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VISCOELASTIC FOURTH-ORDER WAVE EQUATION WITH LOGARITHMIC AND VARIABLE-EXPONENT NONLINEARITIES: BLOW-UP ANALYSIS

MOHAMMED YASSINE TRIGUI[1*] and MOHAMED SAADAOUI[2]

Abstract. In this paper, we study a fourth-order viscoelastic wave equation involving variable-exponent nonlinearities and a logarithmic source term, where the memory kernel is a positive, nonincreasing function, and the variable exponents are measurable functions satisfying certain conditions. By employing the energy method alongside several integral inequality techniques, we establish a finite-time blow-up result for all solutions corresponding to negative initial energy, as well as for a specific class of solutions with positive initial energy.

Keywords. Fourth-order wave equation, Variable exponent, Logarithmic source, Blow up, Viscoelastic
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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $a, b > 0$ are constants. We consider the following initial–boundary value problem posed in $\Omega \times (0, T)$:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(x,s)\mathrm{d}s + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2} \ln|u|, \\ u(x,t) = \frac{\partial u}{\partial \nu} = 0, \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \text{ in } \Omega, \end{cases} \tag{1}$$

where ν is the unit outer normal to $\partial\Omega$, the kernel of memory term $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function such that

$$\int_0^\infty g(s)ds = 1 - l < \frac{(1-\eta)p_1\left((1-\eta)p_1 - 2\right)}{\left((1-\eta)p_1 - 1\right)^2}, \, 0 < \eta < \frac{p_1 - 2}{p_1},$$

and the exponents $m(\cdot)$ and $p(\cdot)$ are given measurable functions on Ω satisfying

$$2 \le q_1 \le q(x) \le q_2 \le \frac{2n}{n-4}, \quad n \ge 5, \tag{2}$$

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* Corresponding author

with

$$q_1 := \operatorname{ess\,inf}_{x\in\Omega} q(x), \quad q_2 := \operatorname{ess\,sup}_{x\in\Omega} q(x),$$

and the log-Hölder continuity condition:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta < 1,\ A > 0. \quad (3)$$

In the case of constant nonlinearities, considerable attention has been devoted to the study of problems associated with (1). Liu [14] investigated the following plate equation with nonlinear damping and a logarithmic source term:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2} u_t = |u|^{p-2} u \log|u|^k, (x,t) \in \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

Using the contraction mapping principle, the author established the local existence of solutions. For cases with subcritical initial energy, the author derived global existence results and provided decay estimates. Additionally, it was shown that solutions with negative initial energy blow up in finite time under suitable conditions. Also, the finite-time blow-up of solutions with arbitrarily high initial energy was demonstrated in the presence of linear damping term (i.e. $m = 2$). In [11], Kafini and Messaoudi studied the following wave equation with delay and logarithmic terms:

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = u|u|^{p-2} \ln|u|^k,$$

the authors utilized semigroup theory to prove the local existence of solutions. Furthermore, they demonstrated that for negative initial energy, the solutions blow up in finite time. This type of problems with logarithmic source term arises in many branches of physics such as nuclear physics, optics, geophysics and quantum field theory. More details on these problems can be found in [4, 9, 8, 5].

In recent years, substantial attention has been directed toward the analysis of nonlinear mathematical models featuring variable exponents of nonlinearity. Such models arise in various physical contexts, including the flow of electro-rheological fluids, fluids with temperature-dependent viscosity, nonlinear viscoelastic behavior, filtration in porous media, and image processing (see [1, 3, 2, 13, 6, 12]). Notable examples of such problems include, for instance, Messaoudi et al. [15] considered the following nonlinear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + a u_t |u_t|^{m(\cdot)-2} = b u |u|^{p(\cdot)-2}, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

where $a, b \geq 0$ are constants and Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Using the Faedo–Galerkin method, the authors established the existence of a unique weak solution under suitable assumptions on the variable exponents m and p. Moreover, the finite-time blow-up of solutions was proved, and a two-dimensional numerical example was provided to illustrate this blow-up behavior. Under Dirichlet boundary conditions, Rahmoune [18] considered a new

type of viscoelastic Petrovsky wave equation involving variable-exponent source terms and a memory effect, given by

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(x,s)\mathrm{d}s + |u_t|^{m(.)-2} u_t = |u|^{p(.)-2}u, \tag{4}$$

where the functions m, p, and the relaxation kernel g satisfy appropriate conditions. The author investigated the finite-time blow-up of solutions with arbitrary positive initial energy, assuming sufficiently large initial data. Moreover, by employing an alternative approach adapted to large bounded positive energy, both the finite-time blow-up and estimates for the upper and lower bounds of the blow-up time are established. In a related study, Hamadouche [10] analyzed the same problem in the case where $g = 0$. Using the Faedo–Galerkin method, the existence of a unique weak solution is proved under suitable assumptions on the variable exponents m and p. In addition, a finite-time blow-up result is obtained for solutions with arbitrary negative initial energy.

