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AN EFFICIENT TECHNIQUE TO APPROXIMATE THE BESSEL FUNCTION OF THE FIRST KIND BASED ON THE HAAR WAVELET BASIS

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Abstract. This paper presents an efficient technique for approximating the Bessel function of the first kind using Haar wavelet basis. This technique provides a suitable formula derived from the Haar wavelet series across different levels. The key benefit of this method lies in its simplicity, speed and more accurate than the standard power series method. Matlab-based illustrative examples highlight the efficiency and effectiveness of the proposed approach.

Keywords. Bessel's Equation, Bessel Functions, Haar Wavelets, Collocation Points.
2020 Mathematics Subject Classification. Primary 33C10; Secondary 34K28

1. Introduction

Bessel functions of the first kind J_η [13] serve a central role in a wide range of applications across physics and engineering. These functions represent solutions of some differential equations, specially they are one of the solutions of the Bessel's equation [13] and constitute some of the most important special functions, that must be evaluated by appropriate mathematical formulae. At the present, the expansion of J_η using a power series still the most frequently used formula to approximate its values. However, a lot of terms of this series are necessary used to get a good approximation for large values of the argument t and each term is complicated by the calculation of the Gamma function. So, the corresponding convergence rate will decrease more slowly. To address this issue and improve the convergence rate, this paper introduces a new approach employing Haar wavelets [3, 6, 7, 8] for the evaluation of Bessel functions of the first kind. This type of wavelets has been applied for solving many problems such as, in image processing [1, 5, 10, 11, 12], in integral equations [2, 4, 9, 14] and in other branches of mathematics. In this work, we investigate the application of Haar wavelets to solve the Bessel's equation of order η with initial conditions, then we

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can easily evaluate the Bessel functions of the first kind. The main advantage of our proposed method is that its algorithm is easy, fast and more accurate.

The structure of this paper is organized as follows: The following section revisits Bessel functions of the first kind, expressed through their power series representation. Section 2 introduces Haar wavelets, which serve as the basis for approximation. Section 3 outlines our proposed method, while Section 4 elaborates further on this approach. Section 5 provides numerical results, showcasing examples of J_η. Finally, this work is concluded with a summary and the key references.

2. Bessel Functions of the First Kind

Recall the Bessel's equation of order η [13]:

$$t^2 x^{''} + t x^{'} + \left(t^2 - \eta^2\right) x = 0. \tag{2.1}$$

This equation has two linearly independent solutions, one of the solutions is that can be obtained by using the Frobenius method, and it is called a Bessel function of the first kind that is denoted by J_η. This solution is regular at $t = 0$ and defined by the following formula:

$$J_\eta(t) = \sum_{k \geq 0} \frac{(-1)^k}{k! \Gamma(k + \eta + 1)} \left(\frac{t}{2}\right)^{\eta + 2k}, \tag{2.2}$$

where Γ is the Gamma function.

This series converges for all $t \in \mathbb{R}$ and it has the following recurrence formulas [13]:

$$\begin{array}{rcl} t\dot{J}_\eta(t) + \eta J_\eta(t) & = & tJ_{\eta-1}(t), \\ t\dot{J}_\eta(t) - \eta J_\eta(t) & = & -tJ_{\eta+1}(t), \\ J_{\eta-1}(t) + J_{\eta+1}(t) & = & \frac{2\eta}{t} J_\eta(t), \\ J_{\eta-1}(t) - J_{\eta+1}(t) & = & 2\dot{J}_\eta(t). \end{array}$$

3. Haar Wavelets

The Haar wavelet basis, defined on the interval $[\alpha, \beta[$, forms a family of functions on subintervals of $[\alpha, \beta[$, generated from the scaling and wavelet functions through dilation and translation [8], these functions are derived from the scaling functions:

$$f_1(t) = \begin{cases} 1, & for\ t \in [\alpha, \beta[, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

and the wavelet function:

$$f_2(t) = \begin{cases} 1, & for\ t \in \left[\alpha, \frac{\alpha+\beta}{2}\right[, \\ -1, & for\ t \in \left[\frac{\alpha+\beta}{2}, \beta\right[, \\ 0, & otherwise. \end{cases} \tag{3.2}$$

