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INVESTIGATION OF SOME EXACT SOLUTIONS AND CONSERVATION LAWS FOR THE TIME FRACTIONAL ZELDOVICH EQUATION

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Abstract. In this paper, the Lie symmetry method is applied to examine the time fractional Zeldovich equation. Based on the obtained infinitesimal generators and for various constants, we show that the studied equation can be transformed into a nonlinear ordinary differential equation in terms of the Erdélyi–Kober fractional operator. Then, we are able to find some of its exact solutions. Using Ibragimov's method, we construct some of its conservation laws.

Keywords. Zeldovich equation, Lie symmetries, Fractional derivative, Erdélyi-Kober fractional operator, power series approach, Exact solutions, Conservation laws

2020 Mathematics Subject Classification. Primary 35B40; Secondary 35L7

1. Introduction

Fractional calculus [9, 8] is a mathematical framework that generalizes the concepts of differentiation and integration to non-integer orders. It allows for the calculation of derivatives and integrals that can take on fractional values, which is useful for modeling processes with memory effects or anomalies that cannot be captured by classical calculus. This approach finds applications across various disciplines, including physics and engineering, where it helps describe complex systems more accurately [7, 4].

Lie symmetry analysis is an effective technique for examining the invariance properties of partial differential equations [10]. It has attracted the interest of many researchers, particularly for its ability to simplify these equations by solving them or reducing their order and nonlinearity. Recently, various researchers [3, 2, 11, 1] introduced an extension formula for an infinitesimal generator associated with a fractional differential equation.

A conservation law asserts that a specific measurable characteristic of an isolated physical system remains constant. They are essential for investigating differential equations' uniqueness, integrability, and fundamental properties. Recently,

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Ibragimov [5] introduced a method for deriving new conservation laws, even for equations that cannot be derived from a Lagrangian.

This research is devoted to the Lie symmetry method and the construction of exact solutions for the time fractional Zeldovich equation:

$$v_t^\alpha + pv_{xx} + qv^2 + rv^3 = 0. \tag{1.1}$$

where v is the temperature of the medium, $v_t^\alpha \quad (0 < \alpha \leq 1)$ is the temporal fractional evolution, pv_{xx}; standard diffusion term and $qv^2 + rv^3$ is the non-linear reaction term, and p, q, and r are arbitrary constants.

This paper is arranged as follows: First, we focuses on the construction of the Lie symmetry algebra for the studied equation. In section (3), we use the invariants of the obtained generators and the Erdélyi–Kober fractional operator to reduce the studied equation to a nonlinear fractional ordinary differential equation. Finally, we derive several conserved quantities based on Ibragimov's conservation theorem.

2. Lie symmetry analysis

To begin with, recall that the classical Zeldovich equation emerges in the study of combustion theory, where it's often used to describe the propagation of flame fronts [13, 14]. In this work, we focus on its time-fractional version given by

$$v_t^\alpha = F(v, v_{xx}), \tag{2.1}$$

where

$$F(v, v_{xx}) = -pv_{xx} - qv^2 - rv^3,$$

and v_t^α is the Riemann-Liouville fractional partial derivative defined by:

$$v_t^\alpha = \begin{cases} \frac{\partial^m v}{\partial t^m}, & \alpha = m, \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t (t-s)^{m-\alpha-1} v(s,x) dr, & m-1 < \alpha < m, \end{cases} \quad m \in \mathbb{N}.$$

We consider a one-parameter group of point transformations

$$\begin{aligned} \tilde{t} &= t + \epsilon\tau + o\left(\epsilon^2\right), \\ \tilde{x} &= x + \epsilon\chi + o\left(\epsilon^2\right), \\ \tilde{v} &= v + \epsilon\eta + o\left(\epsilon^2\right), \end{aligned} \tag{2.2}$$

where ϵ is the group parameter, and τ, χ, and η are infinitesimal functions. Its associated infinitesimal generator is given by

$$V = \tau\frac{\partial}{\partial t} + \chi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial v}. \tag{2.3}$$

Definition 2.1. The $\alpha-$prolongation of V is given by

$$V^{(\alpha)} = V + \eta^x\frac{\partial}{\partial v_x} + \eta^{xx}\frac{\partial}{\partial v_{xx}} + \eta^{\alpha,t}\frac{\partial}{\partial\left(\partial_t^\alpha v\right)}, \tag{2.4}$$

where the extended infinitesimals η^x, η^{xx} and $\eta^{\alpha,t}$ are given by

$$\begin{aligned}
\eta^x &= D_x(\eta) - v_x\left(D_x\chi\right) - v_t\left(D_x\tau\right)\\
\eta^{xx} &= D_x\left(\eta^x\right) - v_{xx}\left(D_x\chi\right) - v_{xt}\left(D_x\tau\right)\\
\eta^{\alpha,t} &= D_t^\alpha(\eta) + \chi D_t^\alpha\left(v_x\right) - D_t^\alpha\left(\chi v_x\right) + D_t^\alpha\left(D_t(\tau)v\right) - D_t^{\alpha+1}(\tau_v) + \tau D_t^{\alpha+1}(v),
\end{aligned} \tag{2.5}$$

where D_x is the total derivative with respect to x and D_t^α is the total time fractional derivative.

