

GRADIENT-TYPE NONLINEAR RESONANCE ELLIPTIC SYSTEMS ON THE SIERPIŃSKI GASKET

ABDELKARIM DAHMANI^{1*}, KAOUTAR LAMRINI UAHABI² and SIHAM EL HABIB³

ABSTRACT. This paper investigates resonant gradient-type elliptic systems defined on the Sierpiński gasket. We prove the existence of infinitely many nontrivial weak solutions involving the weak Laplacian operator with zero Dirichlet boundary conditions. The analysis is based on variational methods and critical point theory, suitably adapted to the fractal framework. We exploit the analytic and geometric properties of the gasket, including compact embeddings of the associated energy space. The nonlinear term satisfies resonance conditions linked to the spectrum of the Laplacian. Due to the non-smooth nature of the fractal domain, classical tools require refinement to ensure compactness and coercivity. Our results extend existing existence theorems for gradient systems to fractal domains, revealing new interactions between nonlinear analysis, variational methods, and geometric measure theory. This work enhances the understanding of elliptic problems on self-similar structures and contributes to the broader theory of PDEs in non-Euclidean settings.

Keywords. Sierpiński gasket, Kajikiya's theorem, critical points, resonant elliptic systems, Dirichlet form.

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1. INTRODUCTION

In this work, we investigate a class of Dirichlet gradient systems governed by the following set of equations:

$$\begin{cases} -\Delta u_1(x) = \lambda u_1(x) + \delta u_2(x) + F_{u_1}(x, u_1(x), u_2(x)), & x \in V \setminus V_0, \\ -\Delta u_2(x) = \delta u_1(x) + \gamma u_2(x) + F_{u_2}(x, u_1(x), u_2(x)), & x \in V \setminus V_0, \\ u_1|_{V_0} = u_2|_{V_0} = 0. \end{cases} \quad (1.1)$$

where V denotes the Sierpiński gasket in $\mathbb{R}^{N-1}$ ($N \geq 3$), V_0 is its intrinsic boundary, and Δ is the weak Laplacian defined via Dirichlet forms on this fractal domain. The parameters $\lambda, \delta, \gamma \in \mathbb{R}$ and $F : V \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuously differentiable with gradient $\nabla F = (F_{u_1}, F_{u_2})$.

The system is resonant when $\sigma(A) \cap \sigma(-\Delta) \neq \emptyset$ where $A = \begin{pmatrix} \lambda & \delta \\ \delta & \gamma \end{pmatrix}$ and $\sigma(-\Delta) = \{\theta_k\}_{k=1}^\infty$ are eigenvalues of the fractal Laplacian. While resonant elliptic systems are well-studied on

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* Corresponding author

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Euclidean domains, extending them to fractal geometries like the Sierpiński gasket presents unique challenges that require novel mathematical adaptations. The polynomial growth of the Laplacian spectrum ($\theta_k \sim k^{\ln N / \ln 2}$ [11] versus linear growth in Euclidean domains) fundamentally alters resonance phenomena, while the absence of classical differentiability properties necessitates new approaches to energy estimates and compactness arguments. Furthermore, Sobolev embeddings on fractals yield $H_0^1(V) \hookrightarrow L^r(V)$ only for finite r [16], imposing stricter growth constraints on nonlinearities.

Our work makes three key contributions to overcome these challenges: First, we develop a modified verification of the Palais-Smale condition by leveraging compact embeddings specific to fractal domains, resolving critical compactness issues in the variational framework. Second, we establish new spectral estimates that adapt the matrix resonance condition to accommodate polynomial eigenvalue growth, controlling interaction terms through precise asymptotic analysis of θ_k . Third, we construct specialized finite-dimensional approximations that capture spectral behavior while satisfying resonance constraints, circumventing the lack of classical regularity. These adaptations enable the first successful treatment of gradient systems at resonance on the Sierpiński gasket.

Physically, solutions correspond to steady states in reaction-diffusion processes on fractal media, with applications in heterogeneous materials and biological systems. Building upon the variational framework developed by Costa and Magalhães [8], we employ Kajikiya's critical point theorem [13] to establish infinitely many solutions. The proof navigates fractal-specific obstacles including non-compactness at resonance and the verification of geometric conditions under spectral constraints.