For the other wavelet functions f_i:

$$f_i(t) = \begin{cases} 1, & for\ t \in [a, b[\,, \\ -1, & for\ t \in [b, c[\,, \\ 0, & otherwise, \end{cases} \tag{3.3}$$

where $a = \alpha + (\beta - \alpha)\frac{k}{m}$, $b = \alpha + (\beta - \alpha)\frac{k+0.5}{m}$, $c = \alpha + (\beta - \alpha)\frac{k+1}{m}$, $i =$

$2, 3, 4, ..., 2M$. And $m = 2^j$, $j = 0, 1, ..., J$, $M = 2^J$, $k = 0, 1, ..., m-1$, $i = m+k+1$.

In [4] E. Babolian and A. Shahsavaran have proved that g_M defined by:

$$g_M(t) = \sum_{i=1}^{2M} \lambda_i f_i(t)\,,\ \ \lambda_i = \langle g, f_i \rangle\,,\ \ i = 1, ..., 2M, \tag{3.4}$$

converges to g in $L^2([\alpha, \beta[)$ norm.

So, any function $g \in L^2([\alpha, \beta[)$ can be decomposed as:

$$g(t) = \sum_{i \geq 1} \lambda_i f_i(t)\,. \tag{3.5}$$

4. Our Proposed Approach

Consider the following second order linear differential equation:

$$t^2 x^{''} + t x^{'} + \left(t^2 - \eta^2\right) x = 0,\ \ t \in [\alpha, \beta[\,,\ \ \eta \in \mathbb{R}, \tag{4.1}$$

with initial conditions $x(\alpha) = d$, $x^{'}(\alpha) = e$.

To solve the eq. (4.1) using Haar wavelets, we must follow the following steps:

Step 1. We assume that

$$x^{''}(t) = \sum_{i=1}^{2M} \lambda_i f_i(t)\,,\ \ t \in [\alpha, \beta[\,,\ \text{for a given level } M = 2^J. \tag{4.2}$$

Step 2. By integrating both sides of eq. (4.2) from α to t, we find:

$$x^{'}(t) = x^{'}(\alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,1}(t)\,, \tag{4.3}$$

where

$$R_{i,1}(t) = \begin{cases} t - a, & for\ t \in [a, b[\,, \\ c - t, & for\ t \in [b, c[\,, \\ 0, & otherwise, \end{cases} \tag{4.4}$$

for $i = 2, ..., 2M$, and $R_{1,1}(t) = t - \alpha$ for all $t \in [\alpha, \beta[\,$.

Step 3. Also by integrating both sides of eq. (4.3) from α to t, we get:

$$x(t) = x(\alpha) + x^{'}(\alpha)(t - \alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,2}(t)\,, \tag{4.5}$$

where

$$R_{i,2}(t) = \begin{cases} 0, & for\ t < a, \\ \frac{1}{2}(t-a)^2, & for\ a \le t < b, \\ \frac{(\beta-\alpha)^2}{4m^2} - \frac{1}{2}(t-c)^2, & for\ b \le t < c, \\ \frac{(\beta-\alpha)^2}{4m^2}, & for\ t \ge c, \end{cases} \tag{4.6}$$

for $i = 2, ..., 2M$, and $R_{1,2}(t) = \frac{1}{2}(t-\alpha)^2$ for all $t \in [\alpha, \beta[$.

Step 4. By substituting the eq. (4.2), eq. (4.3) and eq. (4.5) into eq. (4.1), we obtain:

$$\sum_{i=1}^{2M} g_i(t)\lambda_i = h(t), \tag{4.7}$$

where $g_i(t) = t^2 f_i(t) + tR_{i,1}(t) + (t^2-\eta^2) R_{i,2}(t)$ for $i = 1, ..., 2M$ and $h(t) = -tx'(\alpha) - (t^2-\eta^2)\left(x(\alpha) + x'(\alpha)(t-\alpha)\right)$.

