Gulf Journal of Mathematics
Vol 20, Issue 2 (2025) 194-202

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v20i2.3191

REGULARITY OF THE TRANSFORMATION SEMIGROUPS WITH INVARIANT SETS AND RESTRICTED BY AN EQUIVALENCE RELATION

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ABSTRACT. For a nonempty set X, let $T(X)$ be the full transformation semigroup on X. Let Y be a nonempty subset of X and let ρ be an equivalence relation on X. We define the subset of $T_Y(X,\rho)$ of $T(X)$, consisting of all transformations that map elements of Y into Y and send elements within the same ρ-equivalence class into a single common element. It can be shown that $T_Y(X,\rho)$ forms a subsemigroup of $T(X)$. In this paper, we provide a characterization of the regular, left regular, and right regular elements of $T_Y(X,\rho)$. Furthermore, we identify the necessary and sufficient conditions for the semigroup $T_Y(X,\rho)$ itself to be regular, left regular, or right regular.

Keywords. Transformation semigroups, regular elements, left [right] regular elements, equivalence relations, invariant sets
2020 Mathematics Subject Classification. Primary 46L55; Secondary 44B20

1. Introduction and preliminaries

Let S be a semigroup. An element $a \in S$ is called regular if it can be expressed in the form $a = axa$ for some $x \in S$. If there exists $x \in S$ such that $a = xa^2$, then a is called *left regular*. Similarly, if $a = a^2x$ for some $x \in S$, then a is said to be *right regular*. A semigroup in which every element is regular is called a *regular semigroup*. Similarly, the concepts of *left regular semigroup* and *right regular semigroup* are defined by requiring every element to be left regular and right regular, respectively. For any nonempty set X, the set $T(X)$ of self-maps on X forms a semigroup under composition, known as the *full transformation semigroup*. For $\alpha \in T(X)$, we write $\operatorname{ran}\alpha$ to indicate its image set. If $x \in X$, then $x\alpha$ stands for the output of x under α. The restriction of α to a subset A of X is written as $\alpha\mid_A$. Let A and B be nonempty, disjoint subsets whose union is X and let $a, b \in X$. Sometimes, we use the following notation to represent a function $\alpha \in T(X)$:

$$\alpha = \begin{pmatrix} A & B \\ a & b \end{pmatrix}$$

Date: Received: Jun 27, 2025; Accepted: Jul 27, 2025.
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to indicate that $x\alpha = a$ for all $x \in A$ and $x\alpha = b$ for all $x \in B$.

It is a classical result that $T(X)$ is a regular semigroup, and various studies have focused on identifying regularity properties within its subsemigroups.

For a nonempty subset Y of X, define the subsemigroups:

$$T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\},$$
$$T_Y(X) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

These subsemigroups were initially explored by Magill [2] in 1966. The regular elements within these subsemigroups were characterized by Nenthein, Youngkhong and Kemprasit in [5]. The regularity conditions for $T(X,Y)$ were determined by Sanwong and Sommanee in [9]. In 2013, Choomanee, Honyam and Sanwong [1] provided a characterization of left, right, and intra-regular elements in $T_Y(X)$. Further extensions were made by Phongchan and Honyam [7] in 2023, who detailed the conditions under which $T_Y(X)$ possesses left, right, or complete regularity.

Let ρ be an equivalence relation on a nonempty set X. For each $x \in X$, we use $[x]_\rho$ to denote its equivalence class, and X/ρ represents the set of all equivalence classes under ρ. In 2005, Pei [6] introduced the transformation semigroup

$$T_\rho(X) = \{\, \alpha \in T(X) \mid \forall x, y \in X, (x, y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho \,\}.$$

The author also investigated the regular elements of $T_\rho(X)$ and its Green's relation. In [4], Namnak and Laysilikul identified the exact conditions under which elements of $T_\rho(X)$ are left, right, or completely regular. In 2020, Sangkhanan and Sanwong [8] introduced a subsemigroup of $T(X,Y)$ defined by

$$T_\rho(X,Y) = \{\, \alpha \in T(X,Y) \mid \forall x, y \in X, (x, y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho \,\}.$$