Section 2 formalizes the weak Laplacian and function spaces. Section 3 develops the variational framework for the Euler functional J , with Fréchet differentiability established under:

(SC1) There exist constants $c > 0$ and $0 < \beta < 1$ such that for all $(x, U) \in V \times \mathbb{R}^2$, the gradient of F satisfies the following growth condition:

$$|\nabla F(x, U)| \leq c \left(1 + |U|^\beta\right).$$

(SC2) $F(x, 0) = 0$, for all $x \in V$, and

$$\limsup_{|U| \rightarrow 0} \frac{F(x, U)}{|U|^2} = +\infty \quad \text{uniformly for almost every } x \in V.$$

(SC3) $F(x, U) = F(x, -U)$, $\forall (x, U) \in V \times \mathbb{R}^2$.

Remark 1.1. Condition (SC1) implies that

$$\lim_{|U| \rightarrow \infty} \frac{\nabla F(x, U)}{|U|} = 0.$$

Our main result is presented in the following theorem

Theorem 1.2. *Under resonance condition (R) and assumptions (SC1)-(SC3), system (1.1) admits infinitely many nontrivial weak solutions.*

2. ABSTRACT FRAMEWORK

In this section, we establish essential definitions, propositions, and foundational results required for the subsequent analysis.

The Sierpiński gasket, introduced by the Polish mathematician Waclaw Sierpiński (1882–1969) in [22], is a fundamental object in fractal geometry.

We define the Sierpiński gasket in $\mathbb{R}^{N-1}$ for $N \geq 3$. Let $p_1, p_2, \dots, p_N \in \mathbb{R}^{N-1}$ be points satisfying $|p_i - p_j| = 1$ for $i \neq j$.

For a function $u: V \rightarrow \mathbb{R}$ and each nonnegative integer m , define

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x, y \in V \\ |x-y|=2^{-m}}} (u(x) - u(y))^2,$$

we observe that $W_m(u) \leq W_{m+1}(u)$ for all $m \in \mathbb{N}$, meaning $\{W_m\}$ is increasing sequence. We then define:

$$W(u) = \lim_{m \rightarrow \infty} W_m(u).$$

Let $C(V)$ denote the space of real-valued continuous functions on V equipped with the supremum norm $\|\cdot\|_\infty$. Define:

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}$$

and

$$H_0^1(V) := \{u \in C_0(V) : W(u) < \infty\}.$$

The space $H_0^1(V)$ is a dense linear subspace of $L^2(V, \mu)$, endowed with the norm $\|\cdot\|_2$. For $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$, define the inner product:

$$W_m(u, v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x, y \in V \\ |x-y|=2^{-m}}} (u(x) - u(y))(v(x) - v(y))$$

which induces the norm:

$$\|u\| = \sqrt{W(u)}.$$

From [11], for any $u \in H_0^1(V)$, we have:

$$\|u\|_\infty \leq (2N+3)\|u\|, \quad (2.1)$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty), \quad (2.2)$$

is compact.

Combining this with the continuous embedding of $(C_0(V), \|\cdot\|_\infty)$ into $(L^r(V, \mu), \|\cdot\|_r)$ for $r \in (1, \infty)$, we deduce that:

$$\left(H_0^1(V), \|\cdot\| \right) \hookrightarrow (L^r(V, \mu), \|\cdot\|_r) \quad (2.3)$$

is compact for all $r \in (1, \infty)$ (see [9]).

Let $H^{-1}(V)$ denote the completion of $L^2(V)$ with respect to the pre-norm:

$$\|w\|_{-1} = \sup_{\substack{g \in H_0^1(V) \\ \|g\|=1}} \langle w, g \rangle,$$

where:

$$\langle w, g \rangle = \int_V w g \, d\mu \quad \text{for } w \in L^2(V), g \in H_0^1(V).$$

This gives three Hilbert spaces with embeddings:

$$H_0^1(V) \subset L^2(V, \mu) \subset H^{-1}(V).$$

The weak Laplacian $\Delta: H_0^1(V) \rightarrow H^{-1}(V)$ is uniquely defined by:

$$-W(u, v) = \langle \Delta u, v \rangle \quad \forall v \in H_0^1(V).$$

Remark 2.1. Let $d_H := \frac{\ln N}{\ln 2}$ be the Hausdorff dimension of the Sierpiński gasket. Here, μ denotes the restriction to V of the normalized Hausdorff measure of dimension d_H , that is, $\mu := \mathcal{H}^{d_H}|_V$. The measure $\mathcal{H}^{d_H}(V)$ is finite, and μ is normalized so that $\mu(V) = 1$. Moreover, μ has full support on V ; see [7] for further details.