Theorem 2.2. *The vector field V generates a symmetry of the equation (2.1) if and only if the invariance criterion*

$$V^{(\alpha)}(E)\big|_{E=0} = 0 \quad \text{and} \quad \tau(t,x,v)|_{t=0} = 0, \tag{2.6}$$

is satisfied, where $E = \frac{\partial^\alpha v}{\partial t^\alpha} - F$.

The application of the invariance criterion leads to:

$$\eta^{\alpha,\mathrm{t}} + p\eta^{xx} + 2qv\eta + 3rv^2\eta = 0. \tag{2.7}$$

Following some algebraic calculations, the condition for invariance results in a system of determining equations:

$$\begin{cases}
\tau_x = \tau_v = \chi_v = \chi_t = 0,\\
p(2\eta_{xv} - \chi_{xx}) = 0,\\
p\left(\eta_v - 2\chi_x\right) - p\left(\eta_v - \alpha D_t(\tau)\right) = 0,\\
p\left(\eta_{vv} - 2\chi_{xv}\right) = 0,\\
\frac{\partial^\alpha \eta}{\partial t^\alpha} - v\frac{\partial^\alpha \eta_v}{\partial t^\alpha} + \left(-qv - rv^3\right)\left(\eta_v - \alpha D_t(\tau)\right) + 2qv\eta + 3rv^2\eta + p\eta_{xx} = 0,\\
\begin{pmatrix}\alpha\\ m\end{pmatrix}\frac{\partial^m \eta_v}{\partial t^m} - \begin{pmatrix}\alpha\\ m+1\end{pmatrix} D_t^{m+1}(\tau) = 0, \quad m = 0,1,2...
\end{cases} \tag{2.8}$$

where $\begin{pmatrix}\alpha\\ m\end{pmatrix}$ is the generalized binomial coefficient:

$$\begin{pmatrix}\alpha\\ m\end{pmatrix} = \frac{(-1)^{m-1}\alpha\Gamma(m-\alpha)}{\Gamma(1-\alpha)\Gamma(m+1)}.$$

The solution of the above system is found to be of the following form:

$$\tau = c_1 t, \quad \chi = \frac{\alpha c_1}{2}x + c_2, \quad \eta = -\frac{2q+3rv}{2q+6rv}\alpha v c_1, \tag{2.9}$$

where c_1 and c_2 are constants.
Then, the Lie symmetry algebra of the studied equation is spanned by the following vector fields:

$$\begin{aligned}
V_1 &= t\frac{\partial}{\partial t} + \frac{\alpha x}{2}\frac{\partial}{\partial x} - \frac{2q+3rv}{2q+6rv}\alpha v\frac{\partial}{\partial v},\\
V_2 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{2.10}$$

With the commutator:

$$[V_1, V_2] = -\frac{\alpha}{2} V_2. \tag{2.11}$$

3. Lie symmetry reductions and exact solutions

In what follows, we develop similarity variables and similarity solutions to reduce equation (1.1).

Case 1. For $q = 0$

Reduction with $V_1 = t\frac{\partial}{\partial t} + \frac{\alpha x}{2}\frac{\partial}{\partial x} - \frac{\alpha}{2} v \frac{\partial}{\partial v}$

The similarity variables can be obtained by solving the following characteristic equation

$$\frac{dt}{t} = \frac{dx}{\frac{\alpha x}{2}} = \frac{dv}{-\frac{\alpha v}{2}}. \tag{3.1}$$

The invariants are:

$$v = t^{\frac{-\alpha}{2}} \phi(\xi), \quad \textit{and} \quad \xi = x t^{\frac{-\alpha}{2}}. \tag{3.2}$$

Theorem 3.1. *The similarity transformation $v = t^{\frac{-\alpha}{2}}\phi(\xi)$, where $\xi = xt^{\frac{-\alpha}{2}}$ converts the equation (1.1) to a fractional ordinary differential equation given by:*

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}\phi\right)(\xi) + p\phi_{\xi\xi} + r\phi^3 = 0, \tag{3.3}$$

where $\mathcal{P}_{\sigma}^{\beta,\alpha}\phi$ is the Erdélyi–Kober fractional operator defined by:

$$\begin{aligned} \left(\mathcal{P}_{\sigma}^{\beta,\alpha}\phi\right)(\xi) &= \prod_{j=0}^{m-1}\left(\beta + j - \frac{1}{\sigma}\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_{\sigma}^{\beta+\alpha,m-\alpha}\phi\right)(\xi) \\ \textit{and} \quad m &= \begin{cases} \alpha, & \alpha \in \mathbb{N}, \\ [\alpha]+1, & \alpha \notin \mathbb{N}, \end{cases} \end{aligned} \tag{3.4}$$

with

$$\begin{cases} \left(\mathcal{K}_{\sigma}^{\beta,\alpha}\phi\right)(\xi) = \frac{1}{\Gamma(\alpha)}\int_1^{\infty}(\mu-1)^{\alpha-1}\mu^{-(\beta+\alpha)}\phi\left(\xi\mu^{\frac{1}{\sigma}}\right)d\mu, & \alpha > 0, \\ \left(\mathcal{K}_{\sigma}^{\beta,\alpha}\phi\right)(\xi) = \phi(\xi). & \alpha = 0, \end{cases} \tag{3.5}$$

is the Erdélyi-Kober integral operator[6].