The aim of this work is to investigate the blow-up phenomenon and to identify sufficient conditions on the functions m, p, g and the initial data that lead to blow-up. This paper is structured into three main sections, in addition to the introduction. In Section 2, we recall the definitions of the variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the Sobolev space $W^{k,p(\cdot)}(\Omega)$, together with some of their essential properties, and present a well-posedness result. Section 3 is devoted to the statement and proof of a blow-up theorem for all solutions with negative initial energy. In Section 4, we establish a blow-up result for certain solutions corresponding to positive initial energy.

2. Preliminaries

Let $p : \Omega \subset \mathbb{R}^n \to [1,\infty]$ be a measurable function. We denote by $L^{p(\cdot)}(\Omega)$ all the real measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |\lambda u(x)|^{p(x)}\mathrm{d}x < \infty$, for some $\lambda > 0$. The variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg-type norm

$$\|u\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1\right\},$$

is a Banach space.

The Banach space $W^{k,p(\cdot)}(\Omega)$ $(k \in \mathbb{N})$, known as the variable-exponent Sobolev space, is defined as follows

$$W^{k,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \text{ exists and } |D^\alpha u| \in L^{p(\cdot)}(\Omega) \text{ with } |\alpha| \le k\right\},$$

with respect to the norm

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{0\le|\alpha|\le k} \|D^\alpha u\|_{p(.)}.$$

Furthermore, we set $W_0^{k,p(.)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{k,p(.)}(\Omega)$. Similarly to the classical Sobolev spaces, we define $W^{-k,p'(.)}(\Omega)$ as the dual of $W_0^{k,p(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$.

Lemma 1. *(Poincaré's inequality [7]). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p(\cdot)$ satisfies (3). Then*

$$\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)}, \quad \text{for all } u \in W_0^{1,p(.)}(\Omega),$$

where the positive constant depends on Ω, p_1, and p_2. As a direct result, the space $W_0^{1,p(.)}(\Omega)$ has an equivalent norm given by $\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.)}$.

Lemma 2. *([7]). If $p : \Omega \to [1, \infty)$ is a measurable function and*

$$\begin{aligned} 2 \leq p_1 \leq p(x) \leq p_2 \leq \tfrac{2n}{n-4}, \quad n > 4, \\ 2 \leq p_1 \leq p(x) \leq p_2 < +\infty, \quad n \leq 4. \end{aligned}$$

Then the embedding $H_0^2(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Lemma 3. *(Young's inequality [7]). Let $p, q, s \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.$$

Then for all $a, b \geq 0$,

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{(a)^{p(.)}}{p(.)} + \frac{(b)^{q(.)}}{q(.)}.$$

Lemma 4. *(Hölder's Inequality [7]). Let $p, q, s \geq 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.$$

If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$ and

$$\|fg\|_s \leq 2\|f\|_p\|g\|_q.$$

The following theorem states the well-posedness result, which can be established by combining the approaches presented in [18] and [17]:

Proposition 1. *Let $m(\cdot)$ and $p(\cdot)$ satisfy (2) and (3). Additionally, assume that*

$$\begin{aligned} 2 < p_1 \leq p(x) \leq p_2 < 2\tfrac{n-2}{n-4}, \quad \text{for } n \geq 5, \\ 2 < p_1 \leq p(x) \leq p_2 < +\infty, \quad \text{for } n \leq 4. \end{aligned} \tag{5}$$

Then, for any given $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, the problem (1) admits a unique local weak solution such that:

$$u \in L^\infty((0,T), H_0^2(\Omega)), \quad u_t \in L^\infty((0,T), L^2(\Omega)) \cap L^{m(.)}(\Omega \times (0,T)). \tag{6}$$

