

EPICONVERGENCE OF A SEQUENCE OF INTEGRAL FUNCTIONALS ON A MEASURE SPACE

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ABSTRACT. This work presents the epiconvergence of nonlinear integral functionals defined on bounded Radon measures, with applications to homogenization in microstructured media. Using variational and measure-theoretic tools, it characterizes the homogenized functional and proves convergence under the vague topology.

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1. INTRODUCTION

Epiconvergence, introduced in the 1970s by [8], is a central concept in modern variational analysis. It has established itself as a powerful tool for studying asymptotic problems and limit processes in sequences of functionals [3, 2, 1]. In contrast to pointwise or uniform convergence, epiconvergence guarantees the convergence of minimizers and minimum values, making it a robust theoretical framework for the analysis of relaxation, homogenization, and approximation methods in the calculus of variations.

Initially introduced to treat functionals defined on spaces of integrable functions, epiconvergence has gradually broadened its scope. It now extends to measure spaces, functions of bounded variation (BV), and, more recently, to vector-valued Radon measure spaces [12, 11]. This notion now plays a central role in several disciplines, notably in the mechanics of heterogeneous media, homogenization theory, plasticity, and image analysis.

Following the approach presented in [4, 5, 6, 7], we study the epiconvergence of a sequence of nonlinear integral functionals defined on $(M_b(A))^n$, where $M_b(A)$ denotes the space of bounded Radon measures on a bounded open set $A \subset \mathbb{R}^n$. These functionals incorporate an underlying periodic structure, represented by a Y -Radon measure, and are parameterized by a small scale $\varepsilon_n \rightarrow 0$, reflecting the presence of a fine microstructure or domain heterogeneity.

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The framework under consideration is based on the analysis of a family of integral-type functionals $F_n(u)$. Our main objective is to characterize the epilimit of this sequence of functionals, denoted by F^{hom} , by identifying a homogenized functional expressed in terms of the convex relaxation, which corresponds to the lower semicontinuous envelope of a cell-type functional defined on the torus Y . The main result shows that F_n epiconverges to F^{hom} with respect to the vague topology on $(M_b(A))^n$, revealing an emerging homogeneous structure in the limit.

This type of convergence has many applications, particularly in the homogenization of composite materials. In this context, oscillating coefficients model physical properties varying at fine scales, and the objective is to describe the macroscopic behavior of the system using a homogenized law.

Note that similar problems have been studied in [4, 5, 6, 7], but assuming that the first variable of the integrand F_n is $\frac{x}{\varepsilon_n}$.

The paper is organized as follows: Section 2 presents definitions and preliminaries about the functionals under study. In Section 3, we establish the main theoretical results of the paper. Finally, a conclusion is drawn in Section 4.

2. FUNDAMENTAL DEFINITIONS AND PRELIMINARIES [9]

Let E be a Banach space and let E^* its dual space. Consider a function $\varphi : E \rightarrow \mathbb{R} \cup \{+\infty\}$.

Definition 2.1. The *subdifferential* of φ at the point $x_0 \in E$ is given by the set

$$\partial\varphi(x_0) := \{f \in E^* : \langle f, y - x_0 \rangle + \varphi(x_0) \leq \varphi(y), \quad \forall y \in E\}.$$

Each continuous linear functional $f \in E^*$ belonging to $\partial\varphi(x_0)$ is called a *subgradient* of φ at x_0 .

2.1. Properties. If for fixed x , the map $\varphi(x, \cdot)$ is convex and bounded, then it admits a subdifferential $\partial\varphi(x, \cdot)$ satisfying:

(1) For every $p \in \mathbb{R}^d$,

$$\varphi(x, p) \geq \varphi(x, y) + \langle \partial\varphi(x, y), p - y \rangle.$$

(2) For all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$|\partial\varphi(x, y)| \leq M(1 + |y|),$$

where M is a positive constant independent of x and y .