Proof. Using the invariants: $v = t^{\frac{-\alpha}{2}}\phi(\xi)$ with $\xi = xt^{\frac{-\alpha}{2}}$,
we have:

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} = \frac{\partial^m}{\partial t^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} s^{\frac{-\alpha}{2}}\phi\left(xs^{\frac{-\alpha}{2}}\right)ds\right). \tag{3.6}$$

Let $\mu = \frac{t}{s}$, then $ds = -\frac{t}{\mu^2}d\mu$.
Thus:

$$\begin{aligned}\frac{\partial^\alpha v}{\partial t^\alpha} =& \frac{\partial^m}{\partial t^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_\infty^1 (t-\frac{t}{\mu})^{m-\alpha-1}\left(\frac{t}{\mu}\right)^{\frac{-\alpha}{2}}\phi\left(x\left(\frac{t}{\mu}\right)^{\frac{-\alpha}{2}}\right)\frac{-t}{\mu^2}d\mu\right)\\ =& \frac{\partial^m}{\partial t^m}\left(\frac{t^{m-\frac{3\alpha}{2}}}{\Gamma(m-\alpha)}\int_1^\infty (\mu-1)^{m-\alpha-1}\mu^{-\left(m-\alpha+1-\frac{\alpha}{2}\right)}\phi\left(\xi\mu^{\frac{\alpha}{2}}\right)d\mu\right).\end{aligned} \tag{3.7}$$

Using the Erdélyi–Kober fractional integral operator formulation and the last equation, we obtain:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^m}{\partial t^m}\left(t^{m-\frac{3\alpha}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right). \tag{3.8}$$

Furthermore, we have:

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\left(t^{m-\frac{3\alpha}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right)\\ &= \frac{\partial^{m-1}}{\partial t^{m-1}}\left[\left(m-\frac{3\alpha}{2}\right)t^{m-\frac{3\alpha}{2}-1}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi) + t^{m-\frac{3\alpha}{2}}\frac{\partial}{\partial t}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha}\phi\right)(\xi)\right].\end{aligned} \tag{3.9}$$

By using the chain rule, we obtain:

$$\begin{aligned}t^{m-\frac{3\alpha}{2}}\frac{\partial}{\partial t}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi) =& t^{m-\frac{3\alpha}{2}}\xi\frac{\partial}{\partial \xi}\left(\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right)\frac{\partial\left(xt^{\frac{-\alpha}{2}}\right)}{\partial t}\\ =& -\frac{\alpha}{2}t^{m-\frac{3\alpha}{2}-1}\xi\frac{\partial}{\partial \xi}\left(\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right),\end{aligned} \tag{3.10}$$

and thus:

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\left(t^{m-\frac{3\alpha}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right)\\ &= \frac{\partial^{m-1}}{\partial t^{m-1}}\left[t^{m-\frac{3\alpha}{2}-1}\left(m-\frac{3\alpha}{2}-\frac{\alpha}{2}\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right].\end{aligned} \tag{3.11}$$

After $(m-1)$ iterations, we get:

$$\begin{aligned}&\frac{\partial^m}{\partial t^m}\left(t^{m-\frac{3\alpha}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right)\\ &= \frac{\partial^{m-1}}{\partial t^{m-1}}\left[t^{m-\frac{3\alpha}{2}-1}\left(m-\frac{3\alpha}{2}-\frac{\alpha}{2}\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right]\\ &\quad\vdots\\ &= t^{-\frac{3\alpha}{2}}\prod_{j=0}^{m-1}\left(1-\frac{3\alpha}{2}+j-\frac{\alpha}{2}\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi).\end{aligned} \tag{3.12}$$

Applying the Erdélyi-Kober fractional differential operator, we obtain:

$$\frac{\partial^m}{\partial t^m}\left(t^{m-\frac{3\alpha}{2}}\left(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},m-\alpha}\phi\right)(\xi)\right)=t^{-\frac{3\alpha}{2}}\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}\phi\right)(\xi). \tag{3.13}$$

By the computation of u_x and u_{xx}, we see that the equation (1.1) simplifies to

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}\phi\right)(\xi)+p\phi_{\xi\xi}+r\phi^3=0.$$

□

In order to determine the equation's solution, we now examine the explicit solution using the power series method.

$$\phi(\xi)=\sum_{m=0}^{\infty}a_m\xi^m. \tag{3.14}$$

So, we have:

$$\phi_\xi=\sum_{m=0}^{\infty}(m+1)a_{m+1}\xi^m,\quad \phi_{\xi\xi}=\sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}\xi^m \tag{3.15}$$

and

$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}\phi\right)(\xi)=\sum_{m=0}^{\infty}a_m\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{n\alpha}{2}\right)}\xi^m. \tag{3.16}$$

Thus:

$$\sum_{m=0}^{\infty}a_m\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}\xi^m+p\sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}\xi^m\quad +r\left(\sum_{m=0}^{\infty}a_m\xi^m\right)^3=0. \tag{3.17}$$

This yields:

$$\begin{aligned}&\sum_{m=0}^{\infty}a_m\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}\xi^m+p\sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}\xi^m\\&+r\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{i=0}^{k}a_{m-k-i}a_{k-i}a_i\xi^m=0.\end{aligned} \tag{3.18}$$

Identifying the coefficients, for $m=0$, we obtain

$$a_2=\frac{\frac{-a_0\Gamma\left(2-\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}\right)}-ra_0^3}{2p}, \tag{3.19}$$

and for $m\geq 0$,

$$a_{m+2}=\frac{-1}{p(m+1)(m+2)}\left(\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}a_m+r\sum_{k=0}^{m}\sum_{i=0}^{k}a_{m-k-i}a_{k-i}a_i\right). \tag{3.20}$$

