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FIXED POINT THEOREMS FOR CONTRACTIONS OF INTEGRAL TYPE THROUGH GENERALIZED OMEGA DISTANCE

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Abstract. This paper introduces a novel fixed-point theorem for integral-type contractions using generalized Omega-distance. Our findings broaden and unify multiple well-known results in the literature by weakening contraction requirements and integrating Lebesgue-integrable auxiliary functions. Additionally, we demonstrate the applicability of our theorem by proving the existence and uniqueness of solutions for a class of nonlinear functional equations under minimal assumptions.

Keywords. Fixed point theorem, G_b-metric space, Ω-distance, integral contraction, Cauchy sequence, nonlinear analysis
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1. Introduction and preliminaries

The study of fixed point theorems in generalized metric spaces has gained considerable interest, particularly for their applications in differential equations and functional analysis since Banach's seminal result [5]. Building on foundational work such as [7], which used auxiliary functions in neutrosophic metric spaces, researchers have extended these ideas to various settings. For instance, [6] examined fixed points in b-metric spaces via auxiliary functions. While [9] developed quasi-contraction theorems in neutrosophic fuzzy metric spaces.

Further advances include nonlinear contractions in neutrosophic fuzzy metric spaces [12], Π-quasi contractions in complete neutrosophic spaces [8], and T-distance in b-metric spaces [14]. NF-L contractions were explored in [13]. Samreen et al. [20] introduced the framework of extended b-metric spaces and established novel fixed-point results utilizing generalized b-comparison functions, while Kamran et al. [16] investigated fixed points in graph-structured generalized metric spaces. Chandok and Dutta [10] contributed with results on weakly Chatterjea-type cyclic contractions. Chand et al. [11] introduced fixed point results for paired-Kannan contraction mappings, contributing to the ongoing development of generalized contraction principles.

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In parallel, Karapınar and Fulga [15] analyzed Proinov-type contractions in b-metric spaces. Aydi et al. [4] addressed tripled coincidence points in ordered spaces. Recent work by Malkawi et al. has further enriched the field with results on MR-metric [19], and various weak contraction mappings in b- and Mb-metric spaces [18].

Recent developments have focused on unifying these diverse approaches through generalized distance functions. The concept of Ω-distance, introduced by [2], has emerged as a powerful tool for analyzing contraction mappings across various generalized metric spaces. This work bridges the gap between traditional metric space theory and modern extensions by introducing a flexible framework that accommodates both integral-type contractions and asymmetric distance structures. Our approach builds upon these innovations while addressing the limitations of existing methods, particularly in handling discontinuous mappings and non-uniform convergence. The proposed generalized Ω-distance not only subsumes several known distance functions but also enables sharper existence and uniqueness results under weaker continuity assumptions.

The G_b-metric space, first introduced as a generalization of standard G-metric spaces, provides a more comprehensive framework for studying convergence properties and continuity characteristics. In this paper, we investigate the existence of fixed points for mappings satisfying integral-type contractive conditions through a newly developed generalized Ω-distance function.

The notion of G_b-metric space is described as follows:

Definition 1.1. [1] Let $\widetilde{H}$ be a set that is not devoid of elements and $\vartheta \geq 1$. Consider a mapping $\hat{\mathcal{G}}: \widetilde{H} \times \widetilde{H} \times \widetilde{H} \longrightarrow [0, \infty)$ denotes a function where

(G_b1) $\hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_3) = 0$ if $\aleph_1 = \aleph_2 = \aleph_3$;

(G_b2) $\hat{\mathcal{G}}(\aleph_1, \aleph_1, \aleph_2) > 0$ for all $\aleph_1, \aleph_2 \in \widetilde{H}$, with $\aleph_1 \neq \aleph_2$;

(G_b3) $\hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_2) \leq \hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_3)$, for all $\aleph_1, \aleph_2, \aleph_3 \in \widetilde{H}$ with $\aleph_2 \neq \aleph_3$;

(G_b4) $\hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_3) = \hat{\mathcal{G}}(p\{\aleph_1, \aleph_2, \aleph_3\})$, where p is a permutation of $\aleph_1, \aleph_2, \aleph_3$;

(G_b5) $\hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_3) \leq \vartheta \left[\hat{\mathcal{G}}(\aleph_1, \mu, \mu) \cdot \hat{\mathcal{G}}(\mu, \aleph_2, \aleph_3) \right] \forall \aleph_1, \aleph_2, \aleph_3, \mu \in \widetilde{H}$.

The function $\hat{\mathcal{G}}$ is then referred to as a generalized b-metric, and the pair $(\widetilde{H}, \hat{\mathcal{G}})$ is said to be a G_b-metric space.

Example 1.2. [1] On $(\widetilde{H}, \hat{\mathcal{G}})$ define a new function $\hat{\mathcal{G}}^*(\aleph_1, \aleph_2, \aleph_3) = \left(\hat{\mathcal{G}}(\aleph_1, \aleph_2, \aleph_3)\right)^\alpha$, where $\alpha > 1$. Then $\hat{\mathcal{G}}^*$ defines a G_b-metric on $\widetilde{H}$, with the associated constant $\vartheta = 2^{\alpha-1}$.