Step 5. We introduce the collocation points $t_\ell = \alpha + (\beta-\alpha)\frac{\ell-0.5}{2M}$, $\ell = 1, ..., 2M$, and by replacing them into eq. (4.7), we obtain the following system of linear equations:

$$\sum_{i=1}^{2M} g_i(t_\ell)\lambda_i = h(t_\ell),\ 1 \le \ell \le 2M. \tag{4.8}$$

This system can be written in the matrix form as:

$$GA = H, \tag{4.9}$$

where $G(j,k) = g_k(t_j)$, $1 \le j \le 2M$, $1 \le k \le 2M$ and $A = (\lambda_1, \lambda_2, ..., \lambda_{2M})^T$, $H = (h(t_1), h(t_2), ..., h(t_{2M}))^T$.

Step 6. By solving (4.9), we obtain the wavelet coefficients $A = G^{-1}H$ to replace them into eq. (4.5), then we get the numerical solution of eq. (4.1).

5. Error estimation

Lemma 5.1. *The coefficients λ_i that are given in (4.2) verify*

$$|\lambda_i| = \mathcal{O}\left(\frac{1}{2^{2j}}\right), i = 2^j + k + 1, \forall i \ge 2.$$

Proof. We have, $\forall i \ge 2$:

$$\begin{aligned} \lambda_i &= \int_\alpha^\beta x''(t) f_i(t)\,dt \\ &= \int_a^b x''(t)\,dt - \int_b^c x''(t)\,dt \\ &= (b-a)x''(\xi_1) - (c-b)x''(\xi_2);\xi_1 \in [a,b], \xi_2 \in [b,c]. \end{aligned}$$

Since: $b - a = c - b = \frac{\beta-\alpha}{2m} = \frac{\beta-\alpha}{2^{j+1}}$, then

$$\lambda_i = \frac{\beta-\alpha}{2^{j+1}}(\xi_1-\xi_2)x'''(\xi);\xi \in]\xi_1, \xi_2[.$$

We have

$$a \leq \xi_1 \leq b \text{ and } b \leq \xi_2 \leq c,$$

then

$$0 \leq \xi_2 - \xi_1 \leq c - a = \frac{\beta - \alpha}{m} = \frac{\beta - \alpha}{2^j}.$$

Therefore

$$|\lambda_i| \leq \frac{(\beta - \alpha)^2}{2^{2j+1}} \left\| x^{'''} \right\|_\infty .$$

Consequently

$$|\lambda_i| = \mathcal{O}\left(\frac{1}{2^{2j}}\right).$$

□

Lemma 5.2. *The function $R_{i,2}$ for $i \geq 2$ verifies*

$$\exists \gamma \in \mathbb{R}_+^* : \|R_{i,2}\|_\infty \leq \frac{\gamma}{2^{2j}}; i = 2^j + k + 1.$$

Proof. We have, $R_{i,2}$ is positive and increasing on $[a, b]$, then

$$\|R_{i,2}\|_\infty \leq \frac{(b-a)^2}{2} = \frac{1}{2}\left(\frac{\beta - \alpha}{2m}\right)^2 \leq \frac{\gamma}{2^{2j}}.$$

On the interval $[b, c]$, the function $R_{i,2}$ is also positive and increasing, therefore

$$\|R_{i,2}\|_\infty \leq \frac{(\beta - \alpha)^2}{4m^2} \leq \frac{\gamma}{2^{2j}}.$$

For $t \geq c$, the function $R_{i,2}$ is constant and verifies

$$\|R_{i,2}\|_\infty \leq \frac{\gamma}{2^{2j}}.$$

□

Theorem 5.3. *Let x be the exact solution of problem* (1) *and x_M its approximation formula, which is expressed in* (4.5)*, then*

$$\|x - x_M\|_{L^2([\alpha,\beta[)} = \mathcal{O}\left(\frac{1}{M^2}\right); M = 2^J.$$

Proof. We have

$$x_M(t) = x(\alpha) + x^{'}(\alpha)(t - \alpha) + \sum_{i=1}^{2M} \lambda_i R_{i,2}(t),$$

and

$$x(t) = x(\alpha) + x^{'}(\alpha)(t - \alpha) + \sum_{i \geq 1} \lambda_i R_{i,2}(t),$$

we use the notation $\|.\|_{L^2([\alpha,\beta[)} = \|.\|_2$, we see that

$$\begin{aligned}\|x - x_M\|_2^2 &= \left\| \sum_{i \geq 2M+1} \lambda_i R_{i,2}(t) \right\|_2^2 \\ &= \sum_{l \geq 2M+1} \sum_{i \geq 2M+1} \lambda_i \lambda_l \int_\alpha^\beta R_{i,2}(t) R_{l,2}(t)\, dt.\end{aligned}$$