Now, examine the series solution's convergence. From (3.20), we have

$$|a_{m+2}|\leq M\left[|a_m|+\sum_{k=0}^{m}\sum_{i=0}^{k}|a_{m-k-i}||a_{k-i}||a_i|\right], \tag{3.21}$$

with,

$$M = max\left\{\left|\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{p\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}\right|;|r|\right\}. \tag{3.22}$$

Consider now another power series defined as

$$Q(\xi)=\sum_{m=0}^{\infty} b_m\xi^m, \tag{3.23}$$

where

$$b_i=|a_i|, \quad i=0,1 \tag{3.24}$$

and

$$b_{m+2}=M\left[b_m+\sum_{k=0}^{m}\sum_{i=0}^{k} b_{m-k-i}b_{k-i}b_i\right], \quad b=0,1,2,\ldots \tag{3.25}$$

So, $Q(\xi)$ is the majorant power series of (3.14), in fact:

$$\begin{aligned}
Q(\xi) &= b_0+b_1\xi+\sum_{m=0}^{\infty} b_{m+2}\xi^{m+2}\\
&= b_0+b_1\xi+M\sum_{m=0}^{\infty}\left[b_m\xi^{m+2}+\sum_{k=0}^{m}\sum_{i=0}^{k} b_{m-k-i}b_{k-i}b_i\xi^{m+2}\right]\\
&= b_0+b_1\xi+M\left[Q\xi^2+Q^3\xi^2\right].
\end{aligned}$$

The implicit functional equation with respect to ξ is given by

$$G(\xi,Q)=Q-b_0-b_1\xi-M\left[Q\xi^2+Q^3\xi^2\right]. \tag{3.26}$$

Following to the implicit function theorem [12], we have convergence since G is analytic in the neighborhood of $(0,b_0)$ with $G(0,b_0)=0$ and $\frac{\partial G}{\partial Q}(0,b_0)=1\neq 0$.

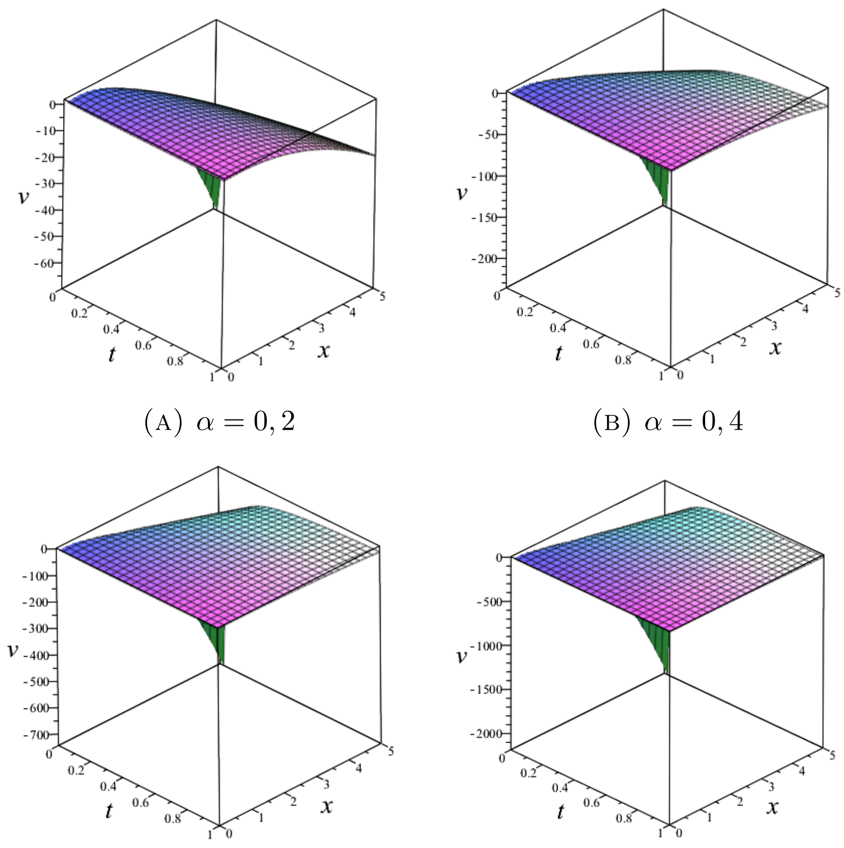
Hence:

$$\phi(\xi)=\sum_{m=0}^{\infty} a_m\xi^m = a_0+a_1\xi+\frac{\frac{-a_0\Gamma\left(2-\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}\right)}-ra_0^3}{2p}\xi^2$$
$$+\sum_{m=1}^{\infty}\frac{-1}{p(m+1)(m+2)}\left(\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}a_m+r\sum_{k=0}^{m}\sum_{i=0}^{k} a_{m-k-i}a_{k-i}a_i\right)\xi^{m+2}. \tag{3.27}$$

Then, the power series solutions are:

$$v(t,x)=a_0t^{\frac{-\alpha}{2}}+a_1xt^{-\alpha}+\frac{\frac{-a_0\Gamma\left(2-\frac{\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}\right)}-ra_0^3}{2p}x^2t^{\frac{-3\alpha}{2}}$$
$$+\sum_{m=1}^{\infty}\frac{-1}{p(m+1)(m+2)}\left(\frac{\Gamma\left(2-\frac{\alpha}{2}+\frac{m\alpha}{2}\right)}{\Gamma\left(2-\frac{3\alpha}{2}+\frac{m\alpha}{2}\right)}a_m+r\sum_{k=0}^{m}\sum_{i=0}^{k} a_{m-k-i}a_{k-i}a_i\right)x^{m+2}t^{-\frac{m\alpha}{2}-\frac{3\alpha}{2}}, \tag{3.28}$$

where a_0 and a_1 are arbitrary constants.
Now, we look to 3D plots of the above solution for some particular cases of the parameters m, a_0, a_1 and different values of α.