Aghanjai *et al.* [1] introduced the notions of G_b-convergence and G_b-Cauchy sequences as outlined below:

Definition 1.3. [1] On $(\widetilde{H}, \hat{\mathcal{G}})$ a sequence $(\aleph_n)$ in $\widetilde{H}$ is called

1) G_b-convergent to $\aleph$ in $\widetilde{H}$ if $\forall\ \hat{\kappa} > 0$, $\exists\ k^* \in \mathbb{N}$ such that

$$\hat{\mathcal{G}}(\aleph, \aleph_n, \aleph_m) < \hat{\kappa},\ \forall\ m,\ n \geq k^*.$$

2) G_b-Cauchy sequence if $\forall\ \hat{\kappa} > 0$, $\exists\ k^* \in \mathbb{N}$ such that

$$\hat{\mathcal{G}}(\aleph_n, \aleph_m, \aleph_l) < \hat{\kappa}.\ \forall l,\ m,\ n \geq k^*$$

Proposition 1.4. [1] *On $(\widetilde{H}, \hat{\mathcal{G}})$ the following are equivalent:*

1) $(\aleph_n)$ *is G_b-convergent to* $\aleph$.
2) $\hat{\mathcal{G}}(\aleph_n,\ \aleph_n,\ \aleph) \to 0$ *as n tends towards infinity.*
3) $\hat{\mathcal{G}}(\aleph_n,\ \aleph,\ \aleph) \to 0$ *as n tends towards infinity.*

Proposition 1.5. [1] *On $(\widetilde{H}, \hat{\mathcal{G}})$ the following are equivalent*

1) *the sequence $(\aleph_n)$ is G_b-Cauchy.*
2) *for each $\hat{\kappa} > 0$, $\exists\ k^* \in \mathbb{N}$ such that $\forall\ n,\ m \geq k^*$, $\hat{\mathcal{G}}(\aleph_n,\ \aleph_m,\ \aleph_m) < \hat{\kappa}$.*

Definition 1.6. [1] A G_b-metric space (X, G_b) is called *G_b-complete* (or simply a *complete G_b-metric space*) if every G_b-Cauchy sequence in $\widetilde{H}$ is G_b-convergent to some point $\widetilde{H}$.

Definition 1.7. [3] On $(\widetilde{H}, \hat{\mathcal{G}})$ a mapping $\Omega : \widetilde{H} \times \widetilde{H} \times \widetilde{H} \to [0, \infty)$ is said to be generalized Omega-distance map or simply Ω_b-distance map on $\widetilde{H}$ if it satisfies the conditions below:

1. For all $\aleph_1, \aleph_2, \aleph_3, \mu \in \widetilde{H}$
$$\Omega(\aleph_1, \aleph_2, \aleph_3) \leq \vartheta\big[\Omega(\aleph_1, \mu, \mu) + \Omega(\mu, \aleph_2, \aleph_3)\big].$$
2. For any $\aleph_1, \aleph_2 \in \widetilde{H}$, the mappings
$$\Omega(\aleph_1, \aleph_2, \cdot),\ \Omega(\aleph_1, \cdot, \aleph_2) : \widetilde{H} \to [0, \infty)$$
are lower semicontinuous.
3. $\forall\ \hat{\kappa} > 0$, there exists $\hat{\delta} > 0$ such that if
$$\Omega(\aleph_1, \mu, \mu) \leq \hat{\delta} \quad \text{and} \quad \Omega(\mu, \aleph_2, \aleph_3) \leq \hat{\delta},$$
then
$$G_b(\aleph_1, \aleph_2, \aleph_3) \leq \hat{\kappa}.$$

Example 1.8. [3] Let $\widetilde{H} = \mathbb{R}$, and consider the G_b-metric $\hat{\mathcal{G}}$ defined by
$$\hat{\mathcal{G}}(\Upsilon, \Lambda, \aleph^*) = \big(|\Upsilon - \Lambda| + |\Lambda - \aleph^*| + |\Upsilon - \aleph^*|\big)^2 \quad \text{for all } \Upsilon, \Lambda, \aleph^* \in \mathbb{R}.$$
Now, define
$$\Omega(\Upsilon, \Lambda, \aleph^*) = \big(|\Upsilon - \Lambda| + |\Upsilon - \aleph^*|\big)^2 \quad \text{for all } \Upsilon, \Lambda, \aleph^* \in \mathbb{R}.$$
Then Ω is an Ω_b-distance mapping on $\mathbb{R}$ with constant $\vartheta = 2$.

