Gulf Journal of Mathematics ISSN: 2309-4966
Vol 20, Issue 2 (2025) 275-287 https://doi.org/10.56947/gjom.v20i2.3169

GAMMA-K DISTANCE AND FIXED POINT THEOREMS

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Abstract. The objective of this manuscript is to present distance mappings, particularly gamma-k distance mappings. By employing these mappings, we can successfully implement considerable contractions on a mapping $f : \mathcal{Z} \to \mathcal{Z}$. To illustrate the significance and practical relevance of our research, we provide a noteworthy example and application. Our findings expand and generalize existing results in the literature.

Keywords. fixed point, gamma distance, gamma-k distance mapping, generalized contraction, fixed point theory, nonlinear analysis.
2020 Mathematics Subject Classification. Primary 47H10; Secondary 54H25

1. Introduction and preliminaries

The Banach principle [1] is a pivotal result in the realm of fixed point theory, asserting that every contraction mapping has a unique fixed point. Following this foundational work, numerous mathematicians have examen various extensions of Banach's theorem across different contexts. For example, Jleli and Samet [2] introduced an innovative concept of generalized metric spaces, which allows for the expansion of several well-established fixed point theorems, including the Banach contraction principle and other types of contractions. The notion of Proinov-C-contraction mapping was initially introduced in [3] and later examined within the framework of b-metric spaces, recognized as one of the most fascinating abstract constructs in mathematics. In fact, the study of fixed point theory is extensive, as highlighted by the numerous references available in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] for those seeking further insights. Moreover, fixed point theory has garnered significant interest in the domain of differential equations due to its numerous applications, as illustrated in works such as [23, 24, 25, 26, 27, 28].

In the study conducted by Bataihah et al. [29], the authors introduce the concept of Gamma distance and illustrate several fixed point theorems that are based on this idea.

The concept of Gamma distance was articulated in the following manner.

Date: Received: Jun 21, 2025; Accepted: Jul 27, 2025.
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Definition 1.1 ([29])**.** Let d be a metric on a set $\mathcal{Z}$. A function

$$\gamma : [0,\infty) \times \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$$

is called a γ-*distance* over $(\mathcal{Z}, d)$ if it satisfies the following conditions:

(γ_1) For all $t \geq 0$ and all $\vartheta, v \in \mathcal{Z}$, it holds that

$$\gamma(t,\vartheta,v) \geq t \cdot d(\vartheta,v).$$

(γ_2) For every sequences $(\vartheta_n), (v_n) \subset \mathcal{Z}$ and $(t_n) \subset (0,\infty)$, if

$$d(\vartheta_n, v_n) \to \xi > 0 \quad \text{and} \quad t_n \to t > 0,$$

then

$$\liminf_{n\to\infty} \gamma(t_n, \vartheta_n, v_n) > \xi t.$$

In the subsequent sections of this manuscript, we will refer to $\mathcal{Z}$ as a non-empty set, and $(\mathcal{Z}, d)$ will denote a metric space.

The subsequent examples were presented in [29].

Example 1.2. [29]Let $\mathcal{Z} = \mathbb{R}$ and let $d : \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ be the usual metric on $\mathcal{Z}$; i.e. $d(\vartheta, v) = |\vartheta - v|$. Then $\gamma : [0, +\infty) \times \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ which defined by

$$\gamma(t,\vartheta,v) = \alpha\ t\ (\ |\vartheta| + |v|\ ),$$

where $\alpha > 1$ is γ-distance on $(\mathcal{Z}, d)$.

In the subsequent example, the functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5,$ and γ_6 are regarded as being defined from $[0, +\infty) \times \mathcal{Z} \times \mathcal{Z}$ to $[0, +\infty)$.

Example 1.3. [29] On $(\mathcal{Z}, d)$, let $\phi : [0,\infty) \to [0,\infty)$ is continuous function with $\phi(a) > 0$ for $a > 0$. Then, $\gamma_1, \gamma_2, \gamma_3, \gamma_4\ \gamma_5$and γ_6 are γ-distance over $(\mathcal{Z}, d)$.