If we put $i = 2^j + k + 1$ and $l = 2^r + s + 1$, we have

$$\begin{aligned}\|x - x_M\|_2^2 &= \sum_{r \geq J+1} \sum_{s=0}^{2^r-1} \sum_{j \geq J+1} \sum_{k=0}^{2^j-1} \lambda_{2^j+k+1} \lambda_{2^r+s+1} \int_\alpha^\beta R_{2^j+k+1,2}(t) R_{2^r+s+1,2}(t)\, dt \\ &\leq \sum_{r \geq J+1} \sum_{j \geq J+1} \left|\lambda_{2^j+1} \lambda_{2^r+1}\right| \int_\alpha^\beta \left|R_{2^j+1,2}(t) R_{2^r+1,2}(t)\right| dt \\ &\quad + \sum_{r \geq J+1} \sum_{s=1}^{2^r-1} \sum_{j \geq J+1} \left|\lambda_{2^j+1} \lambda_{2^r+s+1}\right| \int_\alpha^\beta \left|R_{2^j+1,2}(t) R_{2^r+s+1,2}(t)\right| dt \\ &\quad + \sum_{r \geq J+1} \sum_{j \geq J+1} \sum_{k=1}^{2^j-1} \left|\lambda_{2^j+k+1} \lambda_{2^r+1}\right| \int_\alpha^\beta \left|R_{2^j+k+1,2}(t) R_{2^r+1,2}(t)\right| dt \\ &\quad + \sum_{r \geq J+1} \sum_{s=1}^{2^r-1} \sum_{j \geq J+1} \sum_{k=1}^{2^j-1} \left|\lambda_{2^j+k+1} \lambda_{2^r+s+1}\right| \int_\alpha^\beta \left|R_{2^j+k+1,2}(t) R_{2^r+s+1,2}(t)\right| dt.\end{aligned}$$

By using Lemma 5.1 and Lemma 5.2, we have

$$\begin{aligned}\|x - x_M\|_2^2 &\leq \sum_{r \geq J+1} \sum_{j \geq J+1} \gamma \frac{1}{2^{2j}} . \frac{1}{2^{2r}} \\ &\quad + \sum_{r \geq J+1} \sum_{j \geq J+1} \gamma \frac{1}{2^{2j}} . \frac{1}{2^{2r}} . \frac{(2^r - 1)}{2^{2r}} \\ &\quad + \sum_{r \geq J+1} \sum_{j \geq J+1} \gamma \frac{1}{2^{2j}} . \frac{1}{2^{2r}} . \frac{(2^j - 1)}{2^{2j}} \\ &\quad + \sum_{r \geq J+1} \sum_{j \geq J+1} \gamma \frac{1}{2^{2j}} . \frac{1}{2^{2r}} . \frac{(2^r - 1)}{2^{2r}} . \frac{(2^j - 1)}{2^{2j}} \\ &\leq \gamma \left(\frac{1}{2^{2J}}\right)^2,\end{aligned}$$

where $\gamma \in \mathbb{R}_+^*$, so

$$\|x - x_M\|_2 \leq \frac{\sqrt{\gamma}}{2^{2J}}.$$

Therefore

$$\|x - x_M\|_{L^2([\alpha,\beta[)} \leq \mathcal{O}\left(\frac{1}{M^2}\right);\ M = 2^J.$$

□

6. Numerical Results

This section presents numerical examples to demonstrate the efficiency of the proposed Haar wavelet-based approach, implemented in Matlab. So, the numerical results of examples (5.1, 5.2, 5.3) compare the values of J_η for $\eta = 0, 1, 2$ on $[0, 1]$ by using their tabulated and their Haar wavelet approximations values at different levels J. For example 5.4, we compare the values of $J_{\frac{1}{2}}$ using its analytic and its Haar wavelet approach formula. All results are presented in tables.