Furthermore, the modified energy functional $E(t)$ associated with the problem (1), defined as

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{l}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}(g \circ \Delta u) + \int_\Omega \frac{b|u|^{p(x)}}{p^2(x)}\mathrm{d}x - \int_\Omega \frac{b|u|^{p(x)}\ln|u|}{p(x)}\mathrm{d}x, \tag{7}$$

satisfies

$$E'(t) = \frac{1}{2}(g' \circ \Delta u) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - a\int_\Omega |u_t|^{m(x)}\mathrm{d}x \leq 0, \quad a.e\ t \in [0,T). \tag{8}$$

where $(g \circ u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|_2^2\mathrm{d}s$.

3. Blowup for negative initial energy

In this section, we demonstrate that the solution (6) blows up in finite time if

$$\begin{array}{ll} 2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < 2\frac{n-2}{n-4}, & \text{for } n \geq 5, \\ 2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < +\infty, & \text{for } n \leq 4. \end{array} \tag{9}$$

holds and $E(0) < 0$. To this end, we set $H(t) = -E(t)$. Since $E(t)$ is absolutely continuous, we have the following corollary:

Corollary 1. *Let u be the solution of (1). Then for every t in $[0,T)$, $H'(t) \geq 0$ and*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{b}{p(x)}|u|^{p(x)}\ln|u|\ \mathrm{d}x. \tag{10}$$

Now, we provide some useful lemmas, where C denotes a generic positive constant.

Lemma 5. *([15]). Assume that (9) holds and let u be the solution of (1). Then,*

$$\int_\Omega |u|^{m(x)}\,\mathrm{d}x \leq C\left(\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2}\right).$$

Lemma 6. *For any $u \in H_0^2(\Omega)$, there exists a positive constant $C > 0$ such that*

$$\|u\|_{p_1}^{p_1} \leq C\left[\int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + \|\Delta u\|_2^2\right].$$

Proof. We set

$$\Omega_\infty = \{x \in \Omega \mid |u| \geq e\}, \Omega_e = \{x \in \Omega \mid e > |u| \geq 1\} \text{ and } \Omega_1 = \{x \in \Omega \mid |u| < 1\}.$$

Thus

$$\begin{aligned}
\|u\|_{p_1}^{p_1} &= \int_{\Omega_\infty} |u|^{p_1}\,\mathrm{d}x + \int_{\Omega_e} |u|^{p_1}\,\mathrm{d}x + \int_{\Omega_1} |u|^{p_1}\,\mathrm{d}x \\
&\le \int_{\Omega_\infty} |u|^{p(x)} \ln|u|\,\mathrm{d}x + \int_{\Omega_e} e^{p_1} \left|\frac{u}{e}\right|^{p_1}\,\mathrm{d}x + \int_{\Omega_1} |u|^2\,\mathrm{d}x \\
&\le \int_{\Omega} |u|^{p(x)} \ln|u|\,\mathrm{d}x + \int_{\Omega_1} |u|^{p(x)} |\ln|u||\,\mathrm{d}x + e^{p_1-2}\int_{\Omega_e} |u|^2\,\mathrm{d}x + \int_{\Omega_1} |u|^2\,\mathrm{d}x \\
&\le \int_{\Omega} |u|^{p(x)} \ln|u|\,\mathrm{d}x + \frac{1+e(p_1-2)}{e(p_1-2)}\int_{\Omega_1} |u|^2\,\mathrm{d}x + e^{p_1-2}\int_{\Omega_e} |u|^2\,\mathrm{d}x \\
&\le C\left\{\int_{\Omega} |u|^{p(x)} \ln|u|\,\mathrm{d}x + \|\Delta u\|_2^2\right\}.
\end{aligned}$$

□

Theorem 2. *Suppose the assumptions of Proposition 1 are fulfilled. Additionally, assume that condition (9) holds and the initial energy satisfies $E(0) < 0$. Then, the solution to problem (1) blows up in finite time.*

Proof. For a sufficiently small parameter $\varepsilon > 0$, to be specified later, we introduce the following auxiliary function:

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u\, u_t(x,t)\,\mathrm{d}x, \tag{11}$$

where the parameter α satisfies

$$0 < \alpha \le \min\left\{\frac{p_1-2}{2p_1}, \frac{(p_1-m_2)}{p_1(m_2-1)}\right\}. \tag{12}$$

