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DERIVATIVE OF DRIVING POINT IMPEDANCE FUNCTIONS AT RIGHT HALF PLANE

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Abstract. The purpose of this paper is to provide a result which concerns with the boundary behaviour of positive real functions. $Z(s) = Z(b)+a_1(s-b)+a_2(s-b)^2+...$ is an analytic function defined in the right half of the s-plane. We derive inequalities for the modulus of $Z(s)$ function, $|Z'(c)|$, by assuming the $Z(s)$ function is also analytic at the boundary point $s = c$ on the imaginary axis, where $c = i\Im b$ and finally, the sharpness of these inequalities is proved.

1. Introduction and preliminaries

Driving point impedance functions (DPIFs) are frequently used in electrical engineering to represent spectral characteristics of RL, RC, LC and RLC circuits [6] . Mathematically, DPIFs satisfy the properties of positive real functions (PRFs). Accordingly, the DPIF $Z(s)$, where s represents the complex frequency parameter, $s = \alpha + i\gamma$, is analytic and single valued in $\Re s \geq 0$ except possibly for poles on the axis of imaginaries. Also, $Z(\overline{s}) = \overline{Z(s)}$ and $\Re Z(s) \geq 0$, in $\Re s \geq 0$ [14, 15].

The aim of this study is to perform bound analysis of derivative of positive real functions. In electrical engineering, the derivative of positive real functions is mainly used in network analysis and synthesis. Investigation of the derivative of DPIFs is still a hot topic in the literature where the pioneer studies are back to the 1930s. As one of the pioneer works, Van Der Pol used the derivative of DPIFs to establish a relation between electrical and magnetic energy [13]. In another work, Hazony showed that it is possible to utilize the DPIFs for gyrator design [5]. Theoretical analysis of positive real derivatives of DPIFs is given in [7] where it's proved that the derivative of an RC driving point admittance is positive real under certain coefficient conditions. There are also other studies on the boundary analysis of DPIFs using Schwarz lemma in the literature [11, 12].

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to origin in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma which has quite wide application area

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and is the direct application of the maximum modulus principle is given in the most basic form as follow ([4], p.329):

Let U be the unit disc in the complex plane $\mathbb{C}$. Let $w : U \to U$ be a analytic function with $w(0) = 0$. Under these conditions, $|w(z)| \leq |z|$ for all $z \in U$ and $|w'(0)| \leq 1$. In addition, if the equality $|w(z)| = |z|$ holds for any $z \neq 0$, or $|w'(0)| = 1$, then w is a rotation; that is $w(z) = ze^{i\theta}$, θ real.

Let $Z(s) = Z(b) + a_1(s-b) + a_2(s-b)^2 + ...$ be a Positive Real Function at $H = \{s \in \mathbb{C} : \Re s > 0\}$ and

$$w(z) = \frac{Z\left(\frac{b+\bar{b}z}{1-z}\right) - Z(b)}{Z\left(\frac{b+\bar{b}z}{1-z}\right) + \overline{Z(b)}}, \ \ z = \frac{s-b}{s+\bar{b}}, \ \Re b > 0.$$

$w(z)$ is an analytic function in U, $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$. Therefore, the function $w(z)$ satisfies the conditions of the Schwarz lemma. With simple calculations, we take

$$w(z) = \frac{a_1\left(\frac{(b+\bar{b})z}{1-z}\right) + a_2\left(\frac{(b+\bar{b})z}{1-z}\right)^2 + ...}{Z(b) + \overline{Z(b)} + a_1\left(\frac{(b+\bar{b})z}{1-z}\right) + a_2\left(\frac{(b+\bar{b})z}{1-z}\right)^2 + ...},$$

$$\frac{w(z)}{z} = \frac{a_1\left(\frac{(b+\bar{b})}{1-z}\right) + a_2\left(\frac{(b+\bar{b})}{1-z}\right)^2 z + ...}{Z(b) + \overline{Z(b)} + a_1\left(\frac{(b+\bar{b})z}{1-z}\right) + a_2\left(\frac{(b+\bar{b})z}{1-z}\right)^2 + ...},$$

$$w'(0) = \frac{a_1\left(b+\bar{b}\right)}{Z(b) + \overline{Z(b)}} = a_1\frac{\Re b}{\Re Z(b)}$$

and

$$|a_1| = |Z'(b)| \leq \frac{\Re Z(b)}{\Re b}.$$

Now, we shall show that the last inequality is sharp. Let

$$Z(s) = \frac{s\Re Z(b) + i\Im\left(\bar{b}Z(b)\right)}{\Re b}.$$

Then

$$Z'(s) = \frac{\Re Z(b)}{\Re b}$$

and

$$|Z'(b)| = \frac{\Re Z(b)}{\Re b}$$

In this case, it is get the following lemma.