Remark 2.2. The space $H_0^1(V)$ is separable and reflexive. This follows from Theorem 19.B in [24] (see Remark 4.1 in [6]).

A C^1 functional J satisfies the Palais-Smale (PS) condition if every sequence $\{U_n\} \subset H_0^1(V) \times H_0^1(V)$ with $\{J(U_n)\}$ bounded and $J'(U_n) \rightarrow 0$ contains a convergent subsequence.

Proposition 2.3. Let $\gamma \in L^1(V, \mu)$, and $T: V \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function. The functional $L: V \times C_0(V) \times C_0(V) \rightarrow \mathbb{R}$ defined by:

$$L(x, u, v) = \int_V \gamma(x) T(x, u(x), v(x)) \, d\mu.$$

is C^1 , with differential at (u_0, v_0) given by:

$$L'(x, u_0, v_0)(h_1, h_2) = \int_V \gamma T_s(x, u_0, v_0) h_1 \, d\mu + \int_V \gamma T_t(x, u_0, v_0) h_2 \, d\mu,$$

for all $(h_1, h_2) \in C_0(V) \times C_0(V)$.

Corollary 2.4 (Proof in [5]). Let $\gamma \in L^1(V, \mu)$, and let $T: V \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function. Define the map $\ell: V \times H_0^1(V) \times H_0^1(V) \rightarrow \mathbb{R}$ by

$$\ell(x, u, v) = \int_V \gamma(x) T(x, u(x), v(x)) \, d\mu.$$

Then ℓ is a C^1 functional with a compact derivative, and its differential at $(u_0, v_0) \in H_0^1(V) \times H_0^1(V)$ is

$$\ell'(x, u_0, v_0)(h_1, h_2) = \int_V \gamma T_s(x, u_0, v_0) h_1 \, d\mu + \int_V \gamma T_t(x, u_0, v_0) h_2 \, d\mu$$

for all $(h_1, h_2) \in H_0^1(V) \times H_0^1(V)$

Definition 2.5. Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space. A functional $f: \mathcal{B} \rightarrow \mathbb{R}$ is called coercive if:

$$f(x) \rightarrow +\infty \quad \text{as} \quad \|x\|_{\mathcal{B}} \rightarrow \infty.$$

3. VARIATIONAL FRAMEWORK

In this section, we establish the variational framework for our problem and present essential preliminary results. Our approach adapts techniques from the classical case in domains of $\mathbb{R}^N$ (see [17]) and extends the reasoning developed in [8].

Define the space $H := H_0^1(V) \times H_0^1(V)$ equipped with the standard product norm:

$$\|(u, v)\|_H := \left(\|u\|^2 + \|v\|^2 \right)^{\frac{1}{2}}.$$

Throughout the analysis, we frequently use the inequality:

$$\|u\| + \|v\| \geq \|(u, v)\|_H$$

which provides a foundational estimate.

As introduced in the previous section, we assume that the space H is equipped with the norm induced by the following inner product:

$$((u, v), (\tilde{u}, \tilde{v})) \mapsto W(u, \tilde{u}) + W(v, \tilde{v}) = W(U, \tilde{U}).$$

This inner product structure not only endows H with the necessary Hilbert space properties but also facilitates the application of variational methods tailored to the specific characteristics of the fractal domain V . The variational framework outlined here will serve as a basis for analyzing the existence and properties of solutions to our system.

For constants $\tilde{\zeta}, \zeta \in \mathbb{R}$, we define the following subspaces of $H_0^1(V)$:

- **For the constant $\tilde{\zeta}$:**

Let $H_{\tilde{\zeta}}^+$, $H_{\tilde{\zeta}}^-$, and $H_{\tilde{\zeta}}^0$ denote the subspaces where the quadratic form

$$u \mapsto \|u\|^2 - \tilde{\zeta} \|u\|_2^2$$

is positive definite, negative definite, and zero, respectively.

- **For the constant ζ :**

Let H_{ζ}^+ , H_{ζ}^- , and H_{ζ}^0 denote the subspaces where the quadratic form

$$u \mapsto \|u\|^2 - \zeta \|u\|_2^2$$

is positive definite, negative definite, and zero, respectively.