Differentiating the auxiliary function (11) and applying equation (1), we obtain the following identity:

$$\begin{aligned}
L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) &+ \varepsilon\int_\Omega \left[u_t^2 - |\Delta u|^2\right]\mathrm{d}x + \varepsilon b\int_\Omega |u|^{p(x)}\ln|u|\mathrm{d}x \\
&+ \varepsilon\int_0^t g(t-s)\int_\Omega \Delta u(t)\cdot\Delta u(s)\mathrm{d}x\mathrm{d}s - a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}\mathrm{d}x.
\end{aligned} \tag{13}$$

Next, we estimate the fourth term on the right-hand side of (13):

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \Delta u(t)\cdot\Delta u(s)\mathrm{d}x\mathrm{d}s \\
&= \left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\Delta u|^2\mathrm{d}x + \int_0^t g(t-s)\int_\Omega \Delta u(t)\cdot(\Delta u(s)-\Delta u(t))\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

For any $\tau > 0$, the second term above can be estimated using Young's inequality:

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \Delta u(t)\cdot(\Delta u(s)-\Delta u(t))\mathrm{d}x\mathrm{d}s \\
&\le \tau(g\circ\Delta u)(t) + \frac{1}{4\tau}\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\Delta u(t)|^2\mathrm{d}x.
\end{aligned} \tag{14}$$

By subtracting and adding $\varepsilon(1-\eta)p_1H(t)$ to the right-hand side of (13), and applying the preceding estimate, we obtain:

$$\begin{aligned}
L'(t) \geq\ & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1H(t) + \varepsilon\frac{(1-\eta)p_1+2}{2}\|u_t\|_2^2 \\
& + \varepsilon b\eta\int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + \frac{\varepsilon b p_1(1-\eta)}{p_2^2}\int_\Omega |u|^{p(x)}\ \mathrm{d}x \\
& + \varepsilon\left[\left(\frac{(1-\eta)p_1}{2}-1\right) - \left(\frac{(1-\eta)p_1}{2}-1+\frac{1}{4\tau}\right)(1-l)\right]\|\Delta u\|_2^2 \\
& + \varepsilon\left(\frac{(1-\eta)p_1}{2}-\tau\right)(g\circ\Delta u) - a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}\mathrm{d}x,
\end{aligned}$$

For $\tau \leq 1$, it follows that

$$\begin{aligned}
L'(t) \geq\ & (1-\alpha)H^{1-\alpha}(t)H'(t) \\
& + \varepsilon\beta\left[H(t) + \int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + \|u_t\|_2^2 + g\circ\Delta u + \|\Delta u\|_2^2\right] \\
& - a\varepsilon\int_\Omega u u_t|u_t|^{m(x)-2}\ \mathrm{d}x,
\end{aligned} \tag{15}$$

where $\beta > 0$.
Using Young's inequality, we estimate the last term in (15) as follows:

$$\int_\Omega |u_t|^{m-1}|u|\,\mathrm{d}x \leq \frac{1}{m_1}\int_\Omega \theta^m|u|^m\ \mathrm{d}x + \frac{m_2-1}{m_2}\int_\Omega \theta^{-\frac{m}{m-1}}|u_t|^m\,\mathrm{d}x, \forall\theta>0. \tag{16}$$

By choosing θ so that

$$\theta^{-\frac{m(x)}{m(x)-1}} = k\cdot H^{-\alpha}(t), \quad k>0,$$

we obtain

$$\int_\Omega |u_t|^{m(x)-1}|u|\,\mathrm{d}x \leq \int_\Omega \frac{k^{1-m(x)}|u|^m H^{\alpha(m(x)-1)}(t)}{m_1}\mathrm{d}x + \frac{(m_2-1)\,kH^{-\alpha}(t)H'(t)}{am_2}. \tag{17}$$

Combining (15) with (17), we deduce that

$$\begin{aligned}
L'(t) \geq\ & \left[(1-\alpha) - \varepsilon\left(\frac{m_2-1}{m_2}\right)k\right]H^{-\alpha}(t)H'(t) \\
& - \varepsilon\frac{k^{1-m_1}a}{m_1}H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\ \mathrm{d}x \\
& + \varepsilon\beta\left[H(t) + \|u_t\|_2^2 + \int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + g\circ\Delta u + \|\Delta u\|_2^2\right].
\end{aligned} \tag{18}$$