Lemma 1.1. *Let $Z(s)$ be a Positive Real Function then*

$$|Z'(b)| \leq \frac{\Re Z(b)}{\Re b}. \tag{1.2}$$

The result is sharp and the extremal function is

$$Z(s)=\frac{s\Re Z\left(b\right)+i\Im\left(\bar{b}Z\left(b\right)\right)}{\Re b}.$$

It is a consequence of Schwarz lemma that if w extends continuously to some boundary point z_0 with $|z_0|=1$, and if $|w(z_0)|=1$ and $w'(z_0)$ exists, then $|w'(z_0)|\geq 1$, which is known as the Schwarz lemma on the boundary. In [9], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $w:U\rightarrow U$ be analytic function with $w(0)=0$. Assume that there is a $z_0\in\partial U$ so that f extends continuously to z_0, $|w(z_0)|=1$ and $w'(z_0))$ exists. Then

$$|w'(z_0)|\geq\frac{2}{1+|w'(0)|} \tag{1.3}$$

and

$$|w'(z_0)|\geq 1. \tag{1.4}$$

Inequality $(1,3)$ is sharp, with equality possible for each value of $|w'(0)|$. Also, $|w'(z_0)|>1$ unless $w(z)=ze^{i\theta}$, θ real. In addition, some other types of strengthening inequalities are obtained in [1, 2, 3, 8, 9, 10].

In this study, the properties of positive real functions defined in the right half-plane will be examined. In this consideration, it is shown how the derivative of the function will be limited at the $s=c=i\Im b\in\partial H$ point in the right half-plane. In addition, the boundary of the right half plane, $\partial H=\{s\in\mathbb{C}:\Re s=0\}$ will be evaluated from below at $s=c=i\Im b$ which is the point of derivative of the $Z(s)$ function. In all these considerations, inequalities have been strengthened by taking into account the coefficients of Taylor's expansion of $Z(s)$ function around the $s=b$ point. In the previous studies, the derivative of $Z(s)$ was considered only at the origin. This leads us to the problem of how the derivative of $Z(s)$ behaves at the points other than zero in the boundary of the right half- plane, ∂H. That is, general inequalities to be obtained at the points of the imaginary axis for the derivative of $Z(s)$ function. The functions giving the equality case of these inequalities have important applications in electrical and electronics engineering.

2. Main Results

In this section, a boundary analysis results for the derivative of Positive Real function are presented. From the definition of PRFs, we can state that $Z(s)$ is analytic and single-valued on the right half of the s-plane. In the following theorem, we establish lower bounds on the derivative of $Z'(c)$ for positive real functions with $Z(c)=i\Im Z(b)$, where $c=i\Im b\in\partial H$.

Theorem 2.1. *Let $Z(s)$ be a positive real function and it is also an analytic function at the point $s=c=i\Im b\in\partial H$ of the imaginary axis with $Z(c)=i\Im Z(b)$. Then*

$$|Z'(c)|\geq\frac{\Re Z\left(b\right)}{\Re b}. \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$Z(s)=\frac{s\Re Z\left(b\right)+i\Im\left(\overline{b}Z\left(b\right)\right)}{\Re b}.$$

Proof. Let

$$w(z)=\frac{Z\left(\frac{b+\overline{b}z}{1-z}\right)-Z(b)}{Z\left(\frac{b+\overline{b}z}{1-z}\right)+\overline{Z(b)}},\ z=\frac{s-b}{s+\overline{b}}. \tag{2.2}$$

$w(z)$ is an analytic function in U, $f(0)=0$ and $|w(z)|<1$ for $|z|<1$. Also, for $s=c=i\Im b$ (at the point $s=c=i\Im b$ of the imaginary axis), for $z_0=-1\in\partial U$, we take

$$\begin{aligned} w(-1) &= \frac{Z(\frac{b-\overline{b}}{2})-Z(b)}{Z(\frac{b-\overline{b}}{2})+\overline{Z(b)}}=\frac{Z(i\Im b)-Z(b)}{Z(i\Im b)+\overline{Z(b)}} \\ &= \frac{i\Im Z(b)-Z(b)}{i\Im Z(b)+\overline{Z(b)}},\ Z(c)=i\Im Z(b) \end{aligned}$$

and

$$|w(-1)|=1.$$

Therefore, the function $w(z)$ satisfies the conditions of the schwarz lemma at the boundary. If we take the derivative of both sides of equation (2.2), we obtain