Define the corresponding product spaces:

$$H^0 := H_\xi^0 \times H_\zeta^0, \quad H^+ := H_\xi^+ \times H_\zeta^+, \quad H^- := H_\xi^- \times H_\zeta^-.$$

Introduce the operators:

$$A_1 := \text{id} - \xi(-\Delta)^{-1} \quad \text{and} \quad A_2 := \text{id} - \zeta(-\Delta)^{-1},$$

where id is the identity operator on $H_0^1(V)$.

Let $H^* := (H_0^1(V))^* \times (H_0^1(V))^*$ denote the dual space of H . The inverse Laplacian: $(-\Delta)^{-1}: H^* \rightarrow H$ is linear, self-adjoint, continuous, and bijective. By the compact embedding in (2.3), the operator $(-\Delta)^{-1}: H^* \rightarrow H^*$ is also compact and injective (see [26]).

Define the bounded self-adjoint operator:

$$A: H \rightarrow H, \quad A(U) = (A_1 u_1, A_2 u_2) \quad \text{for} \quad U = (u_1, u_2) \in H,$$

with

$$AU = (A_1 u_1, A_2 u_2) \quad \text{for every} \quad U = (u_1, u_2) \in H.$$

The operator A is a bounded, self-adjoint operator on H , with kernel

$$\ker A = H^0 := \ker(\text{id} - \xi(-\Delta)^{-1}) \times \ker(\text{id} - \zeta(-\Delta)^{-1}) \neq \emptyset$$

where $\dim H^0 < \infty$. The decomposition follows from the spectral theorem for self-adjoint operators with discrete spectrum:

$$H = H^- \oplus H^0 \oplus H^+$$

where H^- and H^+ are A -invariant subspaces. There exists $\eta > 0$ such that:

$$\begin{aligned} \langle AU^-, U^- \rangle &\leq -\eta \|U^-\|_H^2 \quad \forall U^- \in H^-, \\ \langle AU^0, U^0 \rangle &= 0 \quad \forall U^0 \in H^0, \\ \langle AU^+, U^+ \rangle &\geq \eta \|U^+\|_H^2 \quad \forall U^+ \in H^+. \end{aligned} \tag{3.1}$$

This decomposition and the properties of A will be essential in the variational analysis of the system. Throughout, we represent any $U \in H$ in terms of its components U^0, U^+ , and U^- , where $U = U^0 + U^- + U^+$. Clearly, the subspaces H^- and H^0 are finite-dimensional. Additionally, the condition $\sigma(A) \cap \sigma(-\Delta) \neq \emptyset$ ensures that $\dim H^0 \neq 0$. With this framework, the system (1.1) can be reformulated as:

$$\begin{cases} -\Delta U = A^* U + \nabla F(x, U) & \text{in } V \setminus V_0, \\ U = 0 & \text{on } V_0, \end{cases}$$

where

$$U = (u_1, u_2), \quad -\Delta U = \begin{pmatrix} -\Delta u_1 \\ -\Delta u_2 \end{pmatrix}, \quad A^* = \begin{pmatrix} \lambda & \delta \\ \delta & \gamma \end{pmatrix}.$$

This reformulation enables us to utilize the operator structure effectively in the subsequent analysis.

As demonstrated by Costa and Magalhães [8], the kernel of $-\Delta - A^*$ is nontrivial if and only if the operator $A^* - \theta_j I$ is non-invertible for some eigenvalue θ_j of the operator $-\Delta$. A

function $U = (u_1, u_2) \in H$ is a weak solution of (1.1) if it is a critical point of the C^1 energy functional

$$J(U) = \frac{1}{2} \|U\|_H^2 - \frac{1}{2} \int_V -RA^*U \cdot U d\mu - \int_V F(x, U) d\mu$$

where R is a bounded linear operator and $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^2$.

For analytical convenience, the functional J can be expressed as the sum of a quadratic form and a Nemytskii operator

$$J(U) = \frac{1}{2} W(AU, U) - \int_V \tilde{F}(x, U) d\mu, \quad U = (u_1, u_2) \in H, \quad (3.2)$$

where $\tilde{F}(x, s, t) = F(x, se_1 + te_2)$.