We call $T_\rho(X,Y)$ the semigroup of transformations with restricted range that preserve an equivalence. They provided criteria for when $T_\rho(X,Y)$ is regular and described its Green's relations. In 2010, Mendes-Gonçalves and Sullivan [3] defined another semigroup

$$E(X,\rho) = \{\, \alpha \in T(X) \mid \forall x, y \in X, (x, y) \in \rho \Rightarrow x\alpha = y\alpha \,\},$$

known as the *semigroup of transformations restricted by an equivalence relation* ρ. In 2018, Sawatraksa, Namnak and Laysirikul [10] identified which elements of $E(X,\rho)$ satisfy various forms of regularity and provided conditions for the entire semigroup to exhibit left, right, or complete regularity.

Let Y be a nonempty subset of X and ρ an equivalence relation on X. Define

$$T_Y(X,\rho) = \{\, \alpha \in T_Y(X) \mid \forall x, y \in X, (x, y) \in \rho \Rightarrow x\alpha = y\alpha \,\}.$$

It can be demonstrated that $T_Y(X,\rho)$ forms a subsemigroup of $T_Y(X)$.

In this paper, we establish necessary and sufficient criteria for elements of $T_Y(X,\rho)$ to be regular, left regular or right regular and use those conditions to establish whether $T_Y(X,\rho)$ is regular, left regular or right regular.

2. Main results

2.1. **Regularity of $T_Y(X,\rho)$.** First, we give a necessary and sufficient condition for an element of $T_Y(X,\rho)$ to be regular.

Theorem 2.1. *Let $\alpha \in T_Y(X,\rho)$. Then α is regular if and only if the following conditions hold.*

(1) $\forall x, y \in \operatorname{ran}\alpha, (x,y) \in \rho \Rightarrow x = y$.
(2) $x\alpha^{-1} \cap Y \neq \varnothing$ *for all* $x \in \operatorname{ran}\alpha$ *such that* $[x]_\rho \cap Y \neq \varnothing$.

Proof. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in T_Y(X,\rho)$. Let $x, y \in \operatorname{ran}\alpha$ such that $(x,y) \in \rho$. Write $x = a\alpha$ and $y = b\alpha$ for some $a, b \in X$. We have

$$x = a\alpha = a\alpha\beta\alpha = x\beta\alpha = y\beta\alpha = b\alpha\beta\alpha = b\alpha = y.$$

Next, let $x \in \operatorname{ran}\alpha$ such that $[x]_\rho \cap Y \neq \varnothing$. Then $(x,y) \in \rho$ for some $y \in Y$ and $x = a\alpha$ for some $a \in X$. We have $a\alpha\beta = x\beta = y\beta \in Y$ and $a\alpha\beta\alpha = a\alpha = x$. This implies that $a\alpha\beta \in x\alpha^{-1} \cap Y$.

Conversely, assume that (1) and (2) hold. For each $x \in X$ such that $(x,z) \in \rho$ for some $z \in \operatorname{ran}\alpha$, choose $y_z \in z\alpha^{-1} \cap Y$ if $[z]_\rho \cap Y \neq \varnothing$ and choose $a_z \in z\alpha^{-1}$ if $[z]_\rho \cap Y = \varnothing$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} y_z & \text{if } (x,z) \in \rho \text{ for some } z \in \operatorname{ran}\alpha \text{ such that } [z]_\rho \cap Y \neq \varnothing, \\ a_z & \text{if } (x,z) \in \rho \text{ for some } z \in \operatorname{ran}\alpha \text{ such that } [z]_\rho \cap Y = \varnothing, \\ x\alpha & \text{otherwise.} \end{cases}$$

By the condition (1), β is well-defined. Let $y \in Y$. If $(y,z) \in \rho$ for some $z \in \operatorname{ran}\alpha$, then $[z]_\rho \cap Y = [y]_\rho \cap Y \neq \varnothing$, so $y\beta = y_z \in Y$. Otherwise, $y\beta = y\alpha \in Y$. This shows that $Y\beta \subseteq Y$. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in \rho$.