Example 5.1: Consider the Bessel function of order $\eta = 0$.

Table 5.1. Comparison of tabulated $J_0(t)$ using power series and wavelets methods

t	$J_0(t)$ using power series	$J_0(t)$ using wavelets		
		$J=5$	$J=6$	$J=7$
0	1	1	1	1
0.1	0.99750	0.99750	0.99750	0.99750
0.2	0.99002	0.99002	0.99002	0.99002
0.3	0.97762	0.97762	0.97762	0.97762
0.4	0.96039	0.96039	0.96039	0.96039
0.5	0.93846	0.93846	0.93846	0.93846
0.6	0.91200	0.91200	0.91200	0.91200
0.7	0.88120	0.88120	0.88120	0.88120
0.8	0.84628	0.84628	0.84628	0.84628
0.9	0.80752	0.80752	0.80752	0.80752
1	0.76519	0.76519	0.76519	0.76519

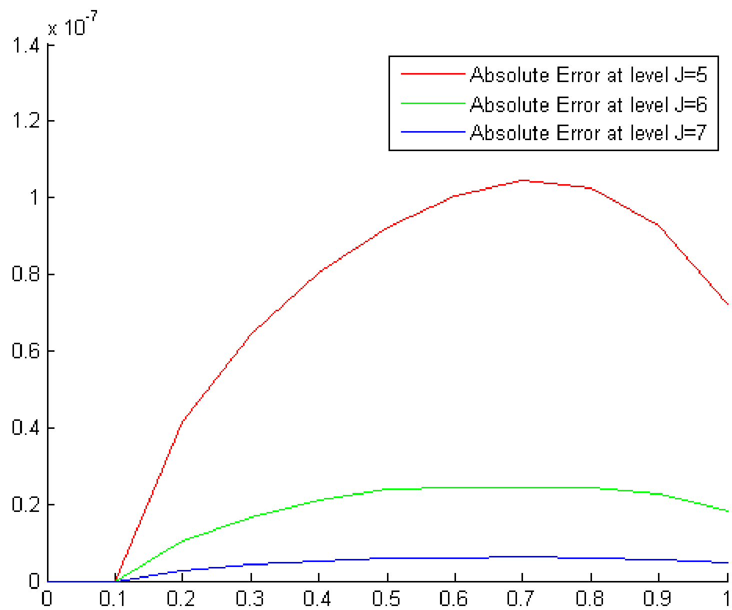

FIGURE 1. Absolute Error for the Bessel function B_0 at levels $J = 5, 6$ and 7

Example 5.2: Consider the Bessel function of order $\eta = 1$.

Table 5.2. Comparison of tabulated $J_1(t)$ using power series and wavelets methods

t	$J_1(t)$ using power series	$J_1(t)$ using wavelets		
		$J=5$	$J=6$	$J=7$
0	0	0	0	0
0.1	0.04993	0.04993	0.04993	0.04993
0.2	0.09950	0.09950	0.09950	0.09950
0.3	0.14831	0.14831	0.14831	0.14831
0.4	0.19602	0.19602	0.19602	0.19602
0.5	0.24226	0.24226	0.24226	0.24226
0.6	0.28670	0.28670	0.28670	0.28670
0.7	0.32899	0.32899	0.32899	0.32899
0.8	0.36884	0.36884	0.36884	0.36884
0.9	0.40594	0.40595	0.40595	0.40594
1	0.44005	0.44005	0.44005	0.44005