(A) $\alpha = 0,2$ (B) $\alpha = 0,4$

(C) $\alpha = 0,6$ (D) $\alpha = 0,8$

FIGURE 1. Representation of the solution (3.28) for $a_0 = a_1 = p = r = 1$, $q = 0$ and different values of α.

Reduction with V_2

The characteristic equation is:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dv}{0}. \tag{3.29}$$

So, the invariant v is a function depends only on time and the equation (1.1) became:

$$v_t^{\alpha} + rv^3 = 0. \tag{3.30}$$

Then, the solution for $r = -1$ is given by:

$$v(t) = \left(\sqrt{\frac{\Gamma\left(1 - \frac{\alpha}{2}\right)}{\Gamma\left(1 - \frac{3\alpha}{2}\right)}} \right) t^{\frac{-\alpha}{2}}. \tag{3.31}$$

Below, we plot the solution for different values of α.

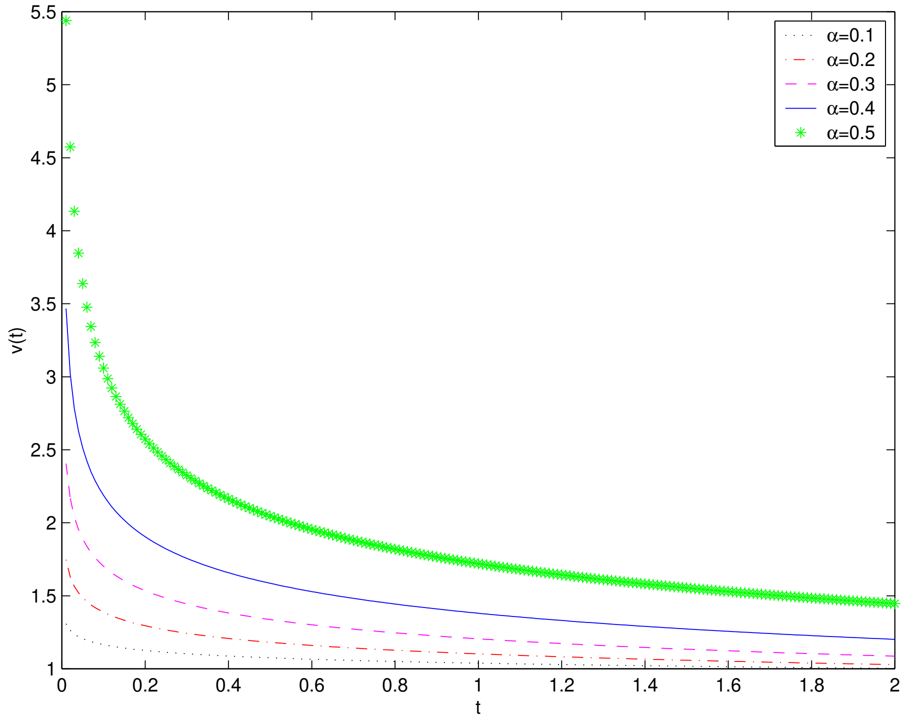


FIGURE 2. Solution of (3.31) for $q = 0$ and different values of α.

Figure (2) illustrates the effect of the fractional order α on the behavior of the solution. We see that if α is large, the solution decreases rapidly, which physically means an increase in temperature.

Case 2. For $r = 0$

Reduction with $V_1 = t\frac{\partial}{\partial t} + \frac{\alpha x}{2}\frac{\partial}{\partial x} - \alpha v\frac{\partial}{\partial v}$
The invariants are given by:

$$v = t^{-\alpha}\phi(\xi) \quad \textit{and} \quad \xi = xt^{\frac{-\alpha}{2}}. \tag{3.32}$$

Theorem 3.2. *With the similarity transformation* $v = t^{-\alpha}\phi(\xi)$, *where* $\xi = xt^{\frac{-\alpha}{2}}$, *the equation (1.1) is transformed to the following fractional ordinary differential equation:*