(1) $\gamma_1(t,\vartheta,v) = t\ d(\vartheta,v) + \phi(d(\vartheta,v))$,
(2) $\gamma_2(t,\vartheta,v) = t\ d(\vartheta,v) + \phi(t)$,
(3) $\gamma_3(t,\vartheta,v) = e^t d(\vartheta,v)$,
(4) $\gamma_4(t,\vartheta,v) = \dfrac{(t+\varepsilon)^2 + [d(\vartheta,v)]^2}{2}$, $\varepsilon > 0$,
(5) $\gamma_5(t,\vartheta,v) = t\ [1 + d(\vartheta,v)]^t d(\vartheta,v)$,
(6) $\gamma_6(t,\vartheta,v) = \dfrac{t^2+1}{2}\ d(\vartheta,v)$,

Bataihah et al. [29] presented the concept of weak non-expansive γ-contractions, subsequently demonstrating that each weak non-expansive γ-contraction possesses a unique fixed point.

Definition 1.4 ([29])**.** Suppose that γ is a γ-distance over the metric space $(\mathcal{Z}, d)$. A self-mapping $f : \mathcal{Z} \to \mathcal{Z}$ is said to be a weak non-expansive γ-contraction** if

$$\vartheta \neq v \implies \gamma(d(f\vartheta, fv), f\vartheta, fv) \leq [d(\vartheta,v)]^2. \tag{1.1}$$

Theorem 1.5. [29] *Suppose that $(\mathcal{Z}, d)$ is complete and there is γ-distance γ on $(\mathcal{Z}, d)$. Assume that $f : \mathcal{Z} \to \mathcal{Z}$ is weak non-expansive γ-contraction. Then $F_f$is singleton.*

The primary aim of this manuscript is to extend the concept of Gamma distance, as introduced by Bataihah et al. [29], and to explore its relevance within fixed point theory. This will be achieved by presenting fixed point results, along with various examples and applications.

While the transition from γ-distance to γ^k-distance may appear incremental, this generalization enables nonlinear scaling of contraction conditions, which is essential for analyzing problems with power-law dependencies. The exponent k provides a unified parameter to interpolate between classical Banach contractions ($k = 0$) and non-expansive mappings ($k \geq 1$), filling a gap in the existing literature on generalized metrics.

2. GAMMA-K DISTANCE

Definition 2.1. Let d be a metric on $\mathcal{Z}$. A function $\gamma : [0, \infty) \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is said to be gamma-k (shortly γ^k) distance over $(\mathcal{Z}, d)$ if γ satisfying:

(γ_1) For all $t \geq 0$, $\gamma(t, \vartheta, v) \geq t^k d(\vartheta, v)$,
(γ_2) for each sequences $(\vartheta_n), (v_n)$ in $\mathcal{Z}$ and (t_n) in $(0, \infty)$, we have $d(\vartheta_n, v_n) \to \xi > 0$ and $t_n \to t > 0$ imply $\liminf_{n\to\infty} \gamma(t_n, \vartheta_n, v_n) > t^k \xi$.

Example 2.2. On $(\mathcal{Z}, d)$, let $\phi : [0, \infty) \to [0, \infty)$ is continuous function with $\phi(a) > 0$ for $a > 0$. Then, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 are γ^k distance over $(\mathcal{Z}, d)$.

(1) $\gamma_1(t, \vartheta, v) = t^k\ d(\vartheta, v) + \phi(d(\vartheta, v))$,
(2) $\gamma_2(t, \vartheta, v) = t^k\ d(\vartheta, v) + \phi(t)$,
(3) $\gamma_3(t, \vartheta, v) = e^{t^k} d(\vartheta, v)$,
(4) $\gamma_4(t, \vartheta, v) = \dfrac{(t + \varepsilon)^{2k} + [d(\vartheta, v)]^2}{2}$, $\varepsilon > 0$,
(5) $\gamma_5(t, \vartheta, v) = \dfrac{t^{2k} + 1}{2}\ d(\vartheta, v)$,

Remark 2.3. The γ^k-distance framework provides a unification of several generalized metric structures. For instance any b-metric space (Z, d) with constant $s \geq 1$, the choice

$$\gamma(t, \vartheta, v) = t^k \cdot d(\vartheta, v) \tag{2.1}$$

satisfies γ^k-distance properties where:

(1) Condition (γ_1) reduces to the b-metric triangle inequality
(2) Condition (γ_2) becomes $\liminf t_n \cdot d(\vartheta_n, v_n) > t\xi$ which holds for b-convergent sequences
(3) The contraction condition (2.3) recovers the standard b-metric contraction when $k = 1$

Remark 2.4. One key advantage over classical frameworks lies in the use of t^k in the γ^k-distance, which introduces a nonlinear scaling mechanism. Unlike traditional linear contractions, this approach allows the contraction strength to vary depending on the magnitude of the distance. This provides finer control over convergence behavior, particularly in situations where uniform contraction is too restrictive.

2.1. **Fixed point results for weak non-expansive γ^k -contractions.**

Definition 2.5. Suppose there is γ over $(\mathcal{Z}, d)$. A self mapping $f : \mathcal{Z} \to \mathcal{Z}$ is said to be weak non-expansive γ^k-contraction if

$$\vartheta \neq v \Longrightarrow \gamma(d(f\vartheta, fv), f\vartheta, fv) \leq d^{k+1}(\vartheta, v). \tag{2.2}$$

Remark 2.6. If $f : \mathcal{Z} \to \mathcal{Z}$ is weak non-expansive γ-contraction, then for each $\vartheta, v \in \mathcal{Z}$, we have

$$d(f\vartheta, fv) \leq d(\vartheta, v).$$

Note that while the condition $d(f\vartheta, fv) \leq d(\vartheta, v)$ resembles non-expansiveness, the γ^k framework allows sharper control via the parameter k. For instance, when $k \in (0, 1)$, we obtain a continuum between Banach and non-expansive contractions, enabling tailored analyses for fractional operators (Corollary 3.2).

Lemma 2.7. *Suppose $f : \mathcal{Z} \to \mathcal{Z}$ is weak non-expansive γ^k-contraction. If $\vartheta, v \in F_f$, then $\vartheta = v$.*

Proof. The proof follows from Remark 2.6. □

Lemma 2.8. *Suppose f is weak γ^k-contraction, and $\vartheta_0 \in \mathcal{Z}$. Then for the Picard sequence (ϑ_n) generated by f at ϑ_0, if $\vartheta_n \neq \vartheta_{n+1}$ for each $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} d(\vartheta_n, \vartheta_{n+1}) = 0.$$

Proof. Since $\vartheta_n \neq \vartheta_{n+1}$, by Condition (2.2)

$$\gamma(d(\vartheta_n, \vartheta_{n+1}), \vartheta_n, \vartheta_{n+1}) \leq d^{k+1}(\vartheta_{n-1}, \vartheta_n)]$$

From Remark 2.6, we have

$$d(\vartheta_n, \vartheta_{n+1}) \leq d(\vartheta_{n-1}, \vartheta_n).$$

Hence $(d(\vartheta_n, \vartheta_{n+1}) : \ n \in \mathbb{N})$ is non-increasing in $(0, \infty)$. So, there is $r \geq 0$ such that $\lim_{n\to\infty} d(\vartheta_n, \vartheta_{n+1}) = r$. Suppose on the contrary $r > 0$. Let $t_n = d(\vartheta_{n-1}, \vartheta_n)$. Then, $t_n \to r$. Therefore, by γ_2 of Definition 1.1, we have

$$r^{k+1} < \liminf_{n\to\infty} \gamma(d(\vartheta_{n-1}, \vartheta_n), \vartheta_n, \vartheta_{n+1}) \leq r^{k+1},$$

a contradiction. Hence we get the result. □

Theorem 2.9. *Suppose that $(\mathcal{Z}, d)$ is complete and there is γ^k-distance γ on $(\mathcal{Z}, d)$. Assume that $f : \mathcal{Z} \to B$ is weak non-expansive γ-contraction. Then F_f consists of only one element.*