Remark 3.1. The functional $J : H \rightarrow \mathbb{R}$ defined in (3.2) is referred to as the energy functional associated with problem (1.1).

To facilitate the subsequent analysis, we state some foundational properties of the energy functional J .

Proposition 3.2. Assume $F \in C^1(V \times \mathbb{R}^2, \mathbb{R})$. Then:

- (i) The functional J defined in (3.2) belongs to $C^1(H, \mathbb{R})$.
- (ii) $U \in H$ is a weak solution of (1.1) if and only if U is a critical point of J .

Proof. We will prove the equivalence between (i) and (ii).

Proof of (i) $\Rightarrow$ (ii)

Let J be the functional defined in (3.2), and assume that $J \in C^1(H, \mathbb{R})$. We want to show that if $J'(u_0, v_0) = 0$, then (u_0, v_0) is a weak solution of (1.1).

The Fréchet derivative of J at (u_0, v_0) is computed as follows:

$$\begin{aligned} J'(u_1, u_2)(h_1, h_2) &= \sum_{i=1}^2 W(u_i, h_i) - \int_V (\lambda u_1 h_1 + \delta(u_1 h_2 + u_2 h_1) + \gamma u_2 h_2) d\mu \\ &\quad - \int_V \nabla F(x, u_1, u_2) \cdot (h_1, h_2) d\mu. \end{aligned}$$

If $J'(u_0, v_0) = 0$, this implies that for all $(h_1, h_2) \in H$, the following system holds:

$$\begin{cases} \int W(u_0, h_1) - \int_V (\lambda u_0 h_1 + \delta v_0 h_1) d\mu - \int_V F_{u_1}(x, u_0, v_0) h_1 d\mu = 0, \\ \int W(v_0, h_2) - \int_V (\gamma v_0 h_2 + \delta u_0 h_2) d\mu - \int_V F_{u_2}(x, u_0, v_0) h_2 d\mu = 0. \end{cases}$$

This is exactly the weak formulation of (1.1). Therefore, (u_0, v_0) is a weak solution of (1.1).

Proof of (ii) $\Rightarrow$ (i)

Now, assume that (u_0, v_0) is a weak solution of (1.1). We need to show that (u_0, v_0) is a critical point of J , i.e., $J'(u_0, v_0) = 0$.

By the definition of a weak solution, we know that for all $(h_1, h_2) \in H$, the following equations hold:

$$\begin{cases} W(u_0, h_1) - \int_V (\lambda u_0 h_1 + \delta v_0 h_1) d\mu - \int_V F_{u_1}(x, u_0, v_0) h_1 d\mu = 0, \\ W(v_0, h_2) - \int_V (\gamma v_0 h_2 + \delta u_0 h_2) d\mu - \int_V F_{u_2}(x, u_0, v_0) h_2 d\mu = 0. \end{cases}$$

These equations exactly match the condition for $J'(u_0, v_0) = 0$, which implies that (u_0, v_0) is a critical point of J .

Thus, we conclude that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i), and therefore, (u_0, v_0) is a weak solution of (1.1) if and only if it is a critical point of J .

Then after that, we define an equivalent inner product $W(.,.)$ and the corresponding norm $\|\cdot\|$ on H given by

$$W(U, V) = W(AU^+, V^+) - W(AU^-, V^-),$$

and

$$\|U\| = \sqrt{W(U, U)},$$

where $U, V \in H^0 \oplus H^- \oplus H^+$ with $U = U^0 + U^- + U^+$ and $V = V^0 + V^- + V^+$. The energy functional becomes:

$$J(U) = \frac{1}{2} \|U^+\|^2 - \frac{1}{2} \|U^-\|^2 - \int_V \tilde{F}(x, U) d\mu.$$

Furthermore, by Proposition 3.2 we have $J \in C^1(H, \mathbb{R})$ and the derivatives are given by

$$J'(U)V = W(U^+, V^+) - W(U^-, V^-) - \int_V (\nabla \tilde{F}(x, U), V) d\mu, \quad \forall U, V \in H.$$

We now introduce the concept of genus.

Definition 3.3. Let E be a real Banach space, and let $\Sigma(E)$ denote the class of closed subsets $E_1 \subset E \setminus \{0\}$ that are symmetric about the origin, meaning that $x \in E_1$ implies $-x \in E_1$. We will use the notion of genus for subsets in $\Sigma(E)$.