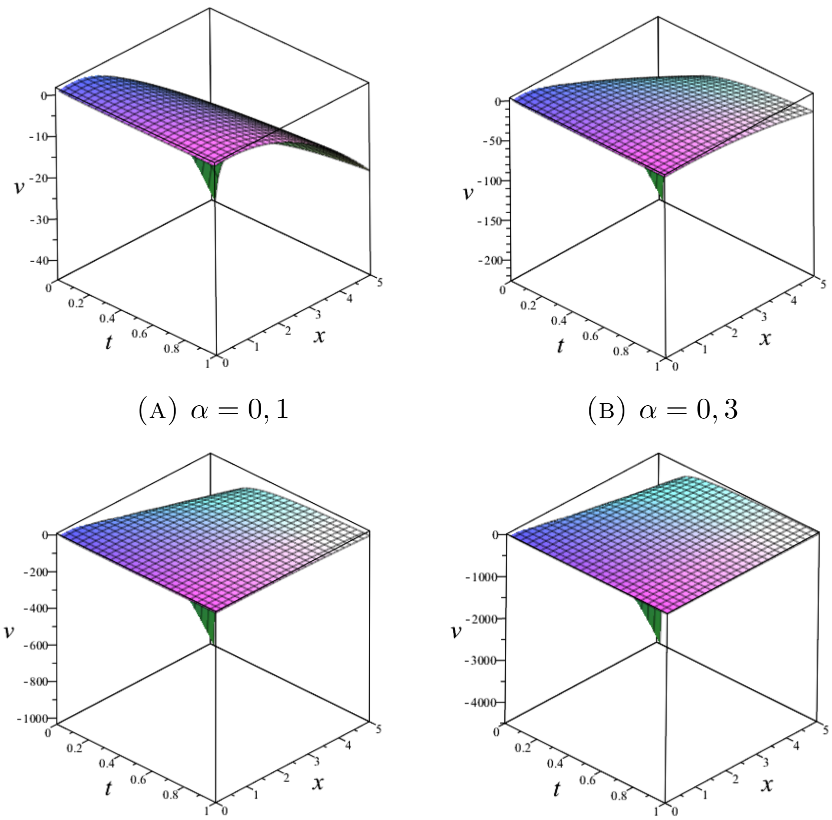
$$\left(\mathcal{P}_{\frac{2}{\alpha}}^{1-2\alpha,\alpha}\phi\right)(\xi) + p\phi_{\xi\xi} + q\phi^2 = 0. \tag{3.33}$$

Using the power series method as before, we obtain the following solutions:

$$\begin{aligned} v(t,x) &= a_0 t^{-\alpha} + a_1 x t^{-\frac{3\alpha}{2}} + \frac{\frac{-a_0\Gamma(2-\alpha)}{\Gamma(2-2\alpha)} - qa_0^2}{2p} x^2 t^{-2\alpha} \\ &+ \sum_{m=1}^{\infty} \frac{-1}{p(m+1)(m+2)} \left(\frac{\Gamma\left(2-\alpha+\frac{m\alpha}{2}\right)}{\Gamma\left(2-2\alpha+\frac{m\alpha}{2}\right)} a_m + q\sum_{k=0}^{m} a_{m-k}a_k \right) x^{m+2} t^{-\frac{m\alpha}{2}-2\alpha}, \end{aligned} \tag{3.34}$$

where a_0 *and* a_1 *are arbitrary constants.*

Now, we examine the 3D plots of the above solution for some particular cases of the parameters m, a_0, a_1 and different parameters α.



(A) $\alpha = 0,1$ (B) $\alpha = 0,3$

(C) $\alpha = 0,5$ (D) $\alpha = 0,7$

FIGURE 3. Representation of the solution (3.34) for $a_0 = a_1 = p = q = 1$, $r = 0$ and different values of α.

Reduction with V_2

In this case, the invariant v is a function depends only on time and the equation (1.1) became:

$$v_t^\alpha + qv^2 = 0. \tag{3.35}$$

With the solution:

$$v(t,x) = \frac{-\Gamma(1-\alpha)}{q\Gamma(1-2\alpha)} t^{-\alpha}. \tag{3.36}$$

In what follows, we plot the solution for $q = -1$ and different values of α.

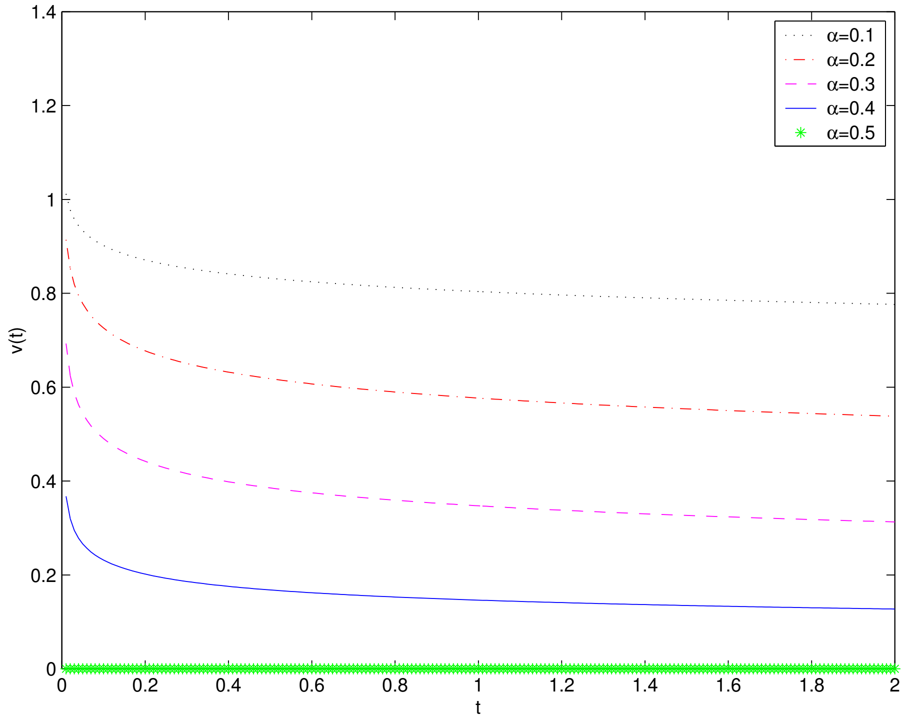


FIGURE 4. Solution of (3.36) with $q = -1$, $r = 0$ and different values of α.