Case 1 : $(x_1, z) \in \rho$ for some $z \in \operatorname{ran}\alpha$. Then $(x_2, z) \in \rho$.

Subcase 1.1 : $[z]_\rho \cap Y \neq \varnothing$. Then $x_1\beta = y_z = x_2\beta$.

Subcase 1.2 : $[z]_\rho \cap Y = \varnothing$. Thus $x_1\beta = a_z = x_2\beta$.

Case 2 : $(x_1, z) \notin \rho$ for all $z \in \operatorname{ran}\alpha$. Then $(x_2, z) \notin \rho$ for all $z \in \operatorname{ran}\alpha$. Thus $x_1\beta = x_1\alpha = x_2\alpha = x_2\beta$.

Then $\beta \in T_Y(X,\rho)$. Let $x \in X$. Then $x\alpha \in \operatorname{ran}\alpha$ and $(x\alpha, x\alpha) \in \rho$. If $[x\alpha]_\rho \cap Y \neq \varnothing$, then $x\alpha\beta\alpha = y_{x\alpha}\alpha = x\alpha$. If $[x\alpha]_\rho \cap Y = \varnothing$, then $x\alpha\beta\alpha = a_{x\alpha}\alpha = x\alpha$. This shows that $\alpha = \alpha\beta\alpha$. Hence, α is regular. □

Next, we apply the above result to determine the regularity of $T_Y(X,\rho)$.

Theorem 2.2. *If $|X/\rho| = 1$, then $T_Y(X,\rho)$ is a regular semigroup.*

Proof. Assume that $|X/\rho| = 1$. Let $\alpha \in T_Y(X,\rho)$. Let $a_1, a_2 \in \operatorname{ran}\alpha$. Then $a_1 = x_1\alpha$, $a_2 = x_2\alpha$ for some $x_1, x_2 \in X$. Since $|X/\rho| = 1$, we have $(x_1, x_2) \in \rho$, so $a_1 = x_1\alpha = x_2\alpha = a_2$. This shows that $|\operatorname{ran}\alpha| = 1$, that is α is a constant map and hence, α is regular. □

Lemma 2.3. *Let $|X/\rho| > 1$. If $T_Y(X,\rho)$ is regular, then all equivalence classes contain at most one element from Y. Moreover, if each equivalence class contains exactly one element from Y, then $T_Y(X,\rho)$ is a regular semigroup.*

Proof. Suppose that $T_Y(X,\rho)$ is regular. Let $P \in X/\rho$ such that $P \cap Y \neq \varnothing$. Let $y_1, y_2 \in P \cap Y$. Define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} P & X-P \\ y_1 & y_2 \end{pmatrix}.$$

Clearly, $\alpha \in T_Y(X,\rho)$. We have $y_1, y_2 \in \operatorname{ran}\alpha$ and $(y_1, y_2) \in \rho$. By Theorem 2.1, we have $y_1 = y_2$. This shows that P contains at most one element from Y. Assume that all equivalence classes contain exactly one element from Y. Let $\alpha \in T_Y(X,\rho)$ and $x = a\alpha \in \operatorname{ran}\alpha$, where $a \in X$. By assumption, $|[a]_\rho \cap Y| = 1$. Then there is $y \in Y$ such that $(a, y) \in \rho$. We have $x = a\alpha = y\alpha$, so $y \in x\alpha^{-1} \cap Y$ and $x \in Y$. This implies that $x\alpha^{-1} \cap Y \neq \varnothing$ and $x \in Y$ for all $x \in \operatorname{ran}\alpha$. Let $x_1, x_2 \in \operatorname{ran}\alpha$ such that $(x_1, x_2) \in \rho$. Then $x_1, x_2 \in [x_1]_\rho \cap Y$. Since $\left|[x_1]_\rho \cap Y\right| = 1$, we get $x_1 = x_2$. Then α is a regular element of $T_Y(X,\rho)$ by Theorem 2.1. Hence, $T_Y(X,\rho)$ is a regular semigroup. □

