

FEKETE–SZEGÖ INEQUALITIES FOR BI-UNIVALENT FUNCTIONS INVOLVING CAUCHY AND BERNOULLI POLYNOMIALS

WAGGAS GALIB ATSHAN^{1,*}, MOHAMMAD EL-ITYAN² AND ADEL SALIM TAYYAH³

ABSTRACT. We present a unified complex-variable extension of both Cauchy and Bernoulli polynomials on the open unit disc. This generalized framework deepens their analytic properties and opens new directions for application in geometric function theory. Within this setting, we introduce and analyze two bi-univalent function classes, for which we establish upper bounds for the initial Taylor coefficients $|a_2|$ and $|a_3|$, along with refined Fekete–Szegő-type inequalities. Starlike and convex subclasses naturally emerge as special cases of the general theory. To the best of our knowledge, this is the first work to jointly apply Cauchy and Bernoulli polynomials in the study of bi-univalent functions, laying the groundwork for future research.

Keywords. Fekete–Szegő inequalities, Cauchy polynomials, Bernoulli polynomials, Open Unit Disc.

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1. INTRODUCTION

In recent times, there has been growing attention toward analytic bi-univalent functions within the scope of geometric function theory. These functions, which are not only univalent in the open unit disk but also have univalent inverses, offer a versatile setup for studying coefficient bounds and tackling classical challenges like the Fekete–Szegő problem. Researchers have investigated how such functions behave in connection with various geometric domains, including those shaped like nephroids, cardioids, symmetric figures, and crescent-like regions.

Let $\mathcal{A}$ be the family of functions that are analytic and univalent inside the open unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and that satisfy the conditions $h(0) = 0$ and $h'(0) = 1$. These functions can be written in the form:

$$h(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k, \quad (\zeta \in \mathbb{D}). \quad (1.1)$$

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* Corresponding author

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We use $\mathbb{S}$ to refer to the subclass of $\mathcal{A}$ containing functions that remain univalent throughout $\mathbb{D}$. Any function $h \in \mathbb{S}$ has an inverse h^{-1} defined on a suitable disk and satisfying:

$$h^{-1}(h(\varsigma)) = \varsigma \quad \text{and} \quad h(h^{-1}(\omega)) = \omega, \quad (\varsigma \in \mathbb{D}, |\omega| < r_0(h) \geq \tfrac{1}{4}).$$

This inverse function has a series representation of the form:

$$h^{-1}(\omega) = g(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (a_4 + 5a_3^2 - 5a_3a_2)\omega^4 + \dots \quad (1.2)$$

We denote by Σ the class of **bi-univalent functions**, which are functions in $\mathbb{D}$ of the form (1.1), and for which both h and its inverse h^{-1} are univalent in their domains [36].

In recent work, there has been a surge of interest in defining subclasses of Σ using generating functions of well-known polynomial families such as Jacobi, Gegenbauer, Horadam, and Fibonacci polynomials. These approaches allow researchers to explore flexible formulations, study coefficient bounds, and uncover new generalizations through tunable parameters [5, 2, 3, 6, 19, 20, 18].

The study of Cauchy numbers, Cauchy polynomials, and their degenerate forms has attracted interest due to their rich combinatorial and arithmetical properties. Researchers have developed generalizations, explicit formulas, and recurrence relations, and explored connections with poly-Bernoulli numbers, Stirling numbers, and Mellin derivatives, leading to new identities and algebraic structures [23, 24, 25].

Cauchy numbers have attracted significant attention due to their appearance across different fields (see, for example, [31]). Several works have introduced generalizations of these numbers (cf. [26, 27, 1, 15]). One important extension involves degenerate versions, first studied by Carlitz [12, 13]. He introduced the degenerate Cauchy polynomials $c_n(x \mid \varrho)$ through the generating function

$$\frac{\varrho t}{(1+t)^\varrho - 1} (1+t)^{\varrho-x} = \sum_{n=0}^{\infty} c_n(x \mid \varrho) \frac{t^n}{n!}.$$

As $x = \varrho \rightarrow 0$, these polynomials reduce to the classical Cauchy numbers c_n , given by [14, 24]

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

Alongside $c_n(x \mid \varrho)$, Carlitz [13] also introduced the degenerate Bernoulli polynomials $\mathfrak{J}_n(x \mid \varrho)$, defined by

$$\frac{t}{(1+\varrho t)^{1/\varrho} - 1} (1+\varrho t)^{x/\varrho} = \sum_{n=0}^{\infty} \mathfrak{J}_n(x \mid \varrho) \frac{t^n}{n!}.$$

As $\varrho \rightarrow 0$, these polynomials approach the classical Bernoulli polynomials. Further studies by Carlitz [13] and Howard [23] expanded the relationships between these degenerate polynomials and other number sequences.