Example 1.9. Consider the space $\widetilde{H} = C([0,1], \mathbb{R})$ of continuous real-valued functions on $[0,1]$ with the G_b-metric $\hat{\mathcal{G}}$ and generalized Ω-distance Ω defined by
$$\hat{\mathcal{G}}(\varphi_1, \varphi_2, \varphi_3) = (\|\varphi_1 - \varphi_2\|_\infty + \|\varphi_2 - \varphi_3\|_\infty + \|\varphi_1 - \varphi_3\|_\infty)^2,$$
$$\Omega(\varphi_1, \varphi_2, \varphi_3) = (\|\varphi_1 - \varphi_2\|_\infty + \|\varphi_1 - \varphi_3\|_\infty)^2,$$

where

$$\|\varphi_1 - \varphi_2\|_\infty = \sup_{t\in[0,1]} |\varphi_1(t) - \varphi_2(t)|.$$

Then, $(\hat{\mathcal{G}}, \widetilde{H})$ is a G_b-metric space with constant $\vartheta = 2$, and Ω is a b-Ω distance on $\widetilde{H}$.

Lemma 1.10. [3] *On $(\widetilde{H}, \hat{\mathcal{G}})$ Suppose Ω is an Ω_b-distance on $\widetilde{H}$. Let $(\aleph_n)$, (τ_n) be sequences in $\widetilde{H}$, (γ_n), (μ_n) be sequences in $[0,\ \infty)$ with $\gamma_n, \mu_n \to 0$, and let $\aleph, \Upsilon, \Lambda, \mu \in \widetilde{H}$. Then*

1) *If $\Omega(\tau_n,\ \aleph_n,\ \aleph_n) \leq \gamma_n$ and $\Omega(\aleph_n,\ \tau_m,\ \aleph) \leq \mu_n$ for any $m > n \in \mathbb{N}$, then $\hat{\mathcal{G}}(\tau_n,\ \tau_m,\ \aleph)$ converges to 0 and hence τ_n converges to $\aleph$;*
2) *If $\Omega(\Lambda,\ \aleph_n,\ \aleph_n) \leq \gamma_n$ and $\Omega(\aleph_n,\ \Lambda,\ \aleph) \leq \mu_n$ for $n \in \mathbb{N}$, then $\hat{\mathcal{G}}(\Lambda,\ \Lambda,\ \aleph) < \hat{\kappa}$ and hence $\Lambda = \aleph$;*
3) *If $\Omega(\aleph_n,\ \aleph_m,\ \aleph_l) \leq \gamma_n$ $\forall m,\ n,\ l \in \mathbb{N}$ with $l \geq m \geq n$, then $(\aleph_n)$ is a G_b-Cauchy sequence;*
4) *If $\Omega(\aleph_n,\ \mu,\ \mu) \leq \gamma_n$ $\forall n \in \mathbb{N}$, then $(\aleph_n)$ is a G_b-Cauchy sequence.*

The concept of generalized Ω-distance was first systematically investigated by [2], who established fundamental fixed-point theorems for nonlinear contractive mappings within this framework.

This work establishes fixed point results in complete G_b-metric spaces equipped with generalized Ω-distance mappings. Beginning with the axiomatic definition of generalized Ω-distance (extending the G_b-metric structure with parameter $\vartheta \geq 1$), we introduce integral-type contraction conditions. Through iterative sequence construction and leveraging the completeness of the space, we prove convergence to unique fixed points. Applications to nonlinear integral equations demonstrate the framework's utility in functional analysis.

2. Main Result

The following two classes of functions are essential for our main result:

$\Phi = \{\hat{\Xi} :\ [0,\infty) \to\ [0,\infty) :\ \hat{\Xi}$ *is Lebesgue integrable with* $0 < \int_0^c \hat{\Xi}(\nu)d\nu\ < \infty,\ \forall c > 0\}$

and

$\tilde{\Pi} = \{\Pi :\ [0,\infty) \to\ [0,\infty) : \Pi$ *is continuous with* $\lim_{n\to\infty} \Pi^n(\nu) = 0, \forall \nu > 0\}$.

Remark 2.1. If $\Pi \in \Pi$, then
$\Pi(\nu) < \nu, for\ \nu > 0,$
$\Pi(0) = 0.$

An important result is the next lemma, that will be helpful in the subsequent sections of our paper.

Lemma 2.2. [17] *If $\hat{\Xi} \in \Phi$ and the sequence (t_n) is made up of positive real numbers under which $\lim\limits_{n\to\infty} t_n = t$, then*

$$\lim_{n\to\infty} \int_0^{t_n} \hat{\Xi}(\nu)\, d\nu = \int_0^t \hat{\Xi}(\nu)\, d\nu.$$

Lemma 2.3. *If $f(t) \geq 0$ for all $t \geq 0$, and $g(c) = \int_0^c f(t)\, dt$, then g is increasing.*

Proof. Let $0 \leq c_1 \leq c_2$. Then,

$$g(c_2) - g(c_1) = \int_{c_1}^{c_2} f(t)\, dt.$$

Since $f(t) \geq 0$ for all $t \in [c_1, c_2]$, the integral is nonnegative. Thus, $g(c_2) \geq g(c_1)$, proving that g is increasing. □

We will now present the concept of an Ω_b-integral contraction.