Theorem 2.4. *Let $|X/\rho| = 2$. Then $T_Y(X,\rho)$ is regular if and only if the following conditions are satisfied.*

(1) *Each equivalence class contains at most one element from Y.*
(2) *If either of the two equivalence classes contains no elements from Y, then $X/\rho = \{X - Y, Y\}$ and $|Y| = 1$.*

Proof. Let $X/\rho = \{P_1, P_2\}$. Assume that (1) and (2) hold. If both P_1 and P_2 contain exactly one element from Y, then by Lemma 2.3, $T_Y(X,\rho)$ is regular. Now, we may assume that P_1 does not contain any elements from Y. Then $P_2 = Y$ and $|P_2| = |Y| = 1$, say $P_2 = \{y\}$. Let $\alpha \in T_Y(X,\rho)$. Since $|X/\rho| = 2$, $|\operatorname{ran}\alpha| \leq 2$. If $|\operatorname{ran}\alpha| = 1$, then α is a constant map and then it is regular. Suppose that $|\operatorname{ran}\alpha| = 2$. Then $y\alpha = y$, so $y\alpha^2 = y\alpha$. Let $x \in P_1$. Since $|\operatorname{ran}\alpha| = 2$, we get $x\alpha \in P_1$. We have $(x, x\alpha) \in \rho$, so $x\alpha^2 = x\alpha$. Then $\alpha = \alpha^2$, so α is regular.

For the converse, assume that $T_Y(X,\rho)$ is regular. By Lemma 2.3, we have the condition (1) holds. For the condition (2), we may assume that $P_1 \cap Y = \varnothing$. This implies that $Y \subseteq P_2$. Let $z \in P_2$ and choose $y \in Y$. Let $\alpha \in T(X)$ be defined by

$$\alpha = \begin{pmatrix} P_1 & P_2 \\ z & y \end{pmatrix}.$$

Clearly, $\alpha \in T_Y(X,\rho)$. We see that $z, y \in \operatorname{ran}\alpha$ and $(z, y) \in \rho$. Since α is regular, we get $z = y$. This shows that $P_2 \subseteq Y$ and $|P_2| = 1$. Therefore, $P_1 = X - Y$, $P_2 = Y$ and $|Y| = 1$. □

Theorem 2.5. *Let $|X/\rho| > 2$. Then $T_Y(X,\rho)$ is regular if and only if the following statements are true.*

(1) *Each equivalence class contains at most one element from Y.*
(2) *If there exists an equivalence class that does not contain any element from Y, then ρ is an identity relation on X and $|Y| = 1$.*

Proof. Assume that (1) and (2) hold. If all equivalence class contain exactly one element from Y, then by Lemma 2.3, $T_Y(X,\rho)$ is regular. Suppose that there exists $P \in X/\rho$ such that P does not contain any elements from Y. Then ρ is an identity relation on X and $|Y| = 1$, say $Y = \{y\}$. Let $\alpha \in T_Y(X, \rho)$. Since ρ is an identity relation on X, we have α satisfies the condition (1) of Theorem 2.1. Let $x \in \operatorname{ran}\alpha$ such that $[x]_\rho \cap Y \neq \varnothing$. This forces $(x, y) \in \rho$, so $x = y$. We have $y \in y\alpha^{-1} \cap Y = x\alpha^{-1} \cap Y$. Then $x\alpha^{-1} \cap Y \neq \varnothing$. Hence, α is regular.