Employing both Lemma 5 and (10), we arrive at:

$$\begin{aligned}
H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\ \mathrm{d}x &\leq CH^{\alpha(m_2-1)}(t)\left[\|u\|_{p_1}^{m_1} + \|u\|_{p_1}^{m_2}\right] \\
&\leq C\left[\sum_{i=1}^{2}(H(t))^{\frac{p_1\alpha(m_2-1)}{p_1-m_i}} + \|u\|_{p_1}^{p_1}\right],
\end{aligned} \tag{19}$$

thanks to Young's inequality.
Now, we utilize the following well-known algebraic inequality

$$z^{\kappa} \leq z+1 \leq \left(1+\frac{1}{d}\right)(z+d), \forall z \geq 0, 0<\kappa \leq 1, d \geq 0, \tag{20}$$

for $z = H(t)$, $d = H(0)$ and $\kappa_{i(i=1,2)} = p_1\alpha(m_2-1)(p_1-m_i)^{-1}$, we get

$$(H(t))^{\kappa_1} + (H(t))^{\kappa_2} \leq C\left[H(0)+H(t)\right] \leq 2CH(t). \tag{21}$$

It follows from (19), (21), and Lemma 6 that:

$$H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\ \mathrm{d}x \leq C\left(\int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + \|\Delta u\|_2^2 + H(t)\right). \tag{22}$$

Combining (18) and (22) yields

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\right]H^{-\alpha}(t)H'(t) \\ & +\varepsilon\left(\beta-\frac{k^{1-m_1}a}{m_1}C\right)\left[H(t)+\|u_t\|_2^2+\int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + g\circ\Delta u + \|\Delta u\|_2^2\right]. \end{aligned} \tag{23}$$

Here, we select k sufficiently large such that

$$\gamma = \beta - \frac{a\cdot k^{1-m_1}}{m_1}C > 0.$$

After k has been fixed we determine ε small enough to ensure

$$(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k \geq 0 \text{ and } L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)\ \mathrm{d}x > 0,$$

Thus, (23) takes the form

$$L'(t) \geq \gamma\varepsilon\left[H(t)+\|u_t\|_2^2+\int_\Omega |u|^{p(x)}\ln|u|\ \mathrm{d}x + g\circ\Delta u + \|\Delta u\|_2^2\right]. \tag{24}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad \text{for all } t \geq 0.$$

On the other hand, we obtain from (11):

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \varepsilon\int_\Omega uu_t(x,t)\ \mathrm{d}x\right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{1/1-\alpha}\left[H(t)+\left(\varepsilon\int_\Omega uu_t(x,t)\ \mathrm{d}x\right)^{\frac{1}{1-\alpha}}\right]. \end{aligned} \tag{25}$$

We observe that

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_{p_1}\|u_t\|_2,$$

which implies

$$\left|\int_\Omega uu_t(x,t)\ \mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq C\|u\|_{p_1}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}}.$$

Again, Young's inequality yields

$$\left|\int_\Omega uu_t(x,t)\,\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq C\left[\|u\|_{p_1}^{\frac{2}{1-2\alpha}} + \|u_t\|_2^2\right], \text{ for } \frac{1-2\alpha}{2(1-\alpha)} + \frac{1}{2(1-\alpha)} = 1.$$

Recalling (12) then applying (20), we find

$$\left|\int_\Omega uu_t(x,t)\,\mathrm{d}x\right|^{\frac{1}{1-\alpha}} \leq C\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right]. \tag{26}$$

Substituting (26) into (25) yields

$$\begin{aligned} L^{\frac{1}{1-\alpha}}(t) &\leq \varepsilon C\left[H(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1}\right] \\ &\leq \varepsilon C\left[H(t) + \|u_t\|_2^2 + \int_\Omega |u|^p \ln|u|\,\mathrm{d}x + g\circ\Delta u + \|\Delta u\|_2^2\right], \end{aligned} \tag{27}$$

by virtue of Lemma 6.
From (24) and (27), we conclude that

$$L'(t) \geq \Lambda L^{\frac{1}{1-\alpha}}(t), \quad \text{for all } t \geq 0. \tag{28}$$

where Λ is a positive constant depending on $\Omega, u_{0,1}, m_{1,2}$ and $p_{1,2}$ only. Performing a direct integration of (28) on $(0,t)$ results in