Definition 2.4. On $(\widetilde{H}, \hat{\mathcal{G}})$ suppose Ω is an Ω_b-distance on $\widetilde{H}$. A mapping $T : \widetilde{H} \to \widetilde{H}$ is called an Ω_b-integral contraction if there is $\hat{\Xi} \in \Phi,\ \Pi \in \tilde{\Pi}$ such that

$$\int_0^{\vartheta\Omega(T\aleph,\ T\Upsilon,\ T\Lambda)} \hat{\Xi}(\nu) d\nu \ \leq \Pi \int_0^{\Omega(\aleph, \Upsilon, \Lambda)} \hat{\Xi}(\nu) d\nu\ ,$$

for all $\aleph, \Upsilon, \Lambda\ \in \widetilde{H}$.

Proposition 2.5. *If T is an Ω_b-integral contraction, then*

$$\Omega(T\aleph,\ T\Upsilon,\ T\Lambda) < \frac{1}{\vartheta} \Omega(\aleph, \Upsilon, \Lambda). \tag{2.1}$$

Proof. From the contraction condition with the property of the function Π , we conclude that

$$\int_0^{\vartheta\Omega(T\aleph,\ T\Upsilon,\ T\Lambda)} \hat{\Xi}(\nu) d\nu \ < \int_0^{\Omega(\aleph, \Upsilon, \Lambda)} \hat{\Xi}(\nu) d\nu.$$

Since $\hat{\Xi} \geq 0$, then Lemma 2.3 implies

$$\Omega(T\aleph,\ T\Upsilon,\ T\Lambda) < \frac{1}{\vartheta} \Omega(\aleph, \Upsilon, \Lambda).$$

□

Theorem 2.6. *Suppose $(\widetilde{H},\ \hat{\mathcal{G}})$ is complete, and suppose Ω is an Ω_b-distance on $\widetilde{H}$. Let T be a self-map on $\widetilde{H}$ that meets the following conditions:*
(1) The function T is an Ω_b-integral contraction.
(2) Assume that one of the following is true:

- *T is continuous, or*
- *For any $u \in \widetilde{H}$ assume that if $Tu \neq u$, then* $\inf\{\Omega(\aleph,\ T\aleph,\ u) :\ \aleph \in \widetilde{H}\} > 0$.

Then T has a unique fixed point in $\widetilde{H}$.

Proof. Let $\aleph_0 \in \widetilde{H}$. We construct a sequence $(\aleph_n)$ in $\widetilde{H}$ such that $\aleph_n = T\aleph_{n-1}$, $n \in \mathbb{N}$.

If there is some $r \in \mathbb{N}$ sch that

$$\Omega(\aleph_r,\ \aleph_{r+1},\ \aleph_{r+1}) = 0, \tag{2.2}$$

then by the contraction condition we have

$$\int_0^{\vartheta\Omega(\aleph_{r+1},\ \aleph_{r+2},\ \aleph_{r+2})} \hat{\Xi}(\nu) d\nu \ \leq \Pi \int_0^{\Omega(\aleph_r,\ \aleph_{r+1},\ \aleph_{r+1})} \hat{\Xi}(\nu) d\nu \ = 0$$

So,

$$\Omega(\aleph_{r+1},\ \aleph_{r+2},\ \aleph_{r+2}) = 0. \tag{2.3}$$

Using Equations (2.2) and (2.3) to apply the definition of Ω, we get $\hat{\mathcal{G}}(\aleph_r,\ \aleph_{r+2},\ \aleph_{r+2}) = 0$ and hence $\aleph_r = \aleph_{r+2}$ or we can write $T^2\aleph_r = \aleph_r$.

Now,

$$\Omega(\aleph_{r+1},\ \aleph_{r+1},\ \aleph_{r+1}) \leq \vartheta\left[\ \Omega(\aleph_{r+1},\ \aleph_r,\ \aleph_r) +\ \Omega(\aleph_r,\ \aleph_{r+1},\ \aleph_{r+1})\right],$$

which implies that

$$\Omega(\aleph_{r+1},\ \aleph_{r+1},\ \aleph_{r+1}) \leq \vartheta\Omega(\aleph_{r+1},\ \aleph_r,\ \aleph_r). \tag{2.4}$$

So Equation (2.3) becomes $\Omega(\aleph_{r+1},\ \aleph_r,\ \aleph_r)$ =0 and Equation (2.4) becomes

$$\Omega(\aleph_{r+1},\ \aleph_{r+1},\ \aleph_{r+1}) \leq \vartheta 0 = 0,$$

and hence

$$\Omega(\aleph_{r+1},\ \aleph_{r+1},\ \aleph_{r+1}) = 0. \tag{2.5}$$

Now, use Equations (2.2) and (2.5) to apply the definition of Ω, we get $\hat{\mathcal{G}}(\aleph_{r+1},\ \aleph_{r+2},\ \aleph_{r+2}) = 0$ and hence $\aleph_{r+1} = \aleph_{r+2}$ or we can say that $T\aleph_r = \aleph_r$.