Conversely, suppose that $T_Y(X, \rho)$ is regular. Then the condition (1) holds by Lemma 2.3. To prove the condition (2), assume that there exists $P \in X/\rho$ such that $P \cap Y = \varnothing$. Let $y_1, y_2 \in Y$ and define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} P & X - P \\ y_1 & y_2 \end{pmatrix}.$$

Clearly, $\alpha \in T_Y(X, \rho)$. Since α is regular, by Theorem 2.1, $y_1\alpha^{-1} \cap Y \neq \varnothing$. If $y_1 \neq y_2$, then $P \cap Y = y_1\alpha^{-1} \cap Y \neq \varnothing$. This is a contradiction. Then $y_1 = y_2$. Thus $|Y| = 1$, say $Y = \{y\}$. Next, let $x_1, x_2 \in X$ with $(x_1, x_2) \in \rho$.

Case 1 : $y \in [x_1]_\rho$. For each $i \in \{1, 2\}$, define $\alpha_i : X \to X$ by

$$\alpha_i = \begin{pmatrix} P & X - P \\ x_i & y \end{pmatrix}.$$

Clearly, $\alpha_1, \alpha_2 \in T_Y(X, \rho)$. For each $i \in \{1, 2\}$, we have $x_i, y \in \operatorname{ran}\alpha_i$ and $(y, x_i) \in \rho$. Since α_1 and α_2 are regular, by Theorem 2.1, we have $x_1 = y = x_2$.

Case 2 : $y \notin [x_1]_\rho$. Then $[x_1]_\rho \cap [y]_\rho = \varnothing$. Since $|X/\rho| > 2$, there exists $x_3 \in X$ such that $[x_3]_\rho \in X/\rho - \big\{[x_1]_\rho, [y]_\rho\big\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} x_1 & \text{if } x \in [x_1]_\rho, \\ x_2 & \text{if } x \in [x_3]_\rho, \\ y & \text{otherwise.} \end{cases}$$

We see that $y \notin [x_1]_\rho \cup [x_3]_\rho$, so $y\alpha = y$. Then $\alpha \in T_Y(X, \rho)$. We have $x_1, x_2 \in \operatorname{ran}\alpha$ and $(x_1, x_2) \in \rho$. Since α is regular, by Theorem 2.1, we get $x_1 = x_2$. Then ρ is an identity relation on X. Therefore, the condition (2) holds. □

2.2. **Left Regularity and Right Regularity of $T_Y(X, \rho)$.** First, we present a necessary and sufficient condition for an element of $T_Y(X, \rho)$ to be left or right regular.

Theorem 2.6. *Let $\alpha \in T_Y(X, \rho)$. Then α is left regular if and only if the following statements are true.*

(1) *$x\alpha^{-1} \cap \operatorname{ran}\alpha \neq \varnothing$ for all $x \in \operatorname{ran}\alpha$.*
(2) *$x\alpha^{-1} \cap Y\alpha \neq \varnothing$ for all $x \in Y\alpha$.*

Proof. Assume that $\alpha = \beta\alpha^2$ for some $\beta \in T_Y(X, \rho)$. Let $x \in \operatorname{ran}\alpha$, say $x = a\alpha$ for some $a \in X$. Then $x = a\alpha = a\beta\alpha^2$, so $a\beta\alpha \in x\alpha^{-1} \cap \operatorname{ran}\alpha$. This proves (1). To prove (2), let $x \in Y\alpha$. Then $x = y\alpha$ for some $y \in Y$. Thus $x = y\alpha = y\beta\alpha^2$, so $y\beta\alpha \in x\alpha^{-1} \cap Y\alpha$.