Let τ be a topology on E , and consider a sequence of functions $(F_n)_{n \in \mathbb{N}}$ with $F_n : E \rightarrow \mathbb{R} \cup \{+\infty\}$.

Definition 2.2. Let $x_0 \in E$, we define the lower and upper τ -epilimits of the sequence $(F_n)_{n \in \mathbb{N}}$ at x_0 as follows:

$$\text{epi lim inf}_{n \rightarrow \infty} F_n(x_0) := \sup_{V \in J_\tau(x_0)} \liminf_{n \rightarrow \infty} \inf_{x \in V} F_n(x),$$

$$\text{epi lim sup}_{n \rightarrow \infty} F_n(x_0) := \sup_{V \in J_\tau(x_0)} \limsup_{n \rightarrow \infty} \sup_{x \in V} F_n(x),$$

where $J_\tau(x_0)$ represents the family of neighborhoods of the point x_0 relative to the topology τ .

If these two values coincide, the sequence (F_n) is said to τ -epiconverge at x_0 , and the shared value

$$\text{epi lim inf } F_n(x_0) = \text{epi lim sup } F_n(x_0) = F(x_0)$$

is referred to as the τ -epilimit of (F_n) at x_0 .

Furthermore, when $(F_n)_{n \in \mathbb{N}}$ τ -epiconverges pointwise for every $x \in E$, we say that the sequence τ -epiconverges to the function $F : E \rightarrow \overline{\mathbb{R}}$, denoted by

$$F = \tau\text{-epi lim } F_n.$$

In the case of a normed space, an equivalent definition is given by the following theorem.

Theorem 2.3 ([8]). *The τ -epiconvergence of the sequence $(F_n)_{n \in \mathbb{N}}$ to a function F is characterized by the satisfaction of the two following conditions:*

(i) *For any $x \in E$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ that τ -converges to x such that*

$$F(x) \geq \limsup_{n \rightarrow +\infty} F_n(x_n).$$

(ii) *For any $x \in E$ and for every sequence $(x_n)_{n \in \mathbb{N}}$ that τ -converges to x , it holds that*

$$F(x) \leq \liminf_{n \rightarrow +\infty} F_n(x_n).$$

The connection between epiconvergence and variational problems is highlighted in the subsequent theorem.

Theorem 2.4 ([8]). *Assume that $F = \tau\text{-epi lim } F_n$. Then the following statements hold:*

(1) *If $(x_n)_{n \in \mathbb{N}}$ constitutes a minimizing sequence for the problem*

$$\inf_{x \in E} F_n(x),$$

and if this sequence is τ -relatively compact, then any cluster point $\bar{x}$ of $(x_n)_{n \in \mathbb{N}}$ must be a minimizer of F . Furthermore, we have

$$F(\bar{x}) = \min_{x \in E} F(x) = \lim_{n \rightarrow +\infty} \inf_{x \in E} F_n(x).$$

(2) *For any function $G : E \rightarrow \mathbb{R}$, a stability property under continuous perturbation holds:*

$$\tau\text{-epi lim}(F_n + G) = F + G.$$

Problem Statement

Given a sequence of integral functionals defined on $(M_b(A))^n$ by:

$$F_n(u) = \begin{cases} \int_A f\left(\frac{u}{\varepsilon_n}, \frac{du}{d\mu_{\varepsilon_n}}\right) d\mu_{\varepsilon_n}, & \text{if } |u| \ll \mu_{\varepsilon_n}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.1)$$

Where, A is a bounded open subset of $\mathbb{R}^n$ such that $\int_{\partial A} dx = 0$, μ is a positive Y -periodic Radon measure (with $Y = [0, 1]^n$), ε_n is a strictly positive parameter tending to 0, μ_{ε_n} denotes the restriction to A of the measure $\mu\left(\frac{x}{\varepsilon_n}\right)$, and $f(x, y)$ is a function defined from $\mathbb{R}^n \times \mathbb{R}^n$ to $[0, +\infty[$, μ -measurable, x -periodic, y -convex, and satisfies:

$$\begin{cases} f(\cdot, 0) \in L^1(Y); \\ \alpha|y| \leq f(x, y) \leq \beta(1 + |y|), \quad 0 < \alpha \leq \beta < +\infty. \end{cases} \quad (2.2)$$