Now, suppose that $\Omega(\aleph_r,\ \aleph_{r+1},\ \aleph_{r+1}) \neq 0$ for each $r \in \mathbb{N}$. Thus,

$$\begin{aligned}
\int_0^{\vartheta\Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_{n+1})} \hat{\Xi}(\nu)d\nu \ &= \int_0^{\vartheta\Omega(T\aleph_{n-1},\ T\aleph_n,\ T\aleph_n)} \hat{\Xi}(\nu)d\nu \\
&\leq \Pi \int_0^{\Omega(\aleph_{n-1},\ \aleph_n,\ \aleph_n)} \hat{\Xi}(\nu)d\nu \\
&\vdots \\
&\leq \Pi^n \int_0^{\Omega(\aleph_0,\ \aleph_1,\ \aleph_1)} \hat{\Xi}(\nu)d\nu\ .
\end{aligned}$$

So, by taking the limit as $n \to \infty$ for both sides and using the property of the function Π we get:

$$\lim_{n\to\infty} \int_0^{\vartheta\Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_{n+1})} \hat{\Xi}(\nu)d\nu = 0,$$

and this yields to

$$\int_0^{\lim_{n\to\infty}\vartheta\Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_{n+1})} \hat{\Xi}(\nu)d\nu = 0.$$

So we have $\lim_{n\to\infty}\vartheta\Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_{n+1}) = 0$.

Hence,

$$\lim_{n\to\infty} \Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_{n+1}) = 0. \tag{2.6}$$

Similarly, using the same argument we can show that

$$\lim_{n\to\infty} \Omega(\aleph_n,\ \aleph_n,\ \aleph_{n+1}) = 0. \tag{2.7}$$

And

$$\lim_{n\to\infty} \Omega(\aleph_{n+1},\ \aleph_n,\ \aleph_n) = 0. \tag{2.8}$$

Now, we claim that $(\aleph_n)$ is G_b-Cauchy sequence.

Step1. We claim that $\lim_{n,m\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_m) = 0$.

Assume by contrary that $\lim_{n,m\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_m) \neq 0$. So, there exists an $\hat{\kappa} > 0$ and two subsequences $(\aleph_{n_k})$, $(\aleph_{m_k})$ such that $\Omega(\aleph_{n_k},\ \aleph_{m_k},\ \aleph_{m_k}) \geq \hat{\kappa}$, where $m_k > n_k > k$ with

$$\Omega(\aleph_{n_k},\ \aleph_{m_k-1},\ \aleph_{m_k-1}) < \hat{\kappa}. \tag{2.9}$$

So, by Equation (2.1), Proposition (2.5), and the triangle inequality for Ω

$$\begin{aligned}\hat{\kappa} \leq \Omega(\aleph_{n_k},\ \aleph_{m_k},\ \aleph_{m_k}) &< \frac{1}{\vartheta}\Omega(\aleph_{n_k-1},\ \aleph_{m_k-1},\ \aleph_{m_k-1}) \\ &< \frac{1}{\vartheta}\vartheta\left[\Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k}) + \Omega(\aleph_{m_k},\ \aleph_{m_k-1},\ \aleph_{m_k-1})\right]\end{aligned}$$

Taking the limit infimum as $k \to \infty$, and using Equation (2.8), we get

$$\hat{\kappa} \leq \lim_{k\to\infty} \inf \Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k}). \tag{2.10}$$

Again, by Equations (2.1), (2.9), and the triangle inequality for Ω

$$\begin{aligned}\Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k}) &< \frac{1}{\vartheta}\Omega(\aleph_{n_k-2},\ \aleph_{m_k-1},\ \aleph_{m_k-1}) \\ &< \frac{1}{\vartheta}\vartheta\left[\Omega(\aleph_{n_k-2},\ \aleph_{n_k},\ \aleph_{n_k}) + \Omega(\aleph_{n_k},\ \aleph_{m_k-1},\ \aleph_{m_k-1})\right] \\ &< \left[\vartheta\left[\Omega(\aleph_{n_k-2},\ \aleph_{n_k-1},\ \aleph_{n_k-1}) + \Omega(\aleph_{n_k-1},\ \aleph_{n_k},\ \aleph_{n_k})\right] + \hat{\kappa}\right].\end{aligned}$$

Taking the limit supremum as $k \to \infty$, and using Equation (2.6), we get

$$\lim_{k\to\infty} \sup\ \Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k}) \leq\ \hat{\kappa}. \tag{2.11}$$

Hence, from Equations (2.10) and (2.11) we get

$$\lim_{k\to\infty} \Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k}) =\ \hat{\kappa}. \tag{2.12}$$

Now,

$$\Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1}) < \frac{1}{\vartheta}\Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k})$$

By taking the limit supremum for both sides as $k \to \infty$ and using Equation (2.12), we get

$$\lim_{k\to\infty} \sup \Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1}) \leq \frac{\hat{\kappa}}{\vartheta}. \tag{2.13}$$