$$w'(z)=\frac{\frac{b+\overline{b}}{(1-z)^2}Z'\left(\frac{b+\overline{b}z}{1-z}\right)\left(Z(b)+\overline{Z(b)}\right)}{\left(Z\left(\frac{b+\overline{b}z}{1-z}\right)+\overline{Z(b)}\right)^2}$$

and

$$1\leq|w'(-1)|=\frac{\frac{b+\overline{b}}{4}\left|Z'\left(\frac{b-\overline{b}}{2}\right)\right|\left(Z(b)+\overline{Z(b)}\right)}{\left|Z\left(\frac{b-\overline{b}}{2}\right)+\overline{Z(b)}\right|^2}.$$

Since $Z(c)=i\Im Z(b)$, $c=i\Im b$, we take $Z(i\Im b)=i\Im Z(b)$ and

$$Z(\frac{b-\overline{b}}{2})=i\Im Z(b)=\frac{Z(b)-\overline{Z(b)}}{2}.$$

Thus, we obtain

$$1\leq\frac{b+\overline{b}}{Z(b)+\overline{Z(b)}}\left|Z'\left(i\Im b\right)\right|$$

and

$$|Z'(c)|\geq\frac{\Re Z\left(b\right)}{\Re b}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$Z(s)=\frac{s\Re Z\left(b\right)+i\Im\left(\overline{b}Z\left(b\right)\right)}{\Re b}$$

Then

$$Z'(s)=\frac{\Re Z\left(b\right)}{\Re b}$$

and

$$Z'(c) = \frac{\Re Z\left(b\right)}{\Re b}.$$

□

The inequality (2.1) can be strengthened as below by taking into account $a_1 = Z'(b)$ which is first coefficient in the expansion of the function $Z(s) = Z(b) + a_1\left(s-b\right) + a_2\left(s-b\right)^2 + ...$

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|Z'(c)\right| \geq \frac{1}{\Re b}\left(\frac{2\left(\Re Z\left(b\right)\right)^2}{\Re Z\left(b\right) + \left|Z'\left(b\right)\right|\Re b}\right). \tag{2.3}$$

In addition, the result is sharp and the extremal function is

$$Z(s) = \frac{s\Re Z\left(b\right) + i\Im\left(\bar{b}Z\left(b\right)\right)}{\Re b}.$$

Proof. Let $w(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.3), we take

$$\begin{aligned} \frac{2}{1+\left|w'(0)\right|} &\leq \left|w'(-1)\right| = \frac{b+\overline{b}}{Z(b)+\overline{Z(b)}}\left|Z'\left(i\Im b\right)\right| \\ &= \frac{\Re b}{\Re Z\left(b\right)}\left|Z'\left(c\right)\right|. \end{aligned}$$

Since

$$w'(z) = \frac{\frac{b+\bar{b}}{(1-z)^2}Z'\left(\frac{b+\bar{b}z}{1-z}\right)\left(Z(b)+\overline{Z(b)}\right)}{\left(Z\left(\frac{b+\bar{b}z}{1-z}\right)+\overline{Z(b)}\right)^2},$$

$$w'(0) = \frac{\left(b+\overline{b}\right)Z'\left(b\right)}{Z\left(b\right)+\overline{Z(b)}} = \frac{\Re b}{\Re Z\left(b\right)}Z'\left(b\right)$$

and

$$\left|w'(0)\right| = \frac{\Re b}{\Re Z\left(b\right)}\left|Z'\left(b\right)\right|,$$

we obtain

$$\frac{2}{1+\frac{\Re b}{\Re Z(b)}\left|Z'\left(b\right)\right|} \leq \frac{\Re b}{\Re Z\left(b\right)}\left|Z'\left(c\right)\right|$$

and

$$\left|Z'(c)\right| \geq \frac{1}{\Re b}\left(\frac{2\left(\Re Z\left(b\right)\right)^2}{\Re Z\left(b\right) + \left|Z'\left(b\right)\right|\Re b}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$Z(s) = \frac{s\Re Z\left(b\right) + i\Im\left(\bar{b}Z\left(b\right)\right)}{\Re b}.$$