We associate to f the function denoted by $\int_Y f d\mu$, defined on $\mathbb{R}^n$ by:

$$\int_Y f d\mu(z) = \inf \left\{ \int_Y f(\cdot, w(\cdot)) d\mu : w \in L^1(Y), \int_Y w d\mu = z \right\}, \quad (2.3)$$

where z is a constant.

We now define the integral functional F^{hom} on $(M_b(A))^n$ by:

$$F^{\text{hom}}(u) = \begin{cases} \int_A f^{\text{hom}}\left(\frac{du}{dx}\right) dx + \int_A f^{\text{hom},\infty}\left(\frac{du_s}{d|u_s|}\right) d|u_s|, & \text{if } |u| \ll \mu_{\varepsilon_n}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.4)$$

Where

$$u = \frac{du}{dx} dx + u_s,$$

is the Lebesgue decomposition of u [12], and f^{hom} is the regularized lower semi-continuous convex envelope (i.e., the *s.c.i.*) of $\int_Y f d\mu$, that is,

$$f^{\text{hom}} = \left(\int_Y f d\mu \right)^{**}, \quad (2.5)$$

and

$$f^{\text{hom},\infty} = \limsup_{t \rightarrow +\infty} \frac{f^{\text{hom}}(tx)}{t}.$$

Here, $f^{\text{hom},\infty}$ is the recession function of f^{hom} .

- (1) From (2.2), we deduce that $\int_Y f d\mu$ is finite at 0 and that its regularized lower semi-continuous envelope, f^{hom} , satisfies the coercivity inequality.
- (2) The polar of f^{hom} is equal to the mean of f^* , defined by:

$$\mathfrak{M}_Y(f^*) = \int_Y f^*(\cdot, y) dy.$$

- (3) Hypothesis (2.2) also implies that $\mathfrak{M}_Y(f^*)$ is finite at 0. Moreover, if $\mathfrak{M}_Y(f^*)$ has finite values on $\mathbb{R}^n$, it results from [13] that the infimum in (2.3) is attained, and that the function $\int_Y f d\mu$ is lower semi-continuous (l.s.c.) and therefore equal to f^{hom} .

3. MAIN RESULT

We begin this section with two auxiliary lemmas that will be useful in proving the main result.

Lemma 3.1. *Let $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a μ -measurable and Y -periodic function such that $\int_Y |g| d\mu < +\infty$. Then the sequence $v_n = g\left(\frac{x}{\varepsilon_n}\right) \mu_{\varepsilon_n}$ converges to $v = (\int_Y g d\mu) dx$ in the weak- $*$ topology $\sigma(M_b(A), C_b(A))$ when g is positive. Which implies the narrow convergence of v_n to v .*

Proof. Consider the case where $g \geq 0$; the general case will be deduced from it by considering the decomposition $v_n = v_n^+ - v_n^-$, where $v_n^+ = g^+\left(\frac{x}{\varepsilon_n}\right) \mu_{\varepsilon_n}$ and $v_n^- = g^-\left(\frac{x}{\varepsilon_n}\right) \mu_{\varepsilon_n}$. It is therefore sufficient to prove that v_n converges narrowly to v in $M_b^+(A)$. Let n_0 be an integer such that the open set $A_0 =]-n_0, n_0[^n$ contains A . We check after an elementary calculation that:

$$\lim_{n \rightarrow +\infty} \int_{A_0} g\left(\frac{x}{\varepsilon_n}\right) \mu\left(\frac{x}{\varepsilon_n}\right) d\mu = (2n_0)^n \int_Y f d\mu = \text{meas}(A_0) \int_Y f d\mu. \quad (3.1)$$