Figure (4) shows the effect of α on the solution. We see that as α increases, the solution decreases, which physically translates into a decrease in temperature.

4. Conservation Laws

Since the equation studied cannot be derived from a Lagrangian, we use the Ibragimov's approach (Ibragimov [5]). The formal Lagrangian is given by:

$$\mathcal{L} = u(t,x)\left(v_t^\alpha + pv_{xx} + qv^2 + rv^3\right), \tag{4.1}$$

where u is a new dependent variable.
The adjoint equation of equation (1.1) is defined by:

$$\frac{\delta \mathcal{L}}{\delta v} = 0, \tag{4.2}$$

where $\frac{\delta}{\delta v}$ is the Euler-Lagrange operator which is presented by the expression:

$$\begin{aligned}\frac{\delta}{\delta v} =& \frac{\partial}{\partial v} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial\left(D_t^\alpha v\right)} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}} \\ &+ \sum_{k=3}^{\infty}(-1)^k D_{i_1}\dots D_{i_k}\frac{\partial}{\partial v_{t_1\dots i_k}},\end{aligned} \tag{4.3}$$

where $\left(D_t^\alpha\right)^*$ is the adjoint operator of D_t^α given by:

$$\left(D_t^\alpha\right)^* = (-1)^m \mathcal{P}_T^{m-\alpha}\left(D_t^m\right), \tag{4.4}$$

where

$$\mathcal{P}_T^{m-\alpha} f(t,x) = \frac{1}{\Gamma(m-\alpha)} \int_t^T \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-\pi}} d\tau, \quad m = [\alpha]+1. \tag{4.5}$$

As

$$\frac{\partial \mathcal{L}}{\partial v} = 2quv + 3rv^2 u, \quad (D_t^\alpha)^* \frac{\partial \mathcal{L}}{\partial (D_t^\alpha v)} = (D_t^\alpha)^* u,$$

$$\frac{\partial \mathcal{L}}{\partial v_x} = 0, \quad \frac{\partial \mathcal{L}}{\partial v_{xx}} = -vu.$$

Combining the equations (4.1) and (4.2), we obtain:

$$(D_t^\alpha)^* (u) + 2qvu + 3rv^2 u + D_x^2 (pu) = 0. \tag{4.6}$$

Finally, we get:

$$(D_t^\alpha)^* (u) + pu_{xx} + 2qvu + 3rv^2 u = 0. \tag{4.7}$$

The adjoint equation behaves similarly to the integer-order case because Ibragimov's technique for determining conservation rules for fractional partial differential equations is comparable to that for ordinary partial differential equations. Consequently, we have:

$$V^{(\alpha)}(\mathcal{L}) + D_t(\tau)\mathcal{L} + D_x(\xi)\mathcal{L} = W_i \frac{\delta \mathcal{L}}{\delta v} + D_t \left(C^t\right) + D_x \left(C^x\right), \tag{4.8}$$

where W_i are the characteristic functions given by:

$$W_i = \eta_i - \tau_i v_t - \xi_i v_x. \tag{4.9}$$

The components C^t and C^x of the conserved vectors C are given in [5] by:

$$\begin{gathered} C^t = \sum_{k=0}^{m-1} (-1)^k D_t^{\alpha-1-k} (W_i) D_t^k \frac{\partial \mathcal{L}}{\partial (D_t^\alpha v)} - (-1)^m \mathcal{J}\left(W_i, D_t^m \frac{\partial \mathcal{L}}{\partial (D_t^\alpha v)}\right), \\ C^x = W_i \frac{\delta \mathcal{L}}{\delta v_x} + D_x (W_i) \frac{\delta \mathcal{L}}{\delta v_{xx}}, \end{gathered} \tag{4.10}$$

where:

$$\mathcal{J}(f,g) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x) g(\theta,x)}{(\theta-\tau)^{1+\alpha-m}} \mathrm{d}\theta \mathrm{d}\tau. \tag{4.11}$$

Then:

$$C^t = D_t^{\alpha-1} (W_i) \frac{\partial \mathcal{L}}{\partial (D_t^\alpha v)} + \mathcal{J}\left(W_i, D_t \left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha v)}\right)\right) \tag{4.12}$$

and

$$C^x = W_i \left(\frac{\partial \mathcal{L}}{\partial v_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial v_{xx}}\right)\right) + D_x (W_i) \frac{\partial \mathcal{L}}{\partial v_{xx}}. \tag{4.13}$$

From (2.10), we obtain the following conserved vectors for (1.1):

Case 1. For $V_1 = t\frac{\partial}{\partial t} + \frac{\alpha x}{2}\frac{\partial}{\partial x} - \frac{2q+3rv}{2q+6rv}\alpha v\frac{\partial}{\partial v}$,

the corresponding Lie characteristic function

$$\begin{aligned} W_1 &= \eta_1 - \tau_1 v_t - \xi_1 v_x \\ &= -\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x. \end{aligned} \tag{4.14}$$

