is high enough.

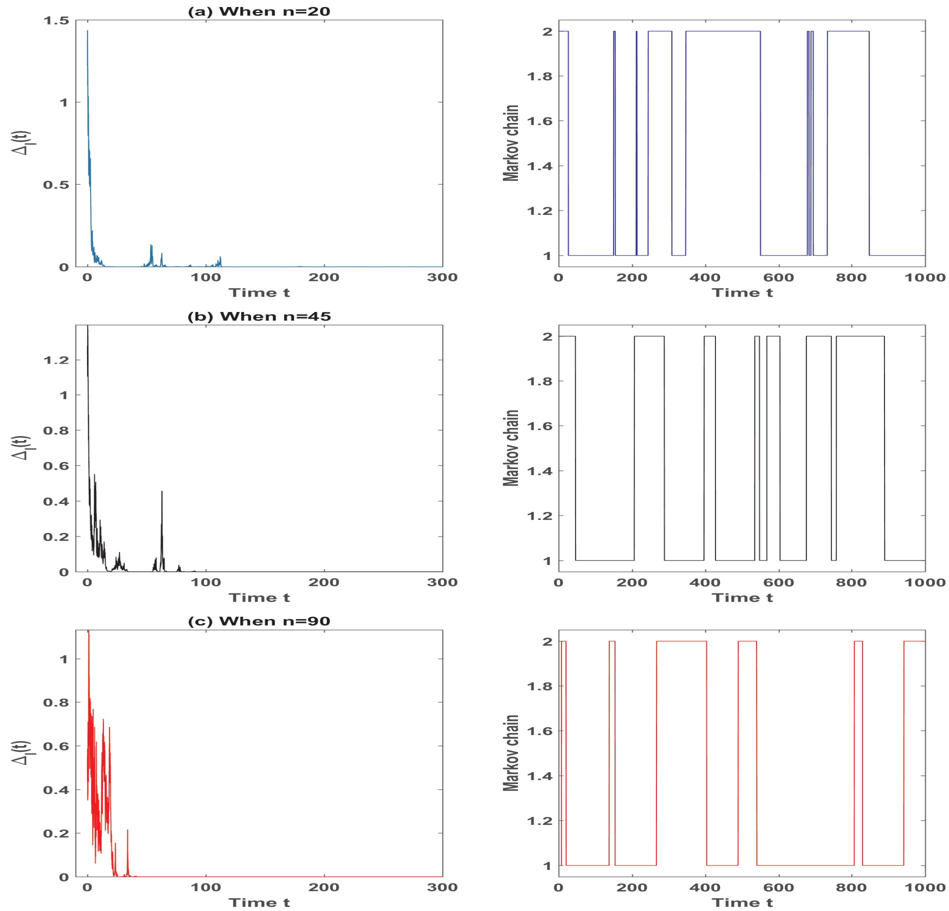

FIGURE 2. The simulations of $\Delta_I(t)$ in system (2.1) based on data from Example 6.2 with n different values of degree. (a) $n = 20$; (b) $n = 45$; (c) $n = 90$.

7. Conclusion

In the current study, we focus on examining the dynamic behaviors of a stochastic regime-switching SIRS epidemic system in complex heterogeneous networks. The first thing we did was establish that the global positive solution exists and is unique. Afterwards, we looked into how our stochastic SIRS epidemic model behaved over the course of time. Our findings meet the requirements for both extinction and persistence in the mean. According to the findings, the level of heterogeneity of the network $(<\jmath^2>/<\jmath>)$, the intensity of the white noises σ_α^2, and the stationary distribution $(\Upsilon_1, \dots, \Upsilon_m)$ of the Markov chain $\{\alpha(t)\}_{t\geq 0}$ all play a significant role in determining whether the epidemic will be eradicated or whether it will continue to persist in the population. In fact, if $\tilde{\mathcal{R}}_1 < 1$ or $\tilde{\mathcal{R}}_2 < 1$ occurs under specific circumstances, the epidemic quickly fades from the population. On the other hand, we demonstrated, on the basis of Theorem 5.3, that the process I_j is persistent in mean if $\tilde{\mathcal{R}}^* > 1$, in other words, the disease will continue in network. Importantly, we run simulations with various parameter values to demonstrate and validate our theoretical findings.