Definition 3.4. A set $E_1 \in \Sigma(E)$ has genus n (denoted by $\gamma[E_1] = n$) if n is the largest integer such that every odd continuous map from E_1 to $\mathbb{R}^{n-1}$ has a zero in E_1 . If E_1 is empty, its genus is defined to be zero.

The genus satisfies the following standard properties:

- (i) Homotopy with spheres: If E_1 is homeomorphic to the $(n-1)$ -dimensional sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, then $\gamma[E_1] = n$, and E_1 contains subsets of every genus less than n .
- (ii) Monotonicity: If $E_1 \subset E_2$, then $\gamma[E_1] \leq \gamma[E_2]$.
- (iii) Equivariance: If ϕ is an odd continuous map of E_1 , then $\gamma[E_1] \leq \gamma[\phi(E_1)]$.
- (iv) Continuity: Every E_1 has a neighborhood N in E such that $\gamma[E_1] = \gamma[N]$.
- (v) Subadditivity: For any two sets E_1 and E_2 , $\gamma[E_1 \cup E_2] \leq \gamma[E_1] + \gamma[E_2]$.

For more details, we refer the reader to [2, 12] and the references therein.

We now present the symmetric mountain pass lemma of Kajikiya [13, Theorem 1].

Theorem 3.5. *Define the family:*

$$\Lambda_k := \{A \in \Sigma(E) : \gamma(A) \geq k\}, \quad k \in \mathbb{N}.$$

Let E be an infinite-dimensional Banach space, and $J \in C^1(E, \mathbb{R})$ satisfy the conditions (K1) and (K2) below.

- (K1) *The functional J is even, bounded from below, satisfies $J(0) = 0$, and fulfills the Palais-Smale (PS) condition.*
- (K2) *For each $k \in \mathbb{N}$, there exists a set $A_k \in \Lambda_k$ such that $\sup_{u \in A_k} J(u) < 0$.
Then J has a sequence of critical points $\{u_k\}$ such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.*

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.2.

We begin by redefining the functional $J(U)$ as follows

$$J(U) = \frac{1}{2} \|U^+\|^2 - \frac{1}{2} \|U^-\|^2 - \int_V \tilde{F}(x, U) d\mu.$$

where $U = (u_1, u_2) \in H$. By Proposition 3.2, J belongs to $C^1(H, \mathbb{R})$ and satisfies $J(0) = 0$, owing to the condition $F(x, 0) = 0$.

In order to apply Kajikiya's critical point theorem (Theorem 3.5) and generate the desired sequence of critical points $\{U_k\}$, it is necessary to demonstrate the existence of a sequence $\{U_k\} \subset H$ satisfying the following properties:

$$J(U_k) \leq 0, \quad U_k \neq 0, \quad \text{and} \quad U_k \rightarrow 0 \text{ in } H \text{ as } k \rightarrow \infty.$$

This sequence will serve to verify the compactness condition required by the theorem.

Since $F(x, U) = F(x, -U)$ for all $(x, U) \in V \times \mathbb{R}^2$ (from assumption (SC3)), it follows that $J(-U) = J(U)$; hence J is an even functional.

Using Remark 1.1 and the continuity of F , for any $\tau > 0$, there exists a positive constant M_τ such that

$$|\nabla F(x, s)| \leq \tau |s| + M_\tau, \quad \forall (x, s) \in V \times \mathbb{R}^2.$$

Applying the embedding (2.3) and noting that $\mu(V) = 1$, we obtain:

$$\left| \int_V \tilde{F}(x, U) d\mu \right| \leq \int_V (\tau |U|^2 + M_\tau |U|) d\mu \leq \tau (2N+3)^2 \|U\|^2 + M_\tau (2N+3) \|U\|.$$

Thus, we have:

$$J(U) \geq \frac{1}{2} \|U\|^2 - \tau (2N+3)^2 \|U\|^2 - M_\tau (2N+3) \|U\|.$$

Setting $\tau = \frac{1}{4(2N+3)^2}$, we get:

$$J(U) \geq \frac{1}{4} \|U\|^2 - M_\tau (2N+3) \|U\|.$$

Since M_τ is a fixed constant (depending only on τ), for $\|U\| \geq R_0$ where:

$$R_0 = \max(1, 4M_\tau(2N+3)),$$

the quadratic term dominates:

$$\frac{1}{4}\|U\|^2 \geq M_\tau(2N+3)\|U\| \quad \text{when} \quad \|U\| \geq R_0.$$

Thus:

$$J(U) \geq \frac{1}{8}\|U\|^2 \quad \text{for} \quad \|U\| \geq R_0.$$

This proves that J is coercive, i.e., $J(U) \rightarrow +\infty$ as $\|U\| \rightarrow \infty$.