When two functions h and g , both belonging to the class $\mathcal{A}$, are related through a composition with a Schwarz function $\mathfrak{w}$, we say that h is *subordinate* to g . This

happens if there is an analytic function $\mathfrak{w}(\varsigma)$, with $\mathfrak{w}(0) = 0$ and $|\mathfrak{w}(\varsigma)| < 1$, such that

$$\hbar(\varsigma) = g(\mathfrak{w}(\varsigma)) \quad \text{for all } \varsigma \in \mathbb{D}.$$

In this case, we write:

$$\hbar \prec g \quad \text{or simply} \quad \hbar(\varsigma) \prec g(\varsigma), \quad (\varsigma \in \mathbb{D}). \quad (1.3)$$

If g is univalent throughout $\mathbb{D}$, the condition $f \prec g$ is equivalent to:

$$\hbar(0) = g(0), \quad \text{and} \quad \hbar(\mathbb{D}) \subset g(\mathbb{D}). \quad (1.4)$$

Between the late 1960s and mid-1980s, researchers began tackling coefficient problems for analytic bi-univalent functions. The first known estimate for $|a_2|$ was put forward by Lewin in 1967 [29]. Later, Brannan and Clunie [10] improved the estimate in 1980. Around 1969, Netanyahu [32] established a maximum of $|a_2| = \frac{4}{3}$, and by 1984, Tan [39] refined the bound, showing $|a_2| \leq 1.485$ for functions in Σ . Additionally, in 1985, Brannan and Taha [11] looked into several subclasses of bi-univalent functions and gave further bounds for their coefficients.

Subsequent research has focused on obtaining coefficient estimates for higher-order terms in bi-univalent functions, enhancing the understanding of their geometric properties and analytic behavior [17, 7, 35, 22, 43, 34, 40].

2. DEFINITIONS AND EXAMPLES

In geometric function theory, starlike and convex functions play an essential role in understanding the structure of analytic functions. A function $\hbar$, analytic in the open unit disk $\mathbb{D}$, is said to belong to the class of *starlike functions of order* $\mathfrak{J}$, denoted by $\mathcal{S}^*(\mathfrak{J})$, if it meets the condition:

$$\operatorname{Re} \left\{ \frac{z\hbar'(\varsigma)}{\hbar(\varsigma)} \right\} > \mathfrak{J}, \quad \varsigma \in \mathbb{D}, \quad 0 \leq \mathfrak{J} < 1.$$

When $\mathfrak{J} = 0$, this reduces to the familiar class of starlike functions $\mathcal{S}^*$. For more insights and results on these functions, see [4, 41, 8].

To go beyond the classical setup, Ma and Minda [30] proposed a broader family of functions through a subordination condition:

$$\mathcal{S}^*(h) = \left\{ \hbar \in \mathcal{A} : \frac{z\hbar'(\varsigma)}{\hbar(\varsigma)} \prec p(\varsigma) \right\},$$

where h is analytic in $\mathbb{D}$, satisfies $p(0) = 1$, has a positive real part throughout the disk, and is symmetric about the real axis. Moreover, the function h is starlike with respect to 1 and satisfies $p'(0) > 0$. Members of this class are often called *Ma–Minda starlike functions*.

These classes generalize many well-known subclasses of univalent functions by choosing different forms for $p(\varsigma)$. Some notable examples include:

This approach provides a unified framework for studying different types of starlike and convex functions using the subordination principle.

Class Name	$p(\varsigma)$	Author(s) and Reference
$\mathcal{S}_L^*$	$\sqrt{1 + \varsigma}$	Sokl et al. [38]
$\mathcal{S}_{\mathcal{C}}^*$	$\varsigma + \sqrt{1 + \varsigma^2}$	Raina et al. [33]
$\mathcal{S}_C^*$	$1 + \frac{4}{3}\varsigma + \frac{2}{3}\varsigma^2$	Sharma et al. [37]
$\mathcal{S}_{\sin}^*$	$1 + \sin \varsigma$	Cho et al. [16]
$\mathcal{S}_h^*$	$1 + \sin^{-1} \varsigma$	Kumar et al. [28]
$\mathcal{S}_{Ne}^*$	$1 + \varsigma - \frac{1}{3}\varsigma^3$	Wani et al. [42]
$\mathcal{S}_H^*$	$1 + \varsigma + \frac{1}{3}\varsigma^2 - \frac{1}{9}\varsigma^3$	Tayyiah and Atshan [41]

TABLE 1. Some subclasses of starlike or bi-univalent functions based on various $p(\varsigma)$ mappings.