Proof. Let $\vartheta_0 \in \mathcal{Z}$ be arbitrary, and consider the sequence (ϑ_n) generated by f at ϑ_0. If there is $s \in \mathbb{N}$ such that $\vartheta_s = \vartheta_{s+1}$, then $\vartheta_s \in F_f$. So assume that for each $n \in \mathbb{N}$, $\vartheta_n \neq \vartheta_{n+1}$. Now, we claim that (ϑ_n) is a Cauchy sequence in $(\mathcal{Z}, d)$. Suppose to the contrary; that is (ϑ_n) is not Cauchy. Therefore, depending on Lemma 2.8 there is $\epsilon > 0$ and two sub-sequences (ϑ_{n_k}) and (ϑ_{m_k}) of (ϑ_n) such that

$$\lim_{k\to\infty} d(\vartheta_{n_k}, \vartheta_{m_k}) = \epsilon. \tag{2.3}$$

and
$$\lim_{k\to\infty} d(\vartheta_{n_k-1}, \vartheta_{m_k-1}) = \epsilon. \tag{2.4}$$
Now, set $t_k = d(\vartheta_{n_k-1}, \vartheta_{m_k-1})$, $a_k = \vartheta_{n_k}$ and $b_k = \vartheta_{m_k}$. Then, $\lim\limits_{k\to\infty} t_k = \lim\limits_{k\to\infty} d(a_k, b_k) = \epsilon > 0$. So, de Lemma 2.8, (γ_2) and condition 2.2, we get
$$\epsilon^{k+1} < \liminf_{k\to\infty} \gamma(t_k, a_k, b_k) \leq \epsilon^{k+1},$$
a contradiction. Hence, (ϑ_n) is Cauchy, so there is $\vartheta \in \mathcal{Z}$ such that (ϑ_n) converges to ϑ. Now, by Remark 2.6, we have
$$d(f\vartheta, \vartheta_{n+1}) \leq d(\vartheta, \vartheta_n).$$
So, if we pass $n \to +\infty$, we get $\vartheta_n \to f\vartheta$, and so, then by uniqueness of the limit, we get $\vartheta = f\vartheta$.

The uniqueness follows from Lemma 2.7. □

Utilizing Example 2.2 and Theorems 2.9, 2.15, we get the following results:

Corollary 2.10. *Suppose $(\mathcal{Z}, d)$ is complete and $\phi : [0,\infty) \to [0,\infty)$ be continuous with $\phi(a) > 0$ for $a > 0$. Suppose $f : \mathcal{Z} \to \mathcal{Z}$ be a self map satisfies*
$$\vartheta \neq v \Longrightarrow d^{k+1}(f\vartheta, fv) \leq d^{k+1}(\vartheta, v) - \phi(d(f\vartheta, fv)). \tag{2.5}$$
Then F_f is singleton.

Corollary 2.11. *Suppose $(\mathcal{Z}, d)$ is complete and $\varepsilon > 0$. Suppose $f : \mathcal{Z} \to \mathcal{Z}$ be a self map satisfies*
$$\vartheta \neq v \Longrightarrow d^{k+1}(f\vartheta, fv) \leq d^{k+1}(\vartheta, v) - \varepsilon. \tag{2.6}$$
Then F_f is singleton.

Corollary 2.12. *Suppose $(\mathcal{Z}, d)$ is complete and $\varepsilon > 0$. Suppose $f : \mathcal{Z} \to \mathcal{Z}$ be a self map satisfies*
$$\vartheta \neq v \Longrightarrow d(f\vartheta, fv)\ (1 + d^{2k}(f\vartheta, fv)) \leq d^{k+1}(\vartheta, v). \tag{2.7}$$
Then F_f is singleton.

Example 2.13. Suppose $\mathcal{Z} = \{0, 1, ..., 10\}$, define a self mapping $f : \mathcal{Z} \to \mathcal{Z}$ by :
$$f\vartheta = \begin{cases} 0, & \vartheta \in \{0, 1, 2\}; \\ 1, & \vartheta \in \{3, 4, \ldots, 7\}; \\ 2, & \vartheta \in \{8, 9, 10\}. \end{cases}$$
Then F_f consists of only one element.
To prove this, we select γ_5 in Example 2.2 as γ^k function i.e. $\forall\ \vartheta_1, \vartheta_2 \in B$, we have:
$$\gamma(d(f\vartheta_1, f\vartheta_2), f\vartheta_1, f\vartheta_2) = \frac{d^{2k}(f\vartheta_1, f\vartheta_2) + 1}{2} d(f\vartheta_1, f\vartheta_2),$$

and $d(\vartheta_1,\vartheta_2)=|\vartheta_1-\vartheta_2|$ with $k\geq 1$, so we want to show:

$$\left(\frac{|f\vartheta_1-f\vartheta_2|^{2k}+1}{2}\right)|f\vartheta_1-f\vartheta_2|\leq|\vartheta_1-\vartheta_2|^{k+1}.$$