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\Lambda t}$$

Therefore, $L(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\Lambda\alpha\left[L(0)\right]^{\frac{\alpha}{1-\alpha}}}.$$

Hence, the proof is finished. □

4. Blowup for positive initial energy

In this section, we establish the blow-up phenomenon for certain solutions possessing positive initial energy. To state and prove our main result, let $B = B_1 l^{1/2}$ denote the optimal constant of the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{p_2+\sigma}(\Omega)$ with

$$0 < \sigma \leq \sigma^*,\ \sigma^* := \begin{cases} \frac{2n}{n-4} - p_2 & \text{if } n \geq 5 \\ +\infty & \text{if } n \leq 4 \end{cases},$$

and set

$$\delta_1 = \left(\frac{ep_1\sigma}{b\,(p_2+\sigma)}\right)^{\frac{2}{p_2+\sigma-2}} (B_1)^{\frac{-2(p_2+\sigma)}{p_2+\sigma-2}}, E_1 = \left(\frac{1}{2} - \frac{1}{p_1}\right)\delta_1,$$

$$F(t) = E_1 - E(t), \tag{29}$$

$$G(t) = F^{1-\alpha}(t) + \varepsilon\int_\Omega u\,u_t(x,t)\,\mathrm{d}x + \varepsilon p_1 E_1 t,\ t \in [0,T), \tag{30}$$

where $\varepsilon > 0$ and $0 < \alpha < 1$ are parameters to be determined later.

Lemma 7. *([16]). Let u be the solution of (1). If*

$$E(0) < E_1 \text{ and } \|\Delta u_0\|_2^2 \geq \delta_1,$$

then there exists $\delta_2 > \delta_1$ such that

$$\|\Delta u(t)\|_2^2 \geq \delta_2, t \geq 0. \tag{31}$$

Lemma 8. *Let the assumptions in Lemma 7 be satisfied, then we have*

$$0 < F(0) \leq F(t) \leq \frac{b}{p_1} \int_\Omega |u|^{p(x)} \ln |u| \, \mathrm{d}x.$$

Proof. Using (7), (8) and (29), we obtain

$$\begin{aligned} 0 &< F(0) \leq F(t) \\ &\leq E_1 + \int_\Omega \frac{b}{p(x)} |u|^{p(x)} \ln |u| \, \mathrm{d}x \\ &\quad - \left[\frac{1}{2} \int_\Omega u_t^2 \, \mathrm{d}x + \frac{l}{2} \int_\Omega |\Delta u|^2 \, \mathrm{d}x + \frac{1}{2} g \circ \Delta u + \int_\Omega \frac{b}{p^2(x)} |u|^{p(x)} \, \mathrm{d}x \right] \end{aligned}$$

and, from (31), we get

$$\begin{aligned} E_1 - \left[\tfrac{1}{2} \textstyle\int_\Omega u_t^2 \, \mathrm{d}x + \tfrac{1}{2} \int_\Omega |\Delta u|^2 \, \mathrm{d}x + \tfrac{1}{2} g \circ \Delta u + \int_\Omega \tfrac{b}{p^2(x)} |u|^{p(x)} \, \mathrm{d}x \right] &\leq E_1 - \tfrac{1}{2} \|\Delta u\|_2^2 \\ \leq E_1 - \tfrac{\delta_2}{2} \leq E_1 - \tfrac{\delta_1}{2} = -\tfrac{\delta_1}{p_1} < 0, \quad \forall t \geq 0. \end{aligned}$$

Hence

$$0 < F(0) \leq F(t) \leq \frac{b}{p_1} \int_\Omega |u|^{p(x)} \ln |u| \, \mathrm{d}x, \quad \forall t \geq 0.$$

□

Theorem 3. *Let the conditions of Lemma 7 be satisfied. Then, the solution of problem (1) given by Proposition 1 blows up in finite time.*

Proof. By differentiating the auxiliary function (30) and applying equation (1), we arrive at:

$$\begin{aligned} G'(t) &= (1-\alpha) F^{-\alpha}(t) F'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\Delta u|^2 \right] \mathrm{d}x + \varepsilon b \int_\Omega |u|^{p(x)} \ln |u| \mathrm{d}x \\ &\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \Delta u(t) \cdot \Delta u(s) \mathrm{d}x \mathrm{d}s - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2} \mathrm{d}x + \varepsilon p_1 E_1. \end{aligned} \tag{32}$$