In this paper, we introduce a new extension of the Cauchy polynomials by incorporating a complex variable ς within the open unit disk $\mathbb{D}$. This extension is designed to study the analytic properties of these polynomials in the complex domain and to explore their applications in geometric function theory. Through this approach, we aim to establish new connections between Cauchy polynomials and important concepts in complex analysis.

In this paper, we consider the degenerate Cauchy polynomials $c_n(x \mid \varrho)$ defined by the generating function

$$\frac{\varrho \varsigma}{(1 + \varsigma)^\varrho - 1} (1 + \varsigma)^{\varrho - x} = \sum_{n=0}^{\infty} c_n(x \mid \varrho) \frac{\varsigma^n}{n!}, \quad \text{for } \varsigma \in \mathbb{D},$$

where $\mathbb{D}$ the open unit disk.

The first few terms of the series are given by (see Figure 1)

$$\begin{aligned} \mathcal{J}(x, \varrho, \varsigma) &= \frac{\varrho \varsigma}{(1 + \varsigma)^\varrho - 1} (1 + \varsigma)^{\varrho - x} = 1 + \frac{1}{2}(1 - 2x + \varrho)\varsigma + \frac{1}{12}(-1 + 6x^2 - 6x\varrho + \varrho^2)\varsigma^2 + \cdots \\ &= 1 + \mathbb{J}_1(x, \varrho)\varsigma + \mathbb{J}_2(x, \varrho)\varsigma^2 + \cdots \end{aligned}$$

In the limit as $x = \varrho \rightarrow 0$, the polynomials reduce to the classical Cauchy numbers c_n , which are generated by:

$$\frac{\varsigma}{\log(1 + \varsigma)} = \sum_{n=0}^{\infty} c_n \frac{\varsigma^n}{n!}, \quad \varsigma \in \mathbb{D}. \quad (1.2)$$

with the expansion

$$\mathcal{M}(\varsigma) = \frac{\varsigma}{\log(1 + \varsigma)} = 1 + \frac{1}{2}\varsigma - \frac{1}{12}\varsigma^2 + \frac{1}{24}\varsigma^3 + \cdots.$$

Similarly, we consider the degenerate Bernoulli polynomials $\mathfrak{J}_n(x \mid \varrho)$ defined by

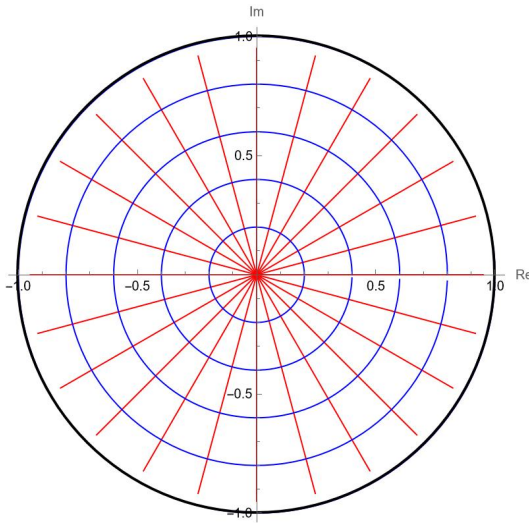
$$\frac{\varsigma}{(1 + \varrho \varsigma)^{1/\varrho} - 1} (1 + \varrho \varsigma)^{x/\varrho} = \sum_{n=0}^{\infty} \mathfrak{J}_n(x \mid \varrho) \frac{\varsigma^n}{n!}, \quad \varsigma \in \mathbb{D}. \quad (1.3)$$