For all $\vartheta\in\mathcal{Z}$ with $\vartheta\neq f\vartheta$ we have the following Cases:
Case (1): If $\vartheta_1,\vartheta_2\in\{0,1,2\}$ *or* $\vartheta_1,\vartheta_2\in\{3,4,\ldots,7\}$ *or* $\vartheta_1,\vartheta_2\in\{8,9,10\}$, then

$$\gamma(d(f\vartheta_1,f\vartheta_2),f\vartheta_1,f\vartheta_2))=0\leq|\vartheta_1-\vartheta_2|^{k+1}.$$

Case (2): If $\vartheta_1\in\{0,1,2\}$ & $\vartheta_2\in\{3,4,\ldots,7\}$, then

$$\gamma(d(f\vartheta_1,f\vartheta_2),f\vartheta_1,f\vartheta_2)=\frac{1^{2k}+1}{2}(1)=1\leq|\vartheta_1-\vartheta_2|^{k+1}.$$

Case (3): If $\vartheta_1\in\{0,1,2\}$ & $\vartheta_2\in\{8,9,10\}$, then

$$\gamma(d(f\vartheta_1,f\vartheta_2),f\vartheta_1,f\vartheta_2)=\frac{2^{2k}+1}{2}(2)\leq 6^{k+1}\leq|\vartheta_1-\vartheta_2|^{k+1}.$$

Case (4): If $\vartheta_1\in\{3,4,\ldots,7\}$ & $\vartheta_2\in\{8,9,10\}$, then

$$\gamma(d(f\vartheta_1,f\vartheta_2),f\vartheta_1,f\vartheta_2)=\frac{1^{2k}+1}{1}(2)=1\leq|\vartheta_1-\vartheta_2|^{k+1}.$$

The remaining cases will follow the same pattern. Hence, all conditions of weak non-expansive γ^k-contraction are met by f. Theorem 2.9 confirms that F_f consists of only one element.

Example 2.14. Let $\mathcal{Z}=[0,1]$ and let $f:\mathcal{Z}\to\mathcal{Z}$ be defined by $f\vartheta=1-\dfrac{\vartheta}{\sqrt{2}B+\vartheta^m}$, where $m<B^2-\sqrt{2}B+1$. Then f has a unique fixed point.

Proof. Let $\mathcal{Z}=[0,1]$ provided with the usual metric $d(\vartheta_1,\vartheta_2)=|\vartheta_1-\vartheta_2|$. Then $f:\mathcal{Z}\to\mathcal{Z}$ which defined by $f\vartheta=1-\dfrac{\vartheta}{\sqrt{2}B+\vartheta^m}$, where $m<B^2-\sqrt{2}B+1$. is a self map. Let $\gamma^k:[0,\infty)\times\mathcal{Z}\times\mathcal{Z}\to[0,\infty)$ be defined by

$$\gamma(t,\vartheta,v)=\frac{(t+\varepsilon)^{2k}+[d(\vartheta,v)]^2}{2},\ \ k,\varepsilon>0.$$

Now, for each $\vartheta_1,\vartheta_2\in[0,1]$ with $\vartheta_1\neq\vartheta_2$, we have