Conversely, suppose that (1) and (2) hold. For each $x \in \operatorname{ran}\alpha$, choose $a_x \in X$ such that $a_x\alpha \in x\alpha^{-1}$ and for each $x \in Y\alpha$, choose $y_x \in Y$ such that $y_x\alpha \in x\alpha^{-1}$. Let $\beta : X \to X$ be defined by

$$x\beta = \begin{cases} y_{x\alpha} & \text{if } [x]_\rho \cap Y \neq \varnothing, \\ a_{x\alpha} & \text{otherwise.} \end{cases}$$

Clearly, $Y\beta \subseteq Y$. Let $x_1, x_2 \in X$ such that $(x_1, x_2) \in \rho$. Then $[x_1]_\rho = [x_2]_\rho$. If $[x_1]_\rho \cap Y \neq \varnothing$, then $[x_2]_\rho \cap Y \neq \varnothing$, so $x_1\beta = y_{x_1\alpha} = y_{x_2\alpha} = x_2\beta$. Otherwise, $x_1\beta = a_{x_1\alpha} = a_{x_2\alpha} = x_2\beta$. Then $\beta \in T_Y(X, \rho)$. Let $x \in X$. If $[x]_\rho \cap Y \neq \varnothing$, then $x\beta\alpha^2 = y_{x\alpha}\alpha^2 = x\alpha$. Otherwise, $x\beta\alpha^2 = a_{x\alpha}\alpha^2 = x\alpha$. Hence, α is left regular. □

Theorem 2.7. *Let $\alpha \in T_Y(X, \rho)$. Then α is right regular if and only if the following statements are true.*

(1) *$\alpha|_{ran\,\alpha}$ is injective.*
(2) *$x\alpha^{-1} \cap ran\,\alpha \cap Y \neq \varnothing$ for all $x \in ran\,(\alpha^2)$ such that $[x]_\rho \cap Y \neq \varnothing$.*

Proof. Assume that $\alpha = \alpha^2\beta$ for some $\beta \in T_Y(X, \rho)$. Let $x_1, x_2 \in \operatorname{ran}\alpha$ such that $x_1\alpha = x_2\alpha$. Then $x_1 = a_1\alpha$ and $x_2 = a_2\alpha$ for some $a_1, a_2 \in X$. We have

$$x_1 = a_1\alpha = a_1\alpha^2\beta = x_1\alpha\beta = x_2\alpha\beta = a_2\alpha^2\beta = a_2\alpha = x_2.$$

Let $x \in \operatorname{ran}(\alpha^2)$ such that $[x]_\rho \cap Y \neq \varnothing$. Then $x = a\alpha^2$ for some $a \in X$ and $(x, y) \in \rho$ for some $y \in Y$. Thus $a\alpha = a\alpha^2\beta = x\beta = y\beta \in Y$. Thus $a\alpha \in x\alpha^{-1} \cap \operatorname{ran}\alpha \cap Y$.

For the converse, assume that (1) and (2) hold. For each $x \in \operatorname{ran}(\alpha^2)$, there is a unique $a_x \in \operatorname{ran}\alpha$ such that $a_x\alpha = x$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a_z & \text{if } (x, z) \in \rho \text{ for some } z \in \operatorname{ran}(\alpha^2), \\ x\alpha & \text{otherwise.} \end{cases}$$

We must show that β is well-defined. Let $x \in X$ such that $(x, z_1) \in \rho$ and $(x, z_2) \in \rho$ for some $z_1, z_2 \in \operatorname{ran}(\alpha^2)$. Then $(z_1, z_2) \in \rho$. It follows that $z_1\alpha = z_2\alpha$. By condition (1), we have $z_1 = z_2$. Then β is well-defined. Next, we will show that $Y\beta \subseteq Y$. Let $y \in Y$. If $(y, z) \notin \rho$ for all $z \in \operatorname{ran}(\alpha^2)$, then $y\beta = y\alpha \in Y$. Suppose that $(y, z) \in \rho$ for some $z \in \operatorname{ran}(\alpha^2)$. Then $[z]_\rho \cap Y \neq \varnothing$. By condition (2), there exists $c \in z\alpha^{-1} \cap \operatorname{ran}\alpha \cap Y$. Then $c\alpha = z = a_z\alpha$. By condition (1), we have $a_z = c \in Y$. Thus $y\beta = a_z \in Y$. Then $\beta \in T_Y(X, \rho)$. Now, let $x \in X$. By the uniqueness of $a_{x\alpha^2}$, we get $x\alpha = a_{x\alpha^2}$. Hence, $x\alpha^2\beta = a_{x\alpha^2} = x\alpha$. Therefore, α is right regular. □

Next, we apply the above result to determine whether $T_Y(X, \rho)$ is left or right regular.