The first few terms are

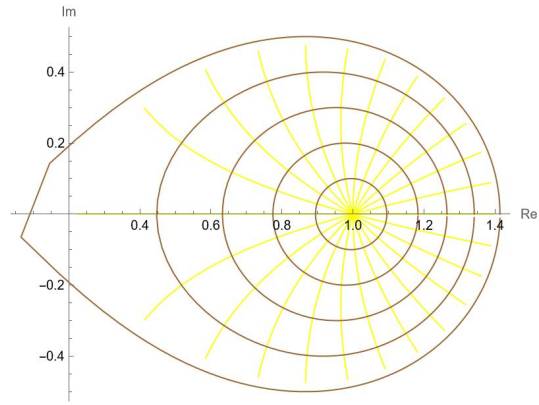
$$\begin{aligned}\mathcal{W}(x, \varrho, \varsigma) &= \frac{\varsigma}{(1 + \varrho\varsigma)^{1/\varrho} - 1} (1 + \varrho\varsigma)^{x/\varrho} = 1 + \left(x + \frac{1}{2}(-1 + \varrho)\right) \varsigma + \frac{1}{12} (1 - 6x + 6x^2 - \varrho^2) \varsigma^2 + \dots \\ &= 1 + \mathbb{M}_1(x, \varrho) \varsigma + \mathbb{M}_2(x, \varrho) \varsigma^2 + \dots\end{aligned}$$

In the limit as $\varrho \rightarrow 0$, we recover the classical Bernoulli polynomials $B_n(x)$ via

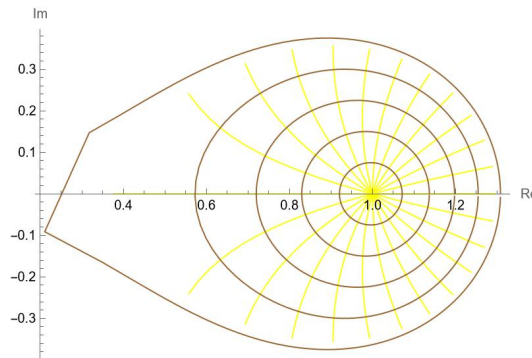
$$B_n(x) = \lim_{\varrho \rightarrow 0} \mathfrak{J}_n(x \mid \varrho).$$



(a) The domain is the open unit disk $\mathbb{D}$.



(b) The range is the image of $\mathbb{D}$ under the mapping $\mathcal{J}(\frac{1}{2}, 1, \varsigma)$.



(c) The range is the image of $\mathbb{D}$ under the mapping $\mathcal{J}(\frac{1}{2}, \frac{3}{4}, \varsigma)$.

FIGURE 1. Plot of the mapping $\mathcal{J}(x, \varrho, \varsigma)$.

Definition 2.1. [9] Suppose $\hbar \in \mathcal{A}$, with $\zeta \geq 1$ being a real constant and $\mathfrak{J}$ a number such that $0 \leq \mathfrak{J} < 1$. Then, the function $\hbar(\varsigma)$ is said to belong to the

class $\mathcal{L}_\varrho(\mathfrak{J})$ of ζ -pseudo-starlike functions of order $\mathfrak{J}$ in the open unit disk $\mathbb{D}$ if and only if it satisfies the inequality:

$$\operatorname{Re} \left(\frac{\varsigma [\tilde{h}'(\varsigma)]^\zeta}{\tilde{h}(\varsigma)} \right) > \mathfrak{J}, \quad \text{for all } \varsigma \in \mathbb{D}.$$

Babalola [9] further pointed out that every function in this category also qualifies as a Bazilevi function of order $\frac{1}{\zeta}$ and type $1 - \frac{1}{\zeta}$, and is univalent within $\mathbb{D}$.

Definition 2.2. A function $f \in \Sigma$ of the form (1.1) is said to belong to the class $\mathcal{ST}_\Sigma(x, \varrho, \zeta)$ if both of the following subordination conditions are satisfied:

$$\frac{\varsigma \tilde{h}'(\varsigma)^\zeta}{\tilde{h}(\varsigma)} \prec \mathcal{J}(x, \varrho, \varsigma) \quad \text{and} \quad \frac{\omega g'(\omega)^\zeta}{g(\omega)} \prec \mathcal{J}(x, \varrho, \omega),$$

where $x, \varrho \in \mathbb{R}$ and $\varsigma, \omega \in \mathbb{D}$.

Example 2.3. A function $f \in \Sigma$ given by (1.1) is a member of the class $\mathcal{ST}_\Sigma(x, \varrho, 1)$ if the following hold true:

$$\frac{\varsigma \tilde{h}'(\varsigma)}{\tilde{h}(\varsigma)} \prec \mathcal{J}(x, \varrho, \varsigma) \quad \text{and} \quad \frac{\omega g'(\omega)}{g(\omega)} \prec \mathcal{J}(x, \varrho, \omega),$$

with $x, \varrho \in \mathbb{R}$ and $\varsigma, \omega \in \mathbb{D}$.