$$\begin{aligned}
d(f\vartheta_1, f\vartheta_2) &= |f\vartheta_1 - f\vartheta_2| \\
&= \left|1 - \frac{\vartheta_1}{\sqrt{2}B + \vartheta_1^m} - (1 - \frac{\vartheta_2}{\sqrt{2}B + \vartheta_2^m})\right| \\
&= \left|\frac{\vartheta_1}{\sqrt{2}B + \vartheta_1^m} - \frac{\vartheta_2}{\sqrt{2}B + \vartheta_2^m}\right| \\
&= \frac{1}{(\sqrt{2}B + \vartheta_1^m)(\sqrt{2}B + \vartheta_2^m)}|\sqrt{2}B(\vartheta_1 - \vartheta_2) + \vartheta_1\vartheta_2(\vartheta_1^{m-1} - \vartheta_1^{m-1})| \\
&\leq \frac{1}{(\sqrt{2}B + \vartheta_1^m)(\sqrt{2}B + \vartheta_2^m)}[\sqrt{2}B|\vartheta_1 - \vartheta_2| + |\vartheta_1^{m-1} - \vartheta_2^{m-1}|] \\
&\leq \frac{\sqrt{2}B + m - 1}{2B^2}|\vartheta_1 - \vartheta_2| \\
&= \frac{\sqrt{2}B + m - 1}{2B^2}d(\vartheta_1, \vartheta_2).
\end{aligned}$$

Since $\vartheta_1 \neq \vartheta_2$, then w.l.o. generality, we may assume $0 < \varepsilon \leq \frac{B^2+1-\sqrt{2}B-m}{2B^2}d(\vartheta_1, \vartheta_2)$. letting $C = \sqrt{2}B + m - 1, d = |\vartheta_1 - \vartheta_2|$, and $C' = (B^2 + 1 - \sqrt{2}B - m)d(\vartheta_1, \vartheta_2)$, we hence have

$$\begin{aligned}
\gamma(d(f\vartheta_1, f\vartheta_2), f\vartheta_1, f\vartheta_2) &= \frac{1}{2}\left([d(f\vartheta_1, f\vartheta_2) + \varepsilon]^{2k} + [d(f\vartheta_1, f\vartheta_2)]^2\right) \\
&= \frac{1}{2}\left([\ |f\vartheta_1 - f\vartheta_2| + \varepsilon]^{2k} + |f\vartheta_1 - f\vartheta_2|^2\right) \\
&\leq \frac{1}{2}\left(\left[\frac{\sqrt{2}B + m - 1}{2B^2}|\vartheta_1 - \vartheta_2| + \varepsilon\right]^{2k} + \left[\frac{\sqrt{2}B + m - 1}{2B^2}|\vartheta_1 - \vartheta_2|\right]^2\right) \\
&= \frac{1}{2}\left(\left[\frac{\sqrt{2}B + m - 1}{2B^2}|\vartheta_1 - \vartheta_2| + \varepsilon\right]^{2k} + \left[\frac{\sqrt{2}B + m - 1}{2B^2}|\vartheta_1 - \vartheta_2|\right]^2\right) \\
&= \frac{1}{2}\left(\left[\frac{C}{2B^2}d + \varepsilon\right]^{2k} + \left[\frac{C}{2B^2}d\right]^2\right) \\
&\leq \frac{1}{2}\left(\left[\frac{C}{2B^2}d + \frac{C'}{2B^2}d\right]^{2k} + \left[\frac{C}{2B^2}d\right]^2\right) \quad \text{where } \varepsilon \leq \frac{C'}{2B^2}d \text{ by our choice} \\
&= \frac{1}{2}\left(\left[\frac{C + C'}{2B^2}d\right]^{2k} + \left[\frac{C}{2B^2}d\right]^2\right) \\
&= \frac{d^{2k}}{2}\left(\frac{(C + C')^{2k}}{(2B^2)^{2k}} + \frac{C^2}{(2B^2)^2}d^{2-2k}\right) \\
&\leq \frac{d^{k+1}}{2}\left(\frac{(C + C')^{2k}}{(2B^2)^{2k}}d^{k-1} + \frac{C^2}{(2B^2)^2}d^{1-k}\right) \quad \text{since } d \leq 1
\end{aligned}$$

$$\leq d^{k+1} \underbrace{\left(\frac{(C+C')^{2k}+C^2}{2(2B^2)^{2k}} \right)}_{K} \quad \text{where } K<1 \text{ when } m<B^2-\sqrt{2}B+1$$
$$< d^{k+1} = [d(\vartheta_1,\vartheta_2)]^{k+1}.$$

Consequently, f fulfills every requirement outlined in of Theorem 2.9 and Theorem 2.9 ensures that F_f has only one element. □

The same reasoning as in Theorem 2.9 leads us to the following outcome.