The sequence $m_n = g\left(\frac{x}{\varepsilon_n}\right) \mu\left(\frac{x}{\varepsilon_n}\right)$ is thus bounded in $M_b^+(A)$ and admits a cluster point m for the vague convergence on A_0 . It suffices to show that $m = (\int_Y g d\mu) dx$. Indeed, from equation (3.1) it follows that $m_n(A_0) \rightarrow m(A_0)$, which implies the narrow convergence of m_n to m , and hence the convergence of $v_n = m_n(A \cap \cdot)$ to $v = m(A \cap \cdot)$, since the boundary of A is m -negligible (i.e., $|\partial A| = 0$ by hypothesis), let $\varphi \in C_0(A_0)$. We consider the family $(x_i^{\varepsilon_n})_{i \in I_n} \subset A_0$, a finite number of points such that each coordinate of $\frac{x_i^{\varepsilon_n}}{\varepsilon_n}$ is an integer. Define the step function φ_n by $\varphi_n(x) := \varphi(x_i^{\varepsilon_n})$ if $x \in x_i^{\varepsilon_n} + \varepsilon Y$.

Clearly, $\|\varphi_n - \varphi\|_\infty$ converges to 0 as $\varepsilon \rightarrow 0$. On the other hand, since m_n is bounded, we have:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{A_0} \varphi dm_n &= \lim_{n \rightarrow +\infty} \int_{A_0} \varphi_n dm_n \\ &= \lim_{n \rightarrow +\infty} \left(\sum_{i \in I_n} \varepsilon_n \varphi(x_i^{\varepsilon_n}) \right) \left(\int_Y f d\mu \right). \end{aligned}$$

This leads to:

$$\lim_{n \rightarrow +\infty} \int_{A_0} \varphi dm_n = \int_{A_0} \varphi dx \int_Y f d\mu = \int_{A_0} \varphi dm.$$

□

Lemma 3.2. *Let $u \in (M_b(A))^d$, and define $m := |u| + dx$. Let Ω_m be the subspace of L_m^∞ consisting of functions v of the form $v = \sum_{i \in I} z_i 1_{A_i}$ where $z_i \in \mathbb{R}^d$, and $(A_i)_{i \in I}$ is a finite family of disjoint open subsets of A such that $\overline{A_i} \subset A_i$ and $\text{meas}(\partial A_i) = 0$. Then we have*

$$F^{\text{hom}} = \sup_{v \in \Omega_m} \left\{ \int_A v \, du - \int_A (f^{\text{hom}})^*(v) \, dx \right\}.$$

Proof. The proof of Lemma 3.2 ([4]) relies on the fact that f^{hom} is coercive. As a result, the integral functional

$$v \in L_m^\infty \mapsto \int_A (f^{\text{hom}})^*(v) \, dx$$

is strongly continuous at 0. □

The main result of our work is given in the following theorem.

Theorem 3.3. *The sequence (F_n) epiconverges to F^{hom} with respect to the vague topology on $(M_b)^n$.*

Proof. To prove Theorem 3.3, it suffices to verify the two assertions (i) and (ii) in Theorem 2.3.

(i) For all $u \in M_b$ and all sequences $(u_n)_n$ narrowly converging to u for the narrow topology, we aim to show

$$\liminf_{n \rightarrow +\infty} F_n(u_n) \geq F^{\text{hom}}(u).$$

According to Lemma 3.1, it suffices to show that for all $v \in \Omega_m$, with $m = |u| + dx$, the following inequality holds:

$$\liminf_{n \rightarrow +\infty} F_n(u_n) \geq \int_A v \, du - \int_A \mathfrak{M}_Y(f^*)(v) \, dx, \quad (3.2)$$

using the identity $(f^{\text{hom}})^* = \mathfrak{M}_Y(f^*)$. We consider a step function v of the form

$$v = \sum_{i \in I} z_i 1_{A_i},$$

where (A_i) is a finite family of disjoint open subsets of A such that $\overline{A_i} \subset A_i$ and $\text{meas}(\partial A_i) = 0$. Since $\text{meas}(A_i) > 0$, we can suppose that $\mathfrak{M}_Y(f^*)(z_i) < +\infty$; otherwise (3.2) is obvious.