Definition 2.4. Let $f \in \Sigma$ as in (1.1). Then we say f belongs to the class $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$ if it satisfies:

$$\frac{\varsigma \tilde{h}'(\varsigma)^\zeta}{\tilde{h}(\varsigma)} \prec \mathcal{W}(x, \varrho, \varsigma) \quad \text{and} \quad \frac{\omega g'(\omega)^\zeta}{g(\omega)} \prec \mathcal{W}(x, \varrho, \omega),$$

where again $x, \varrho \in \mathbb{R}$, and $\varsigma, \omega \in \mathbb{D}$.

Example 2.5. If a function $f \in \Sigma$ of the form (1.1) satisfies:

$$\frac{\varsigma \tilde{h}'(\varsigma)}{\tilde{h}(\varsigma)} \prec \mathcal{W}(x, \varrho, \varsigma) \quad \text{and} \quad \frac{\omega g'(\omega)}{g(\omega)} \prec \mathcal{W}(x, \varrho, \omega),$$

then it is said to be in the class $\mathcal{SN}_\Sigma(x, \varrho, 1)$, where $x, \varrho \in \mathbb{R}$, and $\varsigma, \omega \in \mathbb{D}$.

3. FEKETE–SZEGÖ INEQUALITY AND COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{ST}_\Sigma(x, \varrho, \zeta)$

In this part, we focus on deriving sharp estimates for the initial coefficients $|a_2|$ and $|a_3|$ of functions that fall within the class $\mathcal{ST}_\Sigma(x, \varrho, \zeta)$. We also work out an upper bound for the Fekete–Szegő expression associated with this class, as discussed in [21].

Theorem 3.1. Suppose that $0 \leq \zeta \leq 1$ and $\varsigma, \omega \in \mathbb{D}$. If $\tilde{h}(\varsigma) \in \mathcal{A}$ belongs to the class $\mathcal{ST}_\Sigma(x, \varrho, \zeta)$, then we have:

$$|a_2| \leq \frac{|\mathbb{J}_1(x, \varrho)| \sqrt{\mathbb{J}_1(x, \varrho)}}{\sqrt{(2\zeta - 1)[\zeta \mathbb{J}_1(x, \varrho)]^2 - (2\zeta - 1)[\mathbb{J}_2(x, \varrho)]}},$$

and

$$|a_3| \leq \frac{|\mathbb{J}_1(x, \varrho)|}{|3\zeta - 1|} + \frac{|\mathbb{J}_1(x, \varrho)|^2}{|2\zeta - 1|^2}, \quad (3.1)$$

Proof. If $\hbar(\varsigma) \in \mathcal{ST}_\Sigma(x, \varrho, \zeta)$, then there are two analytic functions s and t defined in the unit disk $\mathbb{D}$, represented as:

$$s(\varsigma) = \sum_{k=1}^{\infty} s_k \varsigma^k, \quad (3.2)$$

$$t(\omega) = \sum_{k=1}^{\infty} t_k \omega^k, \quad (3.3)$$

such that their absolute values remain bounded by 1 in the unit disk:

$$|s(\varsigma)| = |s_1\varsigma + s_2\varsigma^2 + s_3\varsigma^3 + \dots| < 1, \quad (3.4)$$

$$|t(\omega)| = |t_1\omega + t_2\omega^2 + t_3\omega^3 + \dots| < 1. \quad (3.5)$$

From (3.4) and (3.5), it follows that the coefficients satisfy $|s_k| \leq 1$ and $|t_k| \leq 1$ for all $k \in \mathbb{N}$ and $\varsigma, \omega \in \mathbb{D}$. Based on Definition 1.2, the function $\hbar$ and its inverse g can be linked through:

$$\frac{\varsigma \hbar'(\varsigma)}{\hbar(\varsigma)} = \mathcal{J}(x, \varrho, s(\varsigma)), \quad (3.6)$$

$$\frac{\omega g'(\omega)}{g(\omega)} = \mathcal{J}(x, \varrho, t(\omega)). \quad (3.7)$$

Now, applying the series expansions for $\hbar(\varsigma)$ and $g(\omega)$ from equations (1.1) and (1.2), respectively, the expressions (3.6) and (3.7) simplify accordingly.