Theorem 2.15. *Suppose $(\mathcal{Z},d)$ is complete and there is $f:\mathcal{Z}\to\mathcal{Z}$ such that f satisfies*

$$\vartheta \neq v \Longrightarrow \gamma(d(\vartheta,v), f\vartheta, fv) \leq [d(\vartheta,v)]^{k+1}. \tag{2.8}$$

Then F_f is a singleton.

Employing Example 2.2 and Theorems 2.15, we get the following results:

Corollary 2.16. *Suppose $(\mathcal{Z},d)$ is complete and $\phi:[0,\infty)\to[0,\infty)$ be continuous with $\phi(a)>0$ for $a>0$. Suppose $f:\mathcal{Z}\to\mathcal{Z}$ be a self map satisfies*

$$\vartheta \neq v \Longrightarrow d(f\vartheta,fv) \leq d(\vartheta,v) - \frac{\phi(d(f\vartheta,fv))}{d^k(\vartheta,v)}. \tag{2.9}$$

Then F_f is singleton.

Corollary 2.17. *Suppose $(\mathcal{Z},d)$ is complete and $\varepsilon>0$. Suppose $f:\mathcal{Z}\to\mathcal{Z}$ be a self map satisfies*

$$\vartheta \neq v \Longrightarrow d(f\vartheta,fv) \leq d(\vartheta,v) - \frac{\phi(d(\vartheta,v))}{d^k(\vartheta,v)}. \tag{2.10}$$

Then F_f is singleton.

Corollary 2.18. *Suppose $(\mathcal{Z},d)$ is complete and $\varepsilon>0$. Suppose $f:\mathcal{Z}\to\mathcal{Z}$ be a self map satisfies*

$$\vartheta \neq v \Longrightarrow d(f\vartheta,fv) \leq \frac{2d^{k+1}(\vartheta,v)}{1+d^{2k}(\vartheta,v)}. \tag{2.11}$$

Then F_f is singleton.

3. Application

Let $\mathcal{Z}=C[0,1]$ be the set of all real-valued continuous functions on [0,1] and consider the metric d on $\mathcal{Z}$ as follows.

$$d(L_1,L_2)=\|L_1-L_2\|=\max_{\vartheta\in[0,1]}|L_1(\vartheta)-L_2(\vartheta)|, \text{ for all } L_1,L_2\in\mathcal{Z}.$$

Then $(\mathcal{Z},d)$ is a complete metric space.

Consider the following fractional boundary differential equation

$$\begin{cases} D^\alpha L(\vartheta)+H(\vartheta,L(\vartheta))=0, & 0\leq\vartheta\leq1,\ 1<\alpha<2 \\ L(0)=L(1)=0, \end{cases} \tag{3.1}$$

where D^{α} is the Caputo fractional derivative of order α and $H : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The Greens function associated to (3.1) is defined as:

$$G(\vartheta, v) = \begin{cases} \vartheta(1-v)^{\alpha-1} - (\vartheta - v)^{\alpha-1}, \ 0 \le v \le \vartheta \le 1, \\ \dfrac{\vartheta(1-v)^{\alpha-1}}{\Gamma(\alpha)} \qquad\qquad\qquad , \quad 0 \le \vartheta \le v \le 1. \end{cases}$$

Theorem 3.1. *Suppose that there is $B > 1$ such that for all $\vartheta \in [0,1]$ and $s_1, s_2 \in \mathbb{R}$*

$$|H(\vartheta, s_1) - H(\vartheta, s_2)| \le \frac{B^2-1}{2B^2}|s_1 - s_2|.$$

Then,

$$L(\vartheta) = \int_0^1 G(\vartheta, v)H(v, L(v))dv, \tag{3.2}$$

for all $\vartheta \in [0,1]$ and $L \in \mathcal{Z}$ has a unique solution.