Theorem 2.8. *If $|X/\rho| = 1$, then $T_Y(X, \rho)$ is a left and a right regular semigroup.*

Proof. The proof is similar to Theorem 2.2. □

Lemma 2.9. *Let $|X/\rho| > 1$. If $T_Y(X, \rho)$ is left regular or right regular, then $|X/\rho| = 2$.*

Proof. Suppose that $|X/\rho| > 2$.

Case 1 : There is a unique $P \in X/\rho$ such that $P \cap Y \neq \varnothing$. Let $x_0 \in X$ such that $x_0 \notin P$ and $y \in Y \cap P$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} y & \text{if } x \in P \cup [x_0]_\rho, \\ x_0 & \text{otherwise.} \end{cases}$$

By the uniqueness of P, we have $Y\alpha \subseteq Y$. Then $\alpha \in T_Y(X,\rho)$. We see that $x_0 \in \operatorname{ran}\alpha$, but

$$x_0\alpha^{-1} \cap \operatorname{ran}\alpha = \Big[X - \big(P \cup [x_0]_\rho\big)\Big] \cap \{y, x_0\} = \varnothing.$$

By Theorem 2.6, α is not left regular. We have $y, x_0 \in \operatorname{ran}\alpha$ and $y \neq x_0$, but $y\alpha = y = x_0\alpha$. Then $\alpha|_{\operatorname{ran}\alpha}$ is not injective. By Theorem 2.7, α is not right regular.

Case 2 : There exist distinct $P_1, P_2 \in X/\rho$ such that $P_1 \cap Y \neq \varnothing$, $P_2 \cap Y \neq \varnothing$. Let $y_1 \in P_1 \cap Y$ and $y_2 \in P_2 \cap Y$. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} y_1 & \text{if } x \in P_1 \cup P_2 \\ y_2 & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T_Y(X,\rho)$. We have $y_2 \in \operatorname{ran}\alpha$ and

$$y_2\alpha^{-1} \cap \operatorname{ran}\alpha = \Big[X - \big(P_1 \cup P_2\big)\Big] \cap \{y_1, y_2\} = \varnothing.$$

By Theorem 2.6, α is not left regular. We see that $y_1, y_2 \in \operatorname{ran}\alpha$ and $y_1 \neq y_2$, but $y_1\alpha = y_1 = y_2\alpha$. Then $\alpha|_{\operatorname{ran}\alpha}$ is not injective. By Theorem 2.7, α is not right regular. □

Theorem 2.10. *Let $|X/\rho| > 1$. Then the following statements are equivalent.*

(1) *$T_Y(X,\rho)$ is a left regular semigroup.*
(2) *$T_Y(X,\rho)$ is a right regular semigroup.*
(3) *$|X/\rho| = 2$ and the following statement are true.*
   (a) *Each equivalence class contains at most one element from Y.*
   (b) *If either of the two equivalence classes contains no elements from Y, then $X/\rho = \{X - Y, Y\}$ and $|Y| = 1$.*

Proof. First, we will prove (3) ⇒ (1) and (3) ⇒ (2). Assume that (3) holds. Let $X/\rho = \{P_1, P_2\}$.

Case 1 : Both P_1 and P_2 contain exactly one element from Y. Then $P_1 = [y_1]_\rho$ and $P_2 = [y_2]_\rho$, where $y_1, y_2 \in Y$. Then we have only four elements in $T_Y(X,\rho)$, namely :

$$\alpha_1 = \begin{pmatrix} [y_1]_\rho & [y_2]_\rho \\ y_1 & y_1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} [y_1]_\rho & [y_2]_\rho \\ y_2 & y_2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} [y_1]_\rho & [y_2]_\rho \\ y_1 & y_2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} [y_1]_\rho & [y_2]_\rho \\ y_2 & y_1 \end{pmatrix}.$$

The first and second are constant maps. For the third, we have $\alpha_3 = \alpha_3^2$ and for the fourth, $\alpha_4 = \alpha_4^3$. Then $T_Y(X,\rho)$ is left regular and right regular.