Then, we take

$$Z'(c) = \frac{\Re Z\left(b\right)}{\Re b}.$$

On the other hand, we obtain

$$Z(b)+a_1(s-b)+a_2(s-b)^2+...=\frac{s\Re Z(b)+i\Im\left(\bar{b}Z(b)\right)}{\Re b},$$

$$\begin{aligned} a_1(s-b)+a_2(s-b)^2+... &= \frac{s\Re Z(b)+i\Im\left(\bar{b}Z(b)\right)}{\Re b} \\ &\quad -Z(b) \\ &= \frac{(s-b)\left(Z(b)+\overline{Z(b)}\right)}{b+\bar{b}} \end{aligned}$$

and

$$a_1+a_2(s-b)+...=\frac{\Re Z(b)}{\Re b}.$$

Passing to limit in the last equality yields

$$|a_1|=|Z'(b)|=\frac{\Re Z(b)}{\Re b}.$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{\Re b}\left(\frac{2(\Re Z(b))^2}{\Re Z(b)+|Z'(b)|\Re b}\right) &= \frac{1}{\Re b}\left(\frac{2(\Re Z(b))^2}{\Re Z(b)+\frac{\Re Z(b)}{\Re b}\Re b}\right) \\ &= \frac{\Re Z(b)}{\Re b}. \end{aligned}$$

□

In the following theorem, inequality (2.3) has been strenghened by adding the consecutive term a_2 of $Z(s)=Z(b)+a_1(s-b)+a_2(s-b)^2+....$ function.

Theorem 2.3. *Let $Z(s)$ be a positive real function and it is also an analytic function at the point $s=c=i\Im b\in\partial H$ of the imaginary axis with $Z(c)=i\Im Z(b)$. Then*

$$|Z'(c)|\geq\frac{\Re Z(b)}{\Re b}\left(1+\frac{2(\Re Z(b)-|a_1\Re b|)^2}{(\Re Z(b))^2-|a_1\Re b|^2+\Re b\Re Z(b)\left|a_1\Re Z(b)+\Re b\left(2a_2+a_1^2\right)\right|}\right). \tag{2.4}$$

Moreover, the equality in (2.4) occurs for the function

$$Z(s)=\frac{Z(b)-\left(\frac{s-b}{s+\bar{b}}\right)^2\overline{Z(b)}}{1+\left(\frac{s-b}{s+\bar{b}}\right)^2}.$$

Proof. Let $w(z)$ be as in (2.2) and also $m(z)=z$, $z\in U$. By the maximum principle for each $z\in U$, we have $|w(z)|\leq|m(z)|$. Therefore

$$\lambda(z)=\frac{w(z)}{m(z)}$$

is a holomorphic function in U and $|\lambda(z)|<1$ for $|z|<1$. In particular, we have

$$|\lambda(0)|=\frac{|a_1|\Re b}{\Re Z(b)}$$

and

$$|\lambda'(0)| = \frac{\Re b}{\Re Z\left(b\right)}\left|a_1 \Re Z\left(b\right) + \Re b\left(2a_2 + a_1^2\right)\right|.$$

Moreover, it can be seen that

$$\frac{z_0 w'(z_0)}{w(z_0)} = |w'(z_0)| \geq |m'(z_0)| = \frac{z_0 m'(z_0)}{m(z_0)}.$$

Consider the auxiliary function

$$\xi(z) = \frac{\lambda(z) - \lambda(0)}{1 - \overline{\lambda(0)}\lambda(z)}$$

which is analytic in the unit disc U. Furthermore, $|\xi(z)| < 1$ for $z \in U$, $\xi(0) = 0$ and $|\xi(-1)| = 1$, for $z_0 = -1 \in \partial U$. From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1 + |\xi'(0)|} &\leq |\xi'(-1)| = \frac{1 - |\lambda(0)|^2}{\left|1 - \overline{\lambda(0)}\lambda(-1)\right|^2}|\lambda'(-1)| \\
&\leq \frac{1 + |\lambda(0)|}{1 - |\lambda(0)|}\left|\frac{w'(-1)}{m(-1)} - \frac{w(-1)m'(-1)}{\left(m(-1)\right)^2}\right| \\
&\leq \frac{1 + |\lambda(0)|}{1 - |\lambda(0)|}\left(|w'(-1)| - |m'(-1)|\right).
\end{aligned}$$