Proof. The solution of (3.2) is equivalent to the fixed point of the operator $S : \mathcal{Z} \to \mathcal{Z}$ which defined as

$$S(L(\vartheta)) = \int_0^1 G(\vartheta, v)H(v, L(v))dv,$$

For any $L_1, L_2 \in \mathcal{Z}$, with $L_1 \neq L_2$, we get

$$\begin{aligned} |S(L_1(\vartheta)) - S(L_2(\vartheta))| &= \left| \int_0^1 G(\vartheta, v)(H(v, L_1(v)) - H(v, L_2(v)))dv \right| \\ &\le \int_0^1 G(\vartheta, v)|H(v, L_1(v)) - H(v, L_2(v))|dv \\ &\le \int_0^1 G(\vartheta, v)\tfrac{B^2-1}{2B^2}|L_1(v) - L_2(v)|dv \\ &\le \tfrac{B^2-1}{2B^2}\|L_1 - L_2\| \int_0^1 G(\vartheta, v)dv \\ &= \tfrac{B^2-1}{2B^2}\|L_1 - L_2\|. \end{aligned}$$

Hence, $\|SL_1 - SL_2\| \le \frac{B^2-1}{2B^2}\|L_1 - L_2\|$.

Define $\gamma : [0,\infty) \times \mathcal{Z} \times \mathcal{Z} \to [0,\infty)$ by

$$\gamma(t, L_1, L_2) = \begin{cases} \frac{1}{2}\left(\left[\ t + \dfrac{1}{B^2-1}\|L_1 - L_2\|\ \right]^{2k} + \|L_1 - L_2\|^2\right), L_1 \neq L_2 \text{ and } t > 0, \\ \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , o.w. \end{cases}$$

So, for $L_1 \neq L_2$, we have

$$\begin{aligned}
\gamma(d(SL_1,SL_2),SL_1,SL_2) &= \tfrac{1}{2}\left(\left[\ \|SL_1-SL_2\| + \frac{1}{B^2-1}\|SL_1-SL_2\|\right]^{2k} + \|SL_1-SL_2\|^2\right)\\
&\leq \tfrac{1}{2}\left(\left[\ \frac{B^2-1}{2B^2}\|L_1-L_2\| + \frac{1}{2B^2}\|L_1-L_2\|\right]^{2k}\right.\\
&\quad\left. + \left(\tfrac{B^2-1}{2B^2}\|L_1-L_2\|\right)^2\right)\\
&= \tfrac{1}{2}\left(\ \left(\frac{1}{2}\right)^{2k}\|L_1-L_2\|^{2k} + \left(\frac{B^2-1}{2B^2}\|L_1-L_2\|\right)^2\right)\\
&\leq \tfrac{1}{2}\left(\ \left(\frac{1}{2}\right)^{2k}\|L_1-L_2\|^{2k} + \left(\frac{1}{2}\right)^2\|L_1-L_2\|^2\right)\\
&\leq \max\{(\tfrac{1}{2})^{2k},(\tfrac{1}{2})^2\}\frac{\|L_1-L_2\|^{2k}+\|L_1-L_2\|^2}{2}\\
&\leq \|L_1-L_2\|^{k+1}\\
&= (d(L_1,L_2))^{k+1}.
\end{aligned}$$

Hence, the desired result is obtained as a consequence of Theorem 2.9.

□

Corollary 3.2. *Under the conditions of Theorem 3.1, if the fractional order α and γ^k-distance parameter k satisfy $\alpha = k+1$, then the solution operator S satisfies:*

$$\gamma(d(SL_1,SL_2),SL_1,SL_2) \leq d^{\alpha}(L_1,L_2)$$

Moreover, this choice explicitly links:

- *The fractional derivative order α in (3.1)*
- *The contraction power k in Definition 2.1*
- *The existence condition $\frac{B^2-1}{2B^2}$ in Theorem 3.1*

4. Conclusion

Fixed point theory presents a wide range of applications, which has garnered significant attention from researchers in the field. Since the introduction of the Banach contraction principle, numerous scholars have proposed various generalizations. In our research, we have introduced γ^k-distance functions as an extension of the Gamma-distance concept initially proposed by Bataihah et al. Furthermore, we have established fixed point results utilizing this novel concept, accompanied by examples and an application that demonstrate its practical relevance. This advancement has the potential to yield further new findings in fixed point theory and its associated applications.

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