Case 2 : One of the two equivalence classes does not contain any elements from Y, say P_1. By assumption, $P_2 = Y$ and $|P_2| = |Y| = 1$. If $P_2 = \{y\}$, then α must be in the form

$$\alpha = \begin{pmatrix} P_1 & P_2 \\ x & y \end{pmatrix}, \quad \text{for some } x \in X.$$

If $x = y$, then α is constant, and hence, α is both left regular and right regular. Assume that $x \neq y$. Since $|P_2| = 1$, $x \in P_1$. Then $\alpha = \alpha^2$, so α is left regular and right regular.

To prove (1) ⇒ (3) and (2) ⇒ (3), we assume that (3) failed. If $|X/\rho| > 2$, then by Lemma 2.9, $T_Y(X,\rho)$ is neither left regular nor right regular. Now, suppose that $|X/\rho| = 2$. Let $X/\rho = \{P_1, P_2\}$.

Case 1 : There is an equivalence class that does not contain any element from Y, say P_1 and $X/\rho \neq \{X-Y, Y\}$ or $|Y| > 1$. Since $P_1 \cap Y = \varnothing$, we have $Y \subseteq P_2$. This implies that $|P_2| > 1$. Let $y \in P_2 \cap Y$ and $x \in P_2$ such that $x \neq y$. Define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} P_1 & P_2 \\ x & y \end{pmatrix}.$$

Clearly, $\alpha \in T_Y(X,\rho)$. We see that $x \in \operatorname{ran}\alpha$ and

$$x\alpha^{-1} \cap \operatorname{ran}\alpha = P_1 \cap \{x, y\} \subseteq P_1 \cap P_2 = \varnothing.$$

This shows that $x\alpha^{-1} \cap \operatorname{ran}\alpha = \varnothing$, so α is not left regular by Theorem 2.6. We also have $x\alpha = y = y\alpha$, but $x \neq y$. Then $\alpha|_{\operatorname{ran}\alpha}$ is not injective. It follows form Theorem 2.7 that α is not right regular.

Case 2 : There is an equivalence class that contains more than one element from Y, say P_1. Let y_1, y_2 be distinct elements in $P_1 \cap Y$. Define $\alpha : X \to X$ by

$$\alpha = \begin{pmatrix} P_1 & P_2 \\ y_1 & y_2 \end{pmatrix}.$$

Then $y_2 \in \operatorname{ran}\alpha$, but $y_2\alpha^{-1} \cap \operatorname{ran}\alpha = P_2 \cap \{y_1, y_2\} = \varnothing$. Then α is not left regular. We see that $y_1\alpha = y_1 = y_2\alpha$, but $y_1 \neq y_2$, so $\alpha|_{\operatorname{ran}\alpha}$ is not injective and hence, α is not a right regular element of $T_Y(X,\rho)$. $\square$

3. Conclusion

In this work, we established necessary and sufficient conditions under which elements of the semigroup $T_Y(X,\rho)$ are regular, left regular, or right regular, and applied these conditions to study the overall regularity of $T_Y(X,\rho)$. Note that when ρ is the identity relation on X, $T_Y(X,\rho)$ coincides with $T_Y(X)$. In this case, Theorems 2.4 and 2.5 show that $T_Y(X,\rho)$ is regular if and only if $Y = X$ or $|Y| = 1$, which is consistent with the regularity condition for $T_Y(X)$ established in [1]. Therefore, our results generalize and extend the known results for $T_Y(X)$.

Acknowledgement. This work was supported by the PSU-TUYF Charitable Trust Fund, Prince of Songkla University, Contract no. 1-2566-02.

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