For each $\delta > 0$, define the open sets

$$A_i^\delta = \{x \in A_i : d(x, A_i^c) > \delta\},$$

let $\varphi_i^\delta \in C_0(\Omega_i, [0, 1])$ such that $\varphi_i^\delta|_{A_i^\delta} = 1$. Using the convexity, lower semicontinuity and positivity of $f(x, \cdot)$, we obtain

$$\begin{aligned} F_n(u_n) &\geq \sum_{i \in I} \int_{A_i} \varphi_i^\delta f\left(\frac{u_n}{\varepsilon_n}, \frac{du_n}{d\mu_{\varepsilon_n}}\right) d\mu_{\varepsilon_n} \\ &= \sum_{i \in I} \int_{A_i} \varphi_i^\delta f^{**}\left(\frac{u_n}{\varepsilon_n}, \frac{du_n}{d\mu_{\varepsilon_n}}\right) d\mu_{\varepsilon_n} \\ &\geq \sum_{i \in I} \int_{A_i} \varphi_i^\delta \left[z_i \frac{du_n}{d\mu_{\varepsilon_n}} - f^*\left(\frac{u_n}{\varepsilon_n}, z_i\right) \right] d\mu_{\varepsilon_n} \\ &= \sum_{i \in I} \int_{A_i} \langle u_n, z_i \varphi_i^\delta \rangle - \left\langle f^*\left(\frac{u_n}{\varepsilon_n}, z_i\right) \mu_{\varepsilon_n}, \varphi_i^\delta \right\rangle. \end{aligned}$$

Since $\mathfrak{M}_Y(f^*)(z_i)$ is finite and $f^*(\cdot, z_i)$ satisfies the conditions of Lemma 3.1, we deduce

$$\lim_{n \rightarrow +\infty} \left\langle f^*\left(\frac{u_n}{\varepsilon_n}, z_i\right) \mu_{\varepsilon_n}, \varphi_i^\delta \right\rangle = \mathfrak{M}_Y(f^*)(z_i) \int_A \varphi_i^\delta dx.$$

Hence,

$$\liminf_{n \rightarrow +\infty} F_n(u_n) \geq \sum_{i \in I} \left[\int_{A_i} \varphi_i^\delta z_i du - \mathfrak{M}_Y(f^*)(z_i) \int_A \varphi_i^\delta dx \right].$$

Letting $\varphi_i^\delta \rightarrow 1$ almost everywhere on A_i and applying the dominated convergence theorem yields

$$\liminf_{n \rightarrow +\infty} F_n(u_n) \geq \int_A v du - \int_A \mathfrak{M}_Y(f^*)(v) dx.$$

(ii) We must show that for all $u \in (M_b(A))^d$, there exists a sequence (u_n) such that

$$\begin{cases} u_n \rightarrow u, & \text{narrowly in } (M_b(A))^d, \\ \limsup_{n \rightarrow +\infty} F_n(u_n) \leq F^{\text{hom}}(u). \end{cases}$$

Define $F_s(u) := \limsup_{n \rightarrow +\infty} F_n(u_n)$.

First we prove the inequality $F_s(vdx) \leq F^{\text{hom}}(vdx)$ for step functions v , then conclude by the density of step functions in $(L^1(A))^d$. By steps.

Step 1. Let

$$v = \sum_{i \in I} z_i 1_{A_i},$$

where $z_i \in \mathbb{R}^d$ and $(A_i)_{i \in I}$ is a partition of A into disjoint open sets.