On the other hand,

$$\hat{\kappa} \leq \Omega(\aleph_{n_k},\ \aleph_{m_k},\ \aleph_{m_k}) \leq \vartheta\left[\Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1}) + \Omega(\aleph_{m_k+1},\ \aleph_{m_k},\ \aleph_{m_k})\right]$$

By taking the limit infimum for both sides as $k \to \infty$ and using Equation (2.8), we get

$$\frac{\hat{\kappa}}{\vartheta} \leq \lim_{k\to\infty} \inf \Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1}). \tag{2.14}$$

From Equations (2.13) and (2.14), we get

$$\lim_{k\to\infty} \Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1}) = \frac{\hat{\kappa}}{\vartheta}$$

Now by the contraction condition we have,

$$\begin{aligned}\int_0^{\vartheta\Omega(\aleph_{n_k},\ \aleph_{m_k+1},\ \aleph_{m_k+1})} \hat{\Xi}(\nu)d\nu\ &\leq \Pi \int_0^{\Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k})} \hat{\Xi}(\nu)d\nu \\ &< \int_0^{\Omega(\aleph_{n_k-1},\ \aleph_{m_k},\ \aleph_{m_k})} \hat{\Xi}(\nu)d\nu\end{aligned}$$

Take the limit as $k \to \infty$ for both sides we get, $\int_0^{\hat{\kappa}} \hat{\Xi}(\nu)d\nu\ < \int_0^{\hat{\kappa}} \hat{\Xi}(\nu)d\nu$ which is a contradiction. Hence,

$$\lim_{n,m\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_m) = 0. \tag{2.15}$$

Step2. By the same argument in step 1 we can prove that

$$\lim_{m,l\to\infty} \Omega(\aleph_m,\ \aleph_m,\ \aleph_l) = 0. \tag{2.16}$$

Now, we have

$$0 \leq \Omega(\aleph_n,\ \aleph_m,\ \aleph_l) \leq \vartheta\left[\Omega(\aleph_n,\ \aleph_m,\ \aleph_m) + \Omega(\aleph_m,\ \aleph_m,\ \aleph_l)\right]$$

By taking the limit for both sides as $n, m, l \to \infty$ with using Equations (2.15) and (2.16), we get

$$\lim_{n,m,l\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_l) = 0.$$

Hence, $(\aleph_n)$ is G_b-Cauchy sequence.

Now, since G_b is complete, we get $(\aleph_n)$ is G_b-convergent in $\widetilde{H}$, assume it converges to u.

Consider $\hat{\kappa} > 0$. Then there exists $N \in \mathbb{N}$ such that $\Omega(\aleph_n,\ \aleph_m,\ \aleph_l) \leq \hat{\kappa}$, for all $n, m, l \geq N$ and thus $\lim_{l\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_l) \leq \hat{\kappa}$. Note that $\Omega(\aleph_n,\ \aleph_m,\ u) \leq \lim_{p\to\infty} \Omega(\aleph_n,\ \aleph_m,\ \aleph_p) \leq \hat{\kappa}$, for all $n, m \geq N$. Let $m = n+1$. Then $\Omega(\aleph_n,\ \aleph_{n+1},\ u) \leq$

$\lim_{p\to\infty}\Omega(\aleph_n,\ \aleph_{n+1},\ \aleph_p)\leq\hat{\kappa}$ for all $n\geq N$. If $Tu\neq u$, then by the second condition of the theorem we obtain

$$0<\inf\{\Omega_b(\aleph,\ T\aleph,\ u):\ \aleph\in\widetilde{H}\}\leq\inf\{\Omega(\aleph_n,\ \aleph_{n+1},\ u):\ n\geq N\}\leq\hat{\kappa},$$

for each $\hat{\kappa}>0$ which is a contradiction. Therefore $Tu=u$.

Now, assume that there is some element $\widetilde{u}\in\widetilde{H}$ such that $T\widetilde{u}=\widetilde{u}$. Our claim is that $\Omega(u,\ \widetilde{u},\ \widetilde{u})=0$, if not, then

$$\begin{aligned}\int_0^{\vartheta\Omega(u,\ \widetilde{u},\ \widetilde{u})}\hat{\Xi}(\nu)d\nu &= \int_0^{\vartheta\Omega(Tu,\ T\widetilde{u},\ T\widetilde{u})}\hat{\Xi}(\nu)d\nu\\ &\leq \Pi\int_0^{\Omega(u,\ \widetilde{u},\ \widetilde{u})}\hat{\Xi}(\nu)d\nu\\ &< \int_0^{\Omega(u,\ \widetilde{u},\ \widetilde{u})}\hat{\Xi}(\nu)d\nu,\end{aligned}$$

which is a contradiction, so $\Omega(u,\ \widetilde{u},\ \widetilde{u})=0$. By the same argument we can show that $\Omega(u,\ u,\ u)=0$.