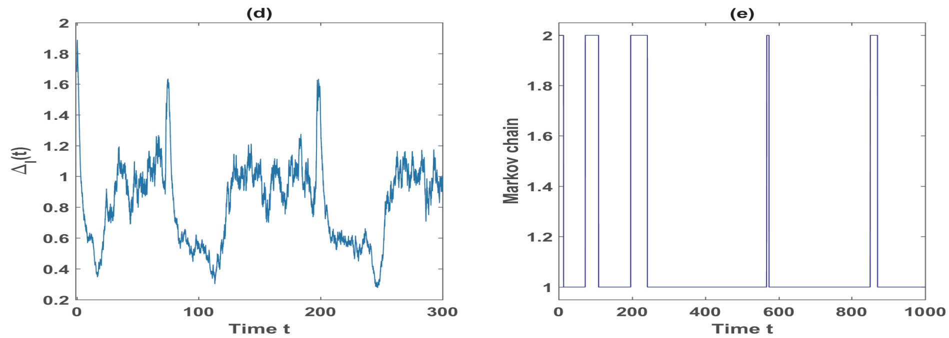

FIGURE 3. The simulations of the persistence of both populations with parameters from Example 6.3 for different initial values. (a) Represents the simulated situations of susceptible individuals $S_1(t), \dots, S_n(t)$; (b) illustrates the simulations of infective individuals $I_1(t), \dots, I_n(t)$; (c) reflects the simulations of recovered individuals $R_1(t), \dots, R_n(t)$; (d) shows the simulations of $\Delta_I(t)$; (e) is the Markov chain $\alpha(t)$.

Our research expands our knowledge of the mechanics of propagation in scale-free networks and gives us more tools to stop epidemics in their tracks. Furthermore, some of the methodologies employed here can also be applied to investigate certain issues in the study of contagious diseases, public perceptions, and electronic virus propagation, due to the numerous parallels between these phenomena and infectious diseases.

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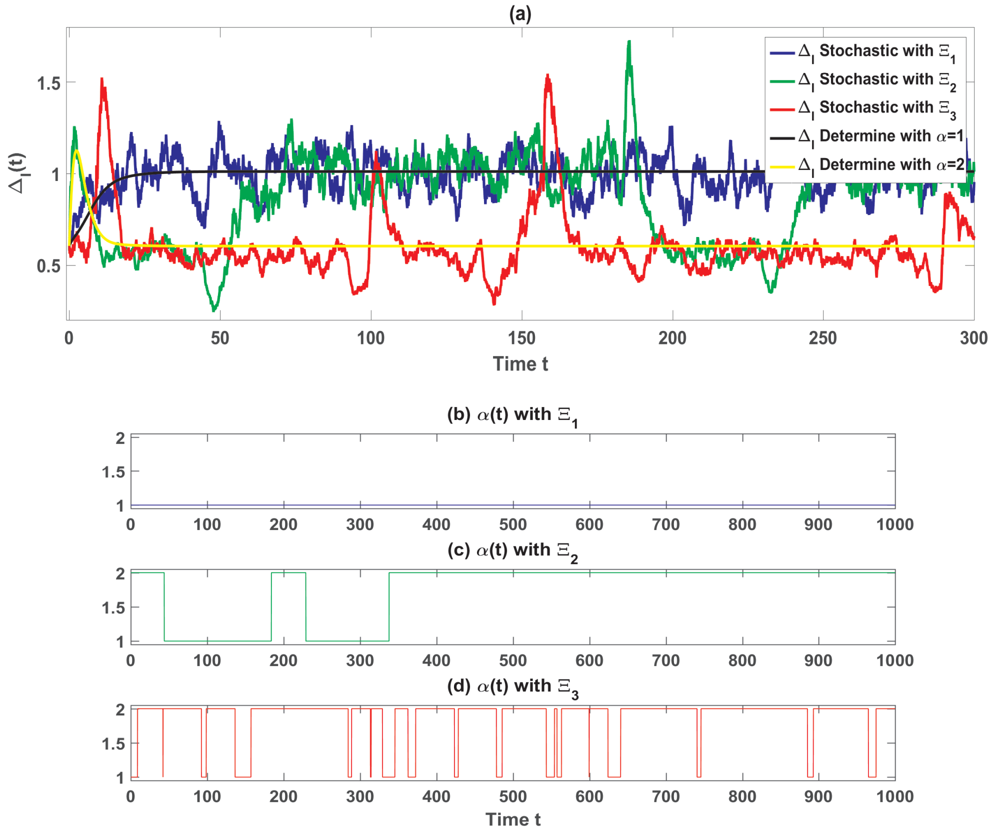

FIGURE 4. The numerical simulations of $\Delta_I(t)$ with initial conditions $\big(U_1(0),\ldots,U_n(0)\big) \in \mathcal{H}$ and accompanying Markov chain $\alpha(t)$ with three different generators Ξ_1, Ξ_2, Ξ_3.

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