$$\begin{aligned} & 1 + (2\zeta - 1)a_2\varsigma + [(3\zeta - 1)a_3 + (2\zeta^2 - 4\zeta + 1)a_2^2] \varsigma^2 + \dots \\ & = 1 + \mathbb{J}_1(x, \varrho)[s(\varsigma)] + \mathbb{J}_2(x, \varrho)[s(\varsigma)]^2 + \dots, \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} & 1 - (2\zeta - 1)a_2\omega + [(-3\zeta + 1)a_3 + (2\zeta^2 + 2\zeta - 1)a_2^2] \omega^2 + \dots \\ & = 1 + \mathbb{J}_1(x, \varrho)[t(\omega)] + \mathbb{J}_2(x, \varrho)[t(\omega)]^2 + \dots \end{aligned} \quad (3.9)$$

Replacing $s(\varsigma)$ and $t(\omega)$ in (3.8) and (3.9), a basic computation yields:

$$\begin{aligned} & 1 + (2\zeta - 1)a_2\varsigma + [(3\zeta - 1)a_3 + (2\zeta^2 - 4\zeta - +1)a_2^2] \varsigma^2 + \dots \\ & = 1 + [\mathbb{J}_1(x, \varrho)]s_1\varsigma + ([\mathbb{J}_1(x, \varrho)]s_2 + [\mathbb{J}_2(x, \varrho)]s_1^2)\varsigma^2 + \dots, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} & 1 - (2\zeta - 1)a_2\omega + [(-3\zeta + 1)a_3 + (2\zeta^2 + 2\zeta - 1)a_2^2] \omega^2 + \dots \\ & = 1 + [\mathbb{J}_1(x, \varrho)]t_1\omega + ([\mathbb{J}_1(x, \varrho)]t_2 + [\mathbb{J}_2(x, \varrho)]t_1^2)\omega^2 + \dots \end{aligned} \quad (3.11)$$

By matching the corresponding terms in the series expansions from (3.10) and (3.11), we arrive at the following identities:

$$(2\zeta - 1)a_2 = [\mathbb{J}_1(x, \varrho)]s_1, \quad (3.12)$$

$$(3\zeta - 1)a_3 + (2\zeta^2 - 4\zeta + 1)a_2^2 = [\mathbb{J}_1(x, \varrho)]s_2 + [\mathbb{J}_2(x, \varrho)]s_1^2, \quad (3.13)$$

and

$$(-2\zeta + 1)a_2 = [\mathbb{J}_1(x, \varrho)]t_1, \quad (3.14)$$

$$(-3\zeta + 1)a_3 + (2\zeta^2 + 2\zeta - 1)a_2^2 = [\mathbb{J}_1(x, \varrho)]t_2 + [\mathbb{J}_2(x, \varrho)]t_1^2. \quad (3.15)$$

By adding equations (3.12) and (3.14), we can easily obtain:

$$s_1 = -t_1. \quad (3.16)$$

By squaring and summing equations (3.12) and (3.14), we arrive at:

$$a_2^2 = \frac{[\mathbb{J}_1(x, \varrho)]^2}{2(2\zeta - 1)^2}(s_1^2 + t_1^2). \quad (3.17)$$

Likewise, adding (3.13) and (3.15) leads to:

$$2\zeta(2\zeta - 1)a_2^2 = [\mathbb{J}_1(x, \varrho)](s_2 + t_2) + [\mathbb{J}_2(x, \varrho)](s_1^2 + t_1^2). \quad (3.18)$$

Now, combining the outcomes from (3.17) and (3.18), we get:

$$a_2^2 = \frac{[\mathbb{J}_1(x, \varrho)]^3(s_2 + t_2)}{2(2\zeta - 1)[\zeta[\mathbb{J}_1(x, \varrho)]^2 - (2\zeta - 1)[\mathbb{J}_2(x, \varrho)]]}. \quad (3.19)$$

Since it's known that $|s_k| \leq 1$ and $|t_k| \leq 1$ for all $k \in \mathbb{N}$, a straightforward estimate gives:

$$|a_2| \leq \frac{|\mathbb{J}_1(x, \varrho)|\sqrt{|\mathbb{J}_1(x, \varrho)|}}{\sqrt{(2\zeta - 1)[\zeta[\mathbb{J}_1(x, \varrho)]^2 - (2\zeta - 1)[\mathbb{J}_2(x, \varrho)]]}}. \quad (3.20)$$