Now by applying the definition of Ω we get $\hat{\mathcal{G}}(u,\ \widetilde{u},\ \widetilde{u})=0$ and so $u=\widetilde{u}$. Hence the result. $\square$

Example 2.7. (see Example 1.9) Consider the space $\widetilde{H}=\{U\in C([0,1],\mathbb{R}):\|u\|\leq 1\}$. Also, define the operator $T:\widetilde{H}\to\widetilde{H}$ by:

$$(T\psi)(t)=\frac{1}{4}\int_0^t\psi^2d\nu$$

From Example 1.9, we have

$$\Omega(T\psi,T\varpi,T\rho)=\left(\|T\psi-T\varpi\|_\infty+\|T\psi-T\rho\|_\infty\right)^2.$$

Now,

$$\begin{aligned}|(T\psi)(t)-(T\varpi)(t)| &= \left|\frac{1}{4}\int_0^t\left(\psi^2-\varpi^2\right)\,d\nu\right|\\ &\leq\frac{1}{4}\int_0^t|(\psi-\varpi)(\psi+\varpi)|\,d\nu.\end{aligned}$$

So,

$$|(T\psi)(t)-(T\varpi)(t)|\leq\frac{1}{4}\|\psi-\varpi\|_\infty\cdot 2\cdot t\leq\frac{1}{2}\|\psi-\varpi\|_\infty.$$

Hence,

$$\|T\psi-T\varpi\|_\infty\leq\frac{1}{2}\|\psi-\varpi\|_\infty,$$

Similarly,

$$\|T\psi - T\rho\|_\infty \leq \frac{1}{2}\|\psi - \rho\|_\infty.$$

Now,

$$\begin{aligned}\Omega(T\psi, T\varpi, T\rho) &\leq \left(\frac{1}{2}\|\psi - \varpi\|_\infty + \frac{1}{2}\|\psi - \rho\|_\infty\right)^2 \\ &= \frac{1}{4}\left(\|\psi - \varpi\|_\infty + \|\psi - \rho\|_\infty\right)^2 \\ &= \frac{1}{4}\Omega(\psi, \varpi, \rho).\end{aligned}$$

Therefore,

$$2\Omega(T\psi, T\varpi, T\rho) \leq \frac{1}{2}\Omega(\psi, \varpi, \rho).$$

For $\hat{\Xi}(t) = 1$ and $\Pi(t) = \frac{t}{2}$, we verify the contraction condition:

$$\begin{aligned}\int_0^{2\Omega(T\psi,T\varpi,T\rho)} \hat{\Xi}(t)dt &= 2\Omega(T\psi, T\varpi, T\rho) \\ &\leq \frac{1}{2}\Omega(\psi, \varpi, \rho) \\ &= \Pi\left(\int_0^{\Omega(\psi,\varpi,\rho)} \hat{\Xi}(\nu)d\nu\right)\end{aligned}$$

Thus, according to Theorem 2.6 T has a unique fixed point $\psi^* \in \widetilde{H}$ satisfying $\psi^*(t) = \frac{1}{4}\int_0^t \psi^{*2} d\nu$.

Each of the following corollaries is established under the assumption that condition (2) of Theorem 2.6 holds.

By defining $\Pi : [0, \infty) \to [0, \infty)$ by $\Pi(\nu) = \beta\nu$, where $\beta \in [0, 1)$, we derive the following result

Corollary 2.8. *Suppose $(\widetilde{H},\ \hat{\mathcal{G}})$ is complete, and suppose Ω is an Ω_b-distance on $\widetilde{H}$. Suppose $T : \widetilde{H} \to \widetilde{H}$ satisfies*

$$\int_0^{\vartheta\Omega(T\aleph,T\Upsilon,T\Lambda)} \hat{\Xi}(\nu)\, d\nu \leq \beta \int_0^{\Omega(\aleph,\Upsilon,\Lambda)} \hat{\Xi}(\nu)\, d\nu,$$

for all $\aleph, \Upsilon, \Lambda \in \widetilde{H}$, and $\hat{\Xi} \in \Phi$.

Then T admits a unique fixed point in $\widetilde{H}$.

By defining $\Pi : [0, \infty) \to [0, \infty)$ by $\Pi(\nu) = (1 + \nu)^q - 1$, where $q \in (0, 1)$, we derive the following result

Corollary 2.9. *Suppose* $(\widetilde{H},\ \hat{\mathcal{G}})$ *is complete, and suppose* Ω *is an* Ω_b*-distance on* $\widetilde{H}$*. Suppose* $T : \widetilde{H} \to \widetilde{H}$ *satisfies*

$$\int_0^{\vartheta\Omega(T\aleph,T\Upsilon,T\Lambda)} \hat{\Xi}(\nu)\,d\nu \le \left(1 + \int_0^{\Omega(\aleph,\Upsilon,\Lambda)} \hat{\Xi}(\nu)\,d\nu\right)^q - 1,$$

for all $\aleph, \Upsilon, \Lambda \in \widetilde{H}$*, and* $\hat{\Xi} \in \Phi$*.*

Then T *admits a unique fixed point in* $\widetilde{H}$*.*

In Corollary 2.8, if we define $\hat{\Xi} : [0,\infty) \to [0,\infty)$ by $\hat{\Xi}(\nu) = e^{-\nu}$, we derive the following result