We proceed to verify that the energy functional J satisfies the Palais–Smale (PS) condition. Let $\{U_j\} \subset H$ be a sequence such that the sequence of real numbers $\{J(U_j)\}$ is bounded, and the sequence of derivatives satisfies $J'(U_j) \rightarrow 0$ in the dual space H^* as $j \rightarrow \infty$. Our goal is to prove that $\{U_j\}$ admits a strongly convergent subsequence in H .

First, we show that $\{U_j\}$ is bounded in H . Suppose, for contradiction, that the sequence $\{U_j\}$ is unbounded in H . Thus, there exists a subsequence, which we relabel as $\{U_j\}$, such that $\|U_j\|_H \rightarrow \infty$ as $j \rightarrow \infty$.

Given the coercivity of the functional J , this implies that:

$$\lim_{\|U_j\|_H \rightarrow \infty} J(U_j) = +\infty,$$

since $J(U_j)$ grows asymptotically as $\|U_j\|_H^2$ for large $\|U_j\|_H$, contradicting the boundedness of $\{J(U_j)\}$ in $\mathbb{R}$. Therefore, $\{U_j\}$ is bounded in H .

Consequently, there exists $u \in H$ such that a subsequence of $\{U_j\}$ converges weakly to u in H . Since the embeddings (2.2) and (2.3) are compact, it follows that $\{U_j\}$ converges strongly to u in both $(C_0(V), \|\cdot\|_\infty)$ and $L^r(V, \mu)$ for all $r \in (1, \infty)$.

We now proceed to show that $\{U_j\}$ converges strongly to u in H . Given that

$$\dim(H^0 \oplus H^-) = \dim H^0 + \dim H^- < \infty,$$

we may assume (by passing to a subsequence if necessary):

$$U_j^0 \rightarrow U^0, \quad U_j^- \rightarrow U^-, \quad U_j^+ \rightharpoonup U^+ \quad \text{and} \quad U_j \rightharpoonup U, \quad \text{as} \quad j \rightarrow \infty, \quad (4.1)$$

for some $U = U^0 + U^- + U^+ \in H = H^0 \oplus H^- \oplus H^+$. By the Riesz Representation Theorem, we can identify H^* with H , allowing us to view the maps $J' : H \rightarrow H^*$ and $G' : H \rightarrow H^*$ as mappings from H to itself, where

$$G(U) := \int_V F(x, U) d\mu.$$

Note that

$$J'(U_j) = U_j^+ - (U_j^- + G'(U_j)) = o(1), \quad \forall j \in \mathbb{N},$$

where $o(1)$ denotes terms converging to zero in H -norm as $j \rightarrow \infty$. Thus, we have

$$U_j^+ = U_j^- + G'(U_j) + o(1), \quad \forall j \in \mathbb{N}. \quad (4.2)$$

The compactness of $G' : H \rightarrow H$ is crucial here and follows from:

- (ii) The compact embedding $H_0^1(V) \hookrightarrow L^r(V)$ for all $r \in (1, \infty)$ (via (2.3))
- (ii) The growth conditions on ∇F ensuring G' maps bounded sets to relatively compact sets
- (iii) Corollary 2.4 which directly establishes this compactness

Since $U_j \rightharpoonup U$ weakly in H and G' is compact, it follows that $G'(U_j) \rightarrow G'(U)$ strongly in H . Therefore, from equation (4.2) and the strong convergence of $G'(U_j)$ in H , we conclude that

$$U_j^+ \rightarrow U^+ \quad \text{strongly in } H \quad \text{as } j \rightarrow \infty.$$

Combining these results with Equation (4.1), we find that

$$U_j^0 \rightarrow U^0, \quad U_j^- \rightarrow U^-, \quad U_j^+ \rightarrow U^+, \quad \text{and} \quad U_j \rightarrow U \quad \text{strongly in } H, \quad \text{as } j \rightarrow \infty.$$