We approximate $u = vdx$ by the sequence

$$u_n = \sum_{i \in I} w_i \left(\frac{x}{\varepsilon_n} \right) 1_{A_i} \mu_{\varepsilon_n},$$

where each w_i is a μ -measurable, Y -periodic function with values in $\mathbb{R}^d$, such that $w_i \in L_\mu^1$ and $\int_Y w_i d\mu = z_i$.

After Lemma 3.1, it is easy to verify that u_n converges vaguely to u . Assuming $f^{\text{hom}}(z_i) < +\infty$ for all $i \in I$ (otherwise $F^{\text{hom}}(vdx) = +\infty$ and there is nothing to

prove), and using the inequality $\int f d\mu \leq f^{\text{hom}}$ (see (2.5)), for any $\eta > 0$ we can choose w_i such that

$$\int_Y f\left(\frac{u}{\varepsilon_n}, w_i\left(\frac{x}{\varepsilon_n}\right)\right) d\mu \leq f^{\text{hom}}(z_i) + \frac{\eta}{\text{card}(I)}.$$

From Lemma 3.2, it follows

$$\begin{aligned} \limsup_{n \rightarrow +\infty} F_n(u_n) &= \lim_{n \rightarrow +\infty} \sum_{i \in I} \int_{A_i} f\left(\frac{u}{\varepsilon_n}, w_i\left(\frac{x}{\varepsilon_n}\right)\right) d\mu_{\varepsilon_n} \\ &= \sum_{i \in I} \int_{A_i} \left(\int f d\mu \right) (z_i) dx \\ &\leq \sum_{i \in I} \int_{A_i} f^{\text{hom}}(z_i) dx + \sum_{i \in I} \int_{A_i} \frac{\eta}{\text{card}(I)} dx \\ &\leq \int_A f^{\text{hom}}(v) dx + \eta \sum_{i \in I} \frac{|A_i|}{\text{card}(I)} < \int_A f^{\text{hom}}(v) dx + \eta. \end{aligned}$$

Letting $\eta \rightarrow 0$, we obtain

$$F_s(v dx) \leq \int_A f^{\text{hom}}(v) dx.$$

Step 2. Let $v \in (L^1(A))^d$, and let $(A_{i,n})_{i \in I_n}$ be a partition of A into open sets of diameter less than $1/n$. For each $i \in I_n$ define

$$z_{i,n} = \frac{1}{|A_{i,n}|} \int_{A_{i,n}} v dx,$$

and construct the piecewise constant function

$$v_n = \sum_{i \in I_n} z_{i,n} 1_{A_{i,n}}.$$

Clearly, v_n converges strongly to v in $(L^1(A))^d$. Indeed,

$$\|v_n - v\|_{(L^1(A))^d} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

By Jensen's inequality [10], we have

$$|A_{i,n}| f^{\text{hom}}(z_{i,n}) \leq \int_{A_{i,n}} f^{\text{hom}}(v) dx,$$

summing over i yields

$$\int_A f^{\text{hom}}(v_n) dx \leq \int_A f^{\text{hom}}(v) dx.$$

Since F_s is σ -lower semicontinuous and $v_n dx \xrightarrow{\sigma} v dx$, from Step 1 we obtain

$$F_s(v dx) \leq \liminf_{n \rightarrow +\infty} F^{\text{hom}}(v_n dx) \leq \limsup_{n \rightarrow +\infty} F^{\text{hom}}(v_n dx) \leq \int_A f^{\text{hom}}(v) dx.$$

Finally,

$$F_s(u) = \begin{cases} \int_A f^{\text{hom}} \left(\frac{du}{dx} \right) dx, & \text{if } |u| \ll dx, \\ +\infty, & \text{otherwise.} \end{cases}$$

□

4. CONCLUSION

In this work, we studied the epiconvergence of nonlinear integral functionals on bounded Radon measures in the presence of periodic microstructures. Using variational and measure-theoretic tools, we described the homogenized functional and its convergence under the vague topology.

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