Corollary 2.10. *Suppose* $(\widetilde{H},\ \hat{\mathcal{G}})$ *is complete, and suppose* Ω *is an* Ω_b*-distance on* $\widetilde{H}$*. Suppose* $T : \widetilde{H} \to \widetilde{H}$ *satisfies*

$$\int_0^{\vartheta\Omega(T\aleph,T\Upsilon,T\Lambda)} e^{-\nu} d\nu \le \beta \int_0^{\Omega(\aleph,\Upsilon,\Lambda)} e^{-\nu} d\nu,$$

for all $\aleph, \Upsilon, \Lambda \in \widetilde{H}$*, Then* T *has a unique fixed point.*

In Corollary 2.8, if we define $\hat{\Xi} : [0,\infty) \to [0,\infty)$ by $\hat{\Xi}(\nu) = 1$, we derive the following result

Corollary 2.11. *Suppose* $(\widetilde{H},\ \hat{\mathcal{G}})$ *is complete, and suppose* Ω *is an* Ω_b*-distance on* $\widetilde{H}$*. If* $T : \widetilde{H} \to \widetilde{H}$ *satisfies:*

$$\Omega(T\aleph, T\Upsilon, T\Lambda) \le \frac{\beta}{\vartheta}\Omega(\aleph, \Upsilon, \Lambda)$$

then T *admits a unique fixed point.*

3. Applications to Nonlinear Integral Equations

In this section, we apply Theorem 2.6 to prove the existence (and uniqueness) of a solution to a class of nonlinear Volterra integral equations in the space $\widetilde{H}$. Consider the space $\widetilde{H} = C([0,1],\mathbb{R})$ as described in Example 1.9.

Consider the Volterra integral equation:

$$u(t) = f(t) + \lambda \int_0^t K(t,\nu,u(\nu))\,d\nu, \quad t \in [0,1],\ f \in \widetilde{H}. \tag{3.1}$$

Define $T : \widetilde{H} \to \widetilde{H}$ by

$$(Tu)(t) = f(t) + \lambda \int_0^t K(t,\nu,u(\nu))\,d\nu$$

Assume that

(1) $K : [0,1] \times [0,1] \times \widetilde{H} \to \mathbb{R}$ is jointly continuous and bounded,
(2) $|K(t,\nu,u(\nu)) - K(t,\nu,v(\nu))| \le L|u(\nu) - v(\nu)|$ for some $L > 0$, with $\lambda \in (0,1)$, and $2\lambda L^2 \le 1$.

For $u, v, w \in \widetilde{H}$, we have

$$\begin{aligned}|Tu(t) - Tv(t)| &\leq \lambda \int_0^t |K(t, \nu, u(\nu)) - K(t, \nu, v(\nu))| \, d\nu \\ &\leq \lambda L \int_0^t |u(\nu) - v(\nu)| \, d\nu \\ &\leq \lambda L \int_0^t \|u - v\| \, d\nu \\ &\leq \lambda L t \|u - v\|.\end{aligned}$$

Taking the supremum over $[0, 1]$, we get

$$\|Tu - Tv\| \leq \lambda L \|u - v\|.$$

Similarly,

$$\|Tu - Tw\| \leq \lambda L \|u - w\|.$$

Hence,

$$(\|Tu - Tv\| + \|Tu - Tw\|)^2 \leq (\lambda L \|u - v\| + \lambda L \|u - w\|)^2.$$

Therefore,

$$(\|Tu - Tv\| + \|Tu - Tw\|)^2 \leq \lambda^2 L^2 (\|u - v\| + \|u - w\|)^2.$$

Hence,

$$2\Omega(Tu, Tv, Tw) \leq 2\lambda^2 L^2 \Omega(u, v, w) \leq \lambda \Omega(u, v, w).$$

Therefore,

$$\int_0^{2\Omega(Tu,Tv,Tw)} 1 d\nu \leq \lambda \int_0^{\Omega(u,v,w)} 1 d\nu$$

Take $\hat{\Xi}(a) = 1$, $\Pi(a) = \lambda a$. Then $\hat{\Xi} \in \Phi$, $\Pi \in \tilde{\Pi}$. Also, we have

$$\int_0^{2\Omega(Tu,Tv,Tw)} \hat{\Xi}(\nu) \, d\nu \leq \Pi \left(\int_0^{\Omega(u,v,w)} \hat{\Xi}(\nu) \, d\nu \right)$$

Therefore, by Theorem 2.6, the operator T admits a unique fixed point in $\widetilde{H}$, which solves the Volterra integral equation (3.1).

Conclusion

This paper established new fixed point theorems for integral-type contractions in complete G_b-metric spaces equipped with generalized Ω-distance. By leveraging Lebesgue-integrable auxiliary functions and relaxed contraction conditions, we generalized existing results in the literature. The applicability of our framework was demonstrated through concrete examples and an analysis of nonlinear integral equations. Future work could explore extensions to multivalued mappings or applications in fractional differential equations.

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