On the other hand, subtracting equation (3.13) from equation (3.15), we get:

$$a_3 - a_2^2 = \frac{[\mathbb{J}_1(x, \varrho)](s_2 - t_2)}{2(3\zeta - 1)}, \quad (3.21)$$

using equation (3.16). Substituting equation (3.17) into (3.21), we obtain:

$$a_3 = \frac{[\mathbb{J}_1(x, \varrho)]}{2(3\zeta - 1)}(s_2 - t_2) + \frac{[\mathbb{J}_1(x, \varrho)]^2}{2(2\zeta - 1)^2}(s_1^2 + t_1^2). \quad (3.22)$$

After applying the triangle rule and simplifying a bit, we reach:

$$|a_3| \leq \frac{|\mathbb{J}_1(x, \varrho)|}{|3\zeta - 1|} + \frac{|\mathbb{J}_1(x, \varrho)|^2}{|2\zeta - 1|^2}. \quad (3.23)$$

□

Theorem 3.2. Suppose $0 \leq \zeta \leq 1$, $\eta \in \mathbb{R}$, and $\varsigma, \omega \in \mathbb{D}$. If $h(\varsigma) \in \mathcal{A}$ and belongs to the class $\mathcal{ST}_\Sigma(x, \varrho, \zeta)$, then we have:

$$|a_3 - \eta a_2^2| \leq \begin{cases} |\mathbb{J}_1(x, \varrho)| \frac{1}{3\zeta-1} & \text{for } |\Delta(\eta)| \leq \left| \frac{1}{2(3\zeta-1)} \right| \\ 2|\mathbb{J}_1(x, \varrho)| |\Delta(\eta)| & \text{for } |\Delta(\eta)| \geq \left| \frac{1}{2(3\zeta-1)} \right|, \end{cases} \quad (3.24)$$

where

$$\Delta(\eta) = \frac{(1-\eta)[\mathbb{J}_1(x, \varrho)]^2(s_2 + t_2)}{2(2\zeta-1)[\zeta[\mathbb{J}_1(x, \varrho)]^2 - (2\zeta-1)[\mathbb{J}_2(x, \varrho)]}.$$

Proof. Suppose that $\eta \in \mathbb{R}$. From equation (3.21), we can write:

$$a_3 - \eta a_2^2 = \frac{[\mathbb{J}_1(x, \varrho)](s_2 - t_2)}{2(3\zeta-1)} + (1-\eta)a_2^2. \quad (3.25)$$

Substituting equation (3.19) into (3.25), we get:

$$a_3 - \eta a_2^2 = \frac{[\mathbb{J}_1(x, \varrho)](s_2 - t_2)}{2(3\zeta-1)} + \frac{(1-\eta)[\mathbb{J}_1(x, \varrho)]^3(s_2 + t_2)}{2(2\zeta-1)[\zeta[\mathbb{J}_1(x, \varrho)]^2 - (2\zeta-1)[\mathbb{J}_2(x, \varrho)]}. \quad (3.26)$$

An equivalent expression is given by:

$$a_3 - \eta a_2^2 = [\mathbb{J}_1(x, \varrho)] \left[\left(\Delta(\eta) + \frac{1}{2(3\zeta-1)} \right) s_2 + \left(\Delta(\eta) - \frac{1}{2(3\zeta-1)} \right) t_2 \right], \quad (3.27)$$

where

$$\Delta(\eta) = \frac{(1-\eta)[\mathbb{J}_1(x, \varrho)]^2(s_2 + t_2)}{2(2\zeta-1)[\zeta[\mathbb{J}_1(x, \varrho)]^2 - (2\zeta-1)[\mathbb{J}_2(x, \varrho)]}. \quad (3.28)$$

Taking the modulus of both sides of equation (3.27) and applying the triangle inequality, with the facts that $|s_2| \leq 1$ and $|t_2| \leq 1$, we obtain the desired inequality in (3.24). $\square$

By substituting $\zeta = 1$ in the previous Theorem, we arrive at the following corollary:

Corollary 3.3. For $\eta \in \mathbb{R}$ and $\varsigma, \omega \in \mathbb{D}$, let $h(\varsigma) \in \mathcal{A}$ be in the class $\mathcal{ST}_\Sigma(x, \varrho, 1)$. Then:

$$\begin{aligned} |a_2| &\leq \frac{|\mathbb{J}_1(x, \varrho)| \sqrt{|\mathbb{J}_1(x, \varrho)|}}{\sqrt{[\mathbb{J}_1(x, \varrho)]^2 - [\mathbb{J}_2(x, \varrho)]}}, \\ |a_3| &\leq \frac{|\mathbb{J}_1(x, \varrho)|}{2} + |\mathbb{J}_1(x, \varrho)|^2, \end{aligned} \quad (3.29)$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\mathbb{J}_1(x, \varrho)|}{2} & \text{for } |\Delta_0(\eta)| \leq \frac{1}{4} \\ 2|\mathbb{J}_1(x, \varrho)| |\Delta_0(\eta)| & \text{for } |\Delta_0(\eta)| \geq \frac{1}{4}, \end{cases} \quad (3.30)$$

where

$$\Delta_0(\eta) = \frac{2(1-\eta)[\mathbb{J}_1(x, \varrho)]^2(s_2 + t_2)}{[\mathbb{J}_1(x, \varrho)]^2 - [\mathbb{J}_2(x, \varrho)]}.$$

4. FEKETE–SZEGÖ INEQUALITY AND COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$

In what follows, we focus on estimating the upper bounds for the coefficients $|a_2|$ and $|a_3|$ of functions that fall within the class $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$. We also explore bounds for the Fekete–Szegő functional tied to this function class.

Theorem 4.1. *Let $0 \leq \zeta \leq 1$ and suppose $\varsigma, \omega \in \mathbb{D}$. Assume $h(\varsigma) \in \mathcal{A}$ and that it belongs to the class $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$. Then the following estimates hold:*

$$|a_2| \leq \frac{|\mathbb{M}_1(x, \varrho)|\sqrt{\mathbb{M}_1(x, \varrho)}}{\sqrt{(2\zeta - 1)[\zeta\mathbb{M}_1(x, \varrho)^2 - (2\zeta - 1)\mathbb{M}_2(x, \varrho)]}},$$

and

$$|a_3| \leq \frac{|\mathbb{M}_1(x, \varrho)|}{|3\zeta - 1|} + \frac{|\mathbb{M}_1(x, \varrho)|^2}{|2\zeta - 1|^2}. \quad (4.1)$$

Proof. The proof of this theorem follows the same steps as the proof of Theorem 3.1 in the previous section and is therefore omitted for brevity. $\square$

Theorem 4.2. *For $0 \leq \zeta \leq 1$, $\eta \in \mathbb{R}$ and $\varsigma, \omega \in \mathbb{D}$, let $h(\varsigma) \in \mathcal{A}$ be in the class $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$. Then:*

$$|a_3 - \eta a_2^2| \leq \begin{cases} [\mathbb{M}_1(x, \varrho)]|\frac{1}{3\zeta-1}| & \text{for } |\Delta(\eta)| \leq |\frac{1}{2(3\zeta-1)}| \\ 2[\mathbb{M}_1(x, \varrho)]|\Delta(\eta)| & \text{for } |\Delta(\eta)| \geq |\frac{1}{2(3\zeta-1)}|, \end{cases} \quad (4.2)$$

where

$$\Delta(\eta) = \frac{(1-\eta)[\mathbb{M}_1(x, \varrho)]^2(s_2 + t_2)}{2(2\zeta - 1)[\zeta[\mathbb{M}_1(x, \varrho)]^2 - (2\zeta - 1)[\mathbb{M}_2(x, \varrho)]}.$$

Proof. The proof of this theorem follows the same steps as the proof of Theorem 3.2 in the previous section and is therefore omitted for brevity. $\square$

CONCLUSION

In this study, we introduced a unified complex-variable extension of Cauchy and Bernoulli polynomials and applied them to define the bi-univalent classes $\mathcal{SI}_\Sigma(x, \varrho, \zeta)$ and $\mathcal{SN}_\Sigma(x, \varrho, \zeta)$. Coefficient bounds and Fekete–Szeg inequalities were established. This framework offers a novel perspective linking special polynomials with geometric function theory, and sets the stage for further exploration.

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¹ DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF AL-QADISIYAH, DIWANIYAH 58002, IRAQ.

Email address: waggas.galib@qu.edu.iq

² DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, AL-BALQA APPLIED UNIVERSITY, SALT, JORDAN.

Email address: mohammad65655vv22@gmail.com, 52107151010@std.bau.edu.jo

³ DEPARTMENT OF COMPUTER SCIENCE, COLLEGE OF COMPUTER SCIENCE AND INFORMATION TECHNOLOGY, UNIVERSITY OF AL-QADISIYAH, DIWANIYAH, 58002, IRAQ.

Email address: adel.tayh@qu.edu.iq