This establishes that J satisfies the Palais-Smale condition. Since both $\dim H^0$ and $\dim H^-$ are finite, we can construct an orthonormal basis of the Hilbert space H as follows. First, let $\{\varphi_m\}_{m=1}^k$ be an orthonormal basis of H^0 , and $\{\varphi_m\}_{m=k+1}^{l_0}$ an orthonormal basis of H^- , where $1 \leq k < \infty$ and $k+1 \leq l_0 < \infty$. We then extend this collection by choosing $\{\varphi_m\}_{m=l_0+1}^\infty$ as an orthonormal basis of H^+ . As a result, the family $\{\varphi_m\}_{m=1}^\infty$ forms a complete orthonormal basis of the space H .

Define

$$H_k = \bigoplus_{j=1}^k X_j,$$

where $k \geq l_0 + 1$ and $X_j = \text{span}(\varphi_j)$ with $\{\varphi_m\}$ an orthonormal basis of H . By the equivalence of norms on finite-dimensional spaces (see [[20], Theorem 1.21]), there exists a constant $c_k > 0$ such that:

$$\|U\|_2 \geq c_k \|U\|, \quad \forall U \in H_k, \quad \forall k \in \mathbb{N}.$$

Using condition (SC2), choose $0 < r < 1$ such that

$$F(x, U) \geq \frac{1}{c_k^2} |U|^2, \quad \forall |U| \leq r, \quad \text{and for almost every } x \in V.$$

This condition implies a lower bound on F that will allow us to control the growth of the functional J . Now, for any $U \in H_k$, we set

$$\|U\| = l_k := \min \left\{ 1, \frac{r}{c_k} \right\}.$$

where c_k is a constant such that $\|U\|_2 \geq c_k \|U\|$ holds for all $U \in H_k$. This choice of l_k ensures that $\|U\| \leq l_k$, thereby maintaining $|U| \leq r$ for almost every $x \in V$, which aligns with the bound provided by condition (SC2). Thus, for all $U \in H_k$ with $\|U\| = l_k$, we obtain the following series of inequalities:

$$\begin{aligned}
J(U) &= \frac{1}{2} \|U^+\|^2 - \frac{1}{2} \|U^-\|^2 - \int_V \tilde{F}(x, U) d\mu \\
&\leq \frac{1}{2} \|U^+\|^2 - \int_V \tilde{F}(x, U) d\mu \\
&\leq \frac{1}{2} \|U\|^2 - \frac{1}{c_k^2} \int_V |U|^2 d\mu \\
&\leq \frac{1}{2} \|U\|^2 - \frac{1}{c_k^2} \|U\|_2^2 \mu(V) \\
&\leq \frac{1}{2} \|U\|^2 - \frac{c_k^2}{c_k^2} \|U\|^2 \\
&= -\frac{1}{2} \|U\|^2 \\
&= -\frac{1}{2} l_k^2.
\end{aligned}$$

Therefore, we conclude that

$$\{U \in H_k : \|U\| = l_k\} \subset \left\{U \in H : J(U) \leq -\frac{1}{2} l_k^2\right\}.$$

Next, we define

$$A_k := \left\{U \in H : J(U) \leq -\frac{1}{2} l_k^2\right\}.$$

By Theorem 3.5 and property (ii) in Definition 3.4, we have

$$\gamma(A_k) \geq \gamma(\{U \in H_k : \|U\| = l_k\}) \geq k,$$

thus showing that $A_k \in \Lambda_k$ and ensuring that

$$\sup_{U \in A_k} J(U) \leq -\frac{1}{2} l_k^2 < 0.$$

This concludes the proof that the sets A_k meet the requirements of the symmetric mountain pass theorem, as stated by Theorem 3.5.

□

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¹ FACULTY OF MULTIDISCIPLINARY STUDIES, UNIVERSITY MOHAMMED FIRST, NADOR, MOROCCO
Email address: karim.dahmani40@gmail.com

² FACULTY OF MULTIDISCIPLINARY STUDIES, UNIVERSITY MOHAMMED FIRST, NADOR, MOROCCO
Email address: ka.lamrini@ump.ac.ma

³ FACULTY OF SCIENCES, UNIVERSITY MOHAMMED FIRST, OUJDA, MOROCCO
Email address: s-elhabib@hotmail.com