Using the definition in (29), we add and subtract $\varepsilon(1-\eta)p_1 F(t)$ on the right-hand side of (32). Then, applying the estimate (14), we obtain:

$$\begin{aligned} G'(t) &\geq (1-\alpha) F^{-\alpha}(t) F'(t) + \varepsilon(1-\eta) p_1 F(t) + \varepsilon \frac{(1-\eta)p_1 + 2}{2} \|u_t\|_2^2 \\ &\quad + \varepsilon b \eta \int_\Omega |u|^{p(x)} \ln |u| \, \mathrm{d}x + \frac{\varepsilon b p_1 (1-\eta)}{p_2^2} \int_\Omega |u|^{p(x)} \, \mathrm{d}x + \varepsilon \eta p_1 E_1 \\ &\quad + \varepsilon \left[\left(\frac{(1-\eta)p_1}{2} - 1 \right) - \left(\frac{(1-\eta)p_1}{2} - 1 + \frac{1}{4\tau} \right) (1-l) \right] \|\Delta u\|_2^2 \\ &\quad + \varepsilon \left(\frac{(1-\eta)p_1}{2} - \tau \right) (g \circ \Delta u) - a\varepsilon \int_\Omega u u_t |u_t|^{m(x)-2} \mathrm{d}x, \end{aligned} \tag{33}$$

To estimate the last term in equation (33), we follow the arguments from (15) to (24) in the proof of Theorem 2, and we arrive at the following estimate:

$$G'(t) \geq \gamma\varepsilon \left[F(t) + \|u_t\|_2^2 + \int_\Omega |u|^{p(x)} \ln|u| \, \mathrm{d}x + g \circ \Delta u + \|\Delta u\|_2^2 \right]. \tag{34}$$

Therefore, we have:

$$G(t) \geq G(0) > 0, \quad \text{for all } t \geq 0.$$

Meanwhile, we have:

$$\begin{aligned} G^{\frac{1}{1-\alpha}}(t) &= \left[F^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t) \, \mathrm{d}x + \varepsilon p_1 E_1 t \right]^{\frac{1}{1-\alpha}} \\ &\leq C \left[F(t) + \left(\varepsilon \int_\Omega u u_t(x,t) \, \mathrm{d}x \right)^{\frac{1}{1-\alpha}} + (\varepsilon p_1 E_1 t)^{\frac{1}{1-\alpha}} \right]. \end{aligned} \tag{35}$$

The estimation of the last term is all that remains. By choosing $\varepsilon > 0$ such that $\varepsilon < \frac{(\delta_2)^{1-\alpha}}{T}$, and recalling Lemma 7, we obtain:

$$|\varepsilon p_1 \mathrm{E}_1 t|^{\frac{1}{1-\alpha}} \leq |\varepsilon p_1 \mathrm{E}_1 T|^{\frac{1}{1-\alpha}} \leq \left(p_1 \mathrm{E}_1 (\delta_2)^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \leq (p_1 \mathrm{E}_1)^{\frac{1}{1-\alpha}} \|\Delta u\|_2^2.$$

Thus, we conclude that

$$\begin{aligned} G^{\frac{1}{1-\alpha}}(t) &\leq \varepsilon C \left[F(t) + \|u_t\|_2^2 + \|u\|_{p_1}^{p_1} + \|\Delta u\|_2^2 \right] \\ &\leq \varepsilon C \left[F(t) + \|u_t\|_2^2 + \int_\Omega |u|^p \ln|u| \, \mathrm{d}x + g \circ \Delta u + \|\Delta u\|_2^2 \right]. \end{aligned} \tag{36}$$

Then, there exists a positive constant $\mu > 0$ such that

$$G'(t) \geq \mu G^{\frac{1}{1-\alpha}}(t), \quad \text{for all } t \geq 0. \tag{37}$$

This implies that $G(t)$ blows up in finite time. □

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[1] Laboratory of Pure and Applied Mathematics, University of Laghouat, Laghouat, Algeria.
Email address: m.trigui@lagh-univ.dz

[2] Laboratory of Pure and Applied Mathematics, University of Laghouat, Laghouat, Algeria.
Email address: saadaouik16@gmail.com