Hence, the conserved quantities:

$$C_1^t = uD_t^{\alpha-1}\left(-\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x\right) + \mathcal{J}\left(-\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x, u_t\right) \tag{4.15}$$

and

$$\begin{aligned} C_1^x &= \left(-\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x\right)(-D_x(pu)) + D_x\left(-\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x\right)(pu) \\ &= \left(-\frac{2q+3rv}{2q+6rv}\alpha v - tv_t - \frac{\alpha x}{2}v_x\right)(-pu_x) \\ &- \left(\frac{18\alpha r^2v^2v_x + 12\alpha qrvv_x + 4\alpha q^2v_x}{(2q+6rv)^2}\right)(pu). \end{aligned} \tag{4.16}$$

Remark

For $q = 0$, the conserved quantities are:

$$C_1^t = uD_t^{\alpha-1}\left(-\frac{\alpha v}{2} - tv_t - \frac{\alpha x}{2}v_x\right) + \mathcal{J}\left(-\frac{\alpha v}{2} - tv_t - \frac{\alpha x}{2}v_x, u_t\right) \tag{4.17}$$

and

$$C_1^x = \left(\frac{\alpha v}{2} + tv_t + \frac{\alpha x}{2}v_x\right)(pu_x) - \frac{\alpha v_x}{2}pu. \tag{4.18}$$

For $r = 0$, we obtain:

$$C_1^t = uD_t^{\alpha-1}\left(-\alpha v - tv_t - \frac{\alpha x}{2}v_x\right) + \mathcal{J}\left(-\alpha v - tv_t - \frac{\alpha x}{2}v_x, u_t\right) \tag{4.19}$$

and

$$C_1^x = \left(\alpha v + tv_t + \frac{\alpha x}{2}v_x\right)(pu_x) - \alpha v_x pu. \tag{4.20}$$

Case 2. For $V_2 = \frac{\partial}{\partial x}$,

The corresponding Lie characteristic function:

$$W_2 = -v_x. \tag{4.21}$$

Then the conserved vectors are:

$$C_2^t = uD_t^{\alpha-1}(-v_x) + \mathcal{J}(-v_x, u_t) \tag{4.22}$$

and

$$\begin{aligned} C_2^x &= -v_x(-pu_x) + D_x(-v_x)(pu) \\ &= pv_xu_x - pv_{xx}vu. \end{aligned} \tag{4.23}$$

5. Conclusion

In this work, we construct Lie symmetries for a time fractional Zeldovich equation applying the invariant criterion. By using the similarity transformations, the Erdélyi–Kober fractional operator and the power series method, we have constructed the class of solutions of the studied equation. The effect of the order of the fractional derivatives is illustrated through some examples. Since the equation cannot derive from a Lagrangian, we establish conservation laws through Ibragimov's nonlocal conservation theorem.

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References

1. El Ansari, B., El Kinani, E. H. and Ouhadan, A. *Symmetry analysis of the time fractional potential-KdV equation*, Computational and Applied Mathematics, Springer, Vol. 42(1) (2025), 44(1), 34.
https://doi.org/10.1007/s40314-024-02991-1
2. El Kinani, E. and Ouhadan, A. *Lie symmetry analysis of some time fractional partial differential equations*, International Journal of Modern Physics: Conference Series, vol. 38, (2015) 1560075–1560083.
https://doi.org/10.1142/S2010194515600757
3. Gazizov, R. K., Kasatkin, A. and Lukashchuk, S. Y. *Continuous transformation groups of fractional differential equations*, Vestnik Usatu, vol. 9, no 3, (2007) 21.
4. Hassouna, M., El Kinani, E. H. and Ouhadan, A. *Fractional calculus: applications in rheology*, Elsevier, (2022) 513-549.
https://doi.org/10.1016/B978-0-12-824293-3.00018-1
5. Ibragimov, N. H. *A new conservation theorem*, Journal of Mathematical Analysis and Applications, vol. 33, (2007) 311–328. https://doi.org/10.1016/j.jmaa.2006.10.078
6. Kiryakova, V. S. *Generalized fractional calculus and applications*, CRC press, (1993).
7. Lakhel, El Hassan, and Salah Hajji. *Neutral stochastic functional differential equation driven by fractional Brownian motion and Poisson point processes.* Gulf journal of mathematics 4.3 (2016).
https://doi.org/10.56947/gjom.v4i3.69
8. Miller, K. S. and Ross, B. *An introduction to the fractional calculus and fractional differential equations*, Wiley, (1993).
9. Oldham, K. and Spanier, J. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Elsevier, (1974).
https://doi.org/978-0-12-525550-9
10. Olver, P. *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1993).
https://doi.org/10.1007/978-1-4684-0274-2
11. Rahioui, Mohamed, El Hassan El Kinani, and Abdelaziz Ouhadan. *Lie symmetry analysis and conservation laws for the time fractional generalized advection–diffusion equation.* Computational and Applied Mathematics 42.1 (2023): 50. https://doi.org/10.1007/s40314-023-02186-0
12. Rudin, W. *Principles of mathematical analysis*, McGraw-Hill, New York, (1953).
13. Ur Rehman, H., Asjad Imran, M., Ullah, N. & Akgül, A. *On solutions of the newell–whitehead–segel equation and zeldovich equation*, Mathematical Methods in the Applied Sciences, vol. 44, (2021) 7134–7149. https://doi.org/10.1002/mma.7249

14. Yusuf, A., Ghanbari, B., Qureshi, S., Mustafa, I. & Baleanu, D. *Symmetry analysis and some new exact solutions of the Newell-Whitehead-Segel and Zeldovich equations*, Results in Nonlinear Analysis, vol. 2, no 4, (2019) 182–192. https://nonlinear-analysis.com/index.php/pub/article/view/38

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