Since

$$\xi'(z) = \frac{1 - |\lambda(0)|^2}{\left(1 - \overline{\lambda(0)}\lambda(z)\right)^2}\lambda'(z),$$

$$\begin{aligned}
|\xi'(0)| &= \frac{|\lambda'(0)|}{1 - |\lambda(0)|^2} = \frac{\frac{\Re b}{\Re Z(b)}\left|a_1 \Re Z\left(b\right) + \Re b\left(2a_2 + a_1^2\right)\right|}{1 - \left|\frac{a_1 \Re b}{\Re Z(b)}\right|^2} \\
&= \Re b \Re Z\left(b\right)\frac{\left|a_1 \Re Z\left(b\right) + \Re b\left(2a_2 + a_1^2\right)\right|}{\left(\Re Z\left(b\right)\right)^2 - |a_1 \Re b|^2},
\end{aligned}$$

$$|w'(-1)| = \frac{\Re b}{\Re Z\left(b\right)}\left|Z'\left(c\right)\right|$$

and

$$|m'(-1)| = 1$$

we obtain

$$\frac{2}{1 + \Re b \Re Z(b)\frac{\left|a_1 \Re Z(b) + \Re b\left(2a_2 + a_1^2\right)\right|}{(\Re Z(b))^2 - |a_1 \Re b|^2}}$$

$$\leq \frac{1 + \frac{|a_1|\Re b}{\Re Z(b)}}{1 - \frac{|a_1|\Re b}{\Re Z(b)}}\left(\frac{\Re b}{\Re Z(b)}\left|Z'\left(c\right)\right| - 1\right),$$

$$\frac{2\left((\Re Z(b))^2 - |a_1 \Re b|^2\right)}{(\Re Z(b))^2 - |a_1 \Re b|^2 + \Re b \Re Z(b)\left|a_1 \Re Z(b) + \Re b\left(2a_2 + a_1^2\right)\right|}$$

$$\leq \frac{\Re Z(b) + |a_1|\Re b}{\Re Z(b) - |a_1|\Re b}\left(\frac{\Re b}{\Re Z(b)}\left|Z'\left(c\right)\right| - 1\right)$$

and

$$|Z'(c)| \geq \frac{\Re Z(b)}{\Re b}$$
$$\left(1 + \frac{2\left(\Re Z(b) - |a_1 \Re b|\right)^2}{\left(\Re Z(b)\right)^2 - |a_1 \Re b|^2 + \Re b \Re Z(b) \left|a_1 \Re Z(b) + \Re b \left(2a_2 + a_1^2\right)\right|}\right).$$

Now, we shall show that the inequality (2.4) is sharp. Let

$$Z\left(\frac{b + \overline{b}z}{1 - z}\right) = \frac{Z(b) - z^2 \overline{Z(b)}}{1 + z^2}, \ z = \frac{s - b}{s + \overline{b}}.$$

If we take the derivative of both sides of last equation, we obtain

$$\frac{b + \overline{b}}{(1 - z)^2} Z'\left(\frac{b + \overline{b}z}{1 - z}\right)$$
$$= \frac{-2z\overline{Z(b)}\left(1 + z^2\right) - 2z\left(Z(b) - z^2\overline{Z(b)}\right)}{\left(1 + z^2\right)^2}.$$

For $z = -1$, we have

$$Z'(c) = 2\frac{\Re Z(b)}{\Re b}, \ c = i\Im b.$$

On the other hand, we obtain

$$Z(b) + a_1\left(\frac{(b + \overline{b})z}{1 - z}\right) + a_2\left(\frac{(b + \overline{b})z}{1 - z}\right)^2 + ...$$
$$= \frac{Z(b) - z^2\overline{Z(b)}}{1 + z^2}, \ z = \frac{s - b}{s + \overline{b}},$$
$$a_1\left(\frac{(b + \overline{b})z}{1 - z}\right) + a_2\left(\frac{(b + \overline{b})z}{1 - z}\right)^2 + ... = -z^2\frac{Z(b) + \overline{Z(b)}}{1 + z^2}$$

and

$$a_1\left(\frac{(b + \overline{b})}{1 - z}\right) + a_2\left(\frac{(b + \overline{b})}{1 - z}\right)^2 z + ... = -z\frac{Z(b) + \overline{Z(b)}}{1 + z^2}.$$

Passing to limit in the last equality yields $a_1 = 0$. Similarly, using straightforward calculations, we take $|a_2| = \frac{\Re Z(b)}{2(\Re b)^2}$. Therefore, we get

$$\frac{\Re Z(b)}{\Re b}\left(1 + \frac{2(\Re Z(b) - |a_1 \Re b|)^2}{(\Re Z(b))^2 - |a_1 \Re b|^2 + \Re b \Re Z(b)\left|a_1 \Re Z(b) + \Re b\left(2a_2 + a_1^2\right)\right|}\right)$$
$$= \frac{\Re Z(b)}{\Re b}\left(1 + \frac{2(\Re Z(b))^2}{(\Re Z(b))^2 + \Re b \Re Z(b)\left|\Re b 2\frac{\Re Z(b)}{2(\Re b)^2}\right|}\right)$$
$$= 2\frac{\Re Z(b)}{\Re b}.$$

□

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