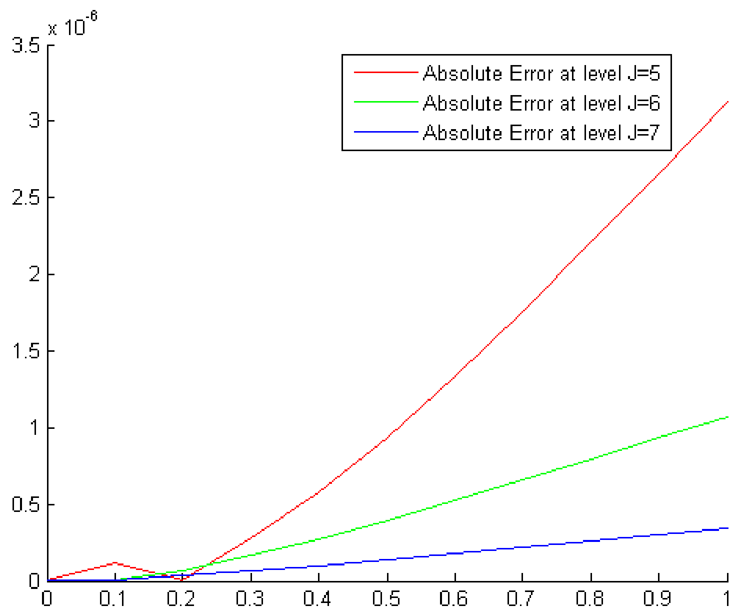

FIGURE 2. Absolute Error for the Bessel function B_1 at levels $J = 5, 6$ and 7

Example 5.3: Consider the Bessel function of order $\eta = 2$.

Table 5.3. Comparison of tabulated $J_2(t)$ using power series and wavelets methods

t	$J_2(t)$ using power series	$J_2(t)$ using wavelets		
		$J = 5$	$J = 6$	$J = 7$
0	0	0	0	0
0.1	0.00124	0.00124	0.00124	0.00124
0.2	0.00498	0.00498	0.00498	0.00498
0.3	0.01116	0.01116	0.01116	0.01116
0.4	0.01973	0.01973	0.01973	0.01973
0.5	0.03060	0.03060	0.03060	0.03060
0.6	0.04366	0.04366	0.04366	0.04366
0.7	0.05878	0.05878	0.05878	0.05878
0.8	0.07581	0.07581	0.07581	0.07581
0.9	0.09458	0.09458	0.09458	0.09458
1	0.11490	0.11490	0.11490	0.11490

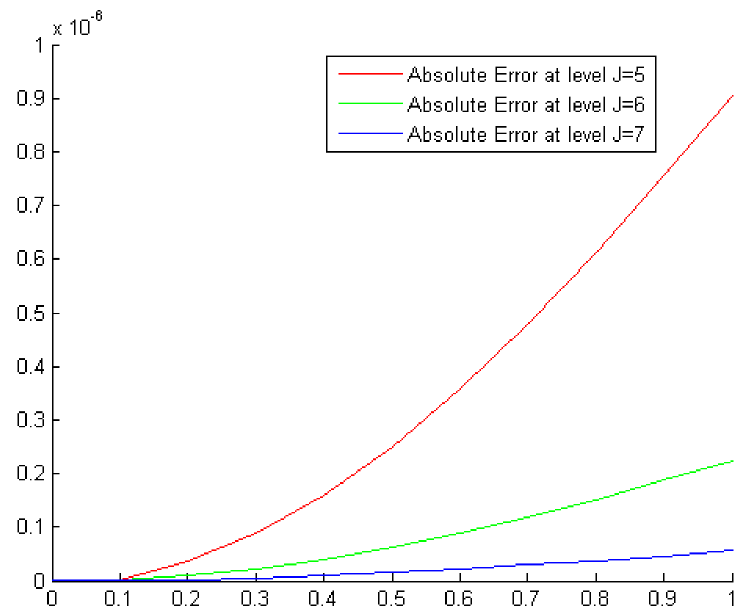

FIGURE 3. Absolute Error for the Bessel function B_2 at levels $J = 5, 6$ and 7

Example 5.4: Consider the Bessel function of order $\eta = \frac{1}{2}$, where its analytic formula is $J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}} \sin t$ for $t > 0$.

Table 5.4. Comparison of tabulated $J_{\frac{1}{2}}(t)$ using power series and wavelets methods

t	$J_{\frac{1}{2}}(t)$ using exact values	$J_{\frac{1}{2}}(t)$ using wavelets		
		$J=5$	$J=6$	$J=7$
0	0	0	0	0
0.1	0.25189	0.25189	0.25189	0.25189
0.2	0.35445	0.35451	0.35446	0.35445
0.3	0.43049	0.43056	0.43051	0.43049
0.4	0.49127	0.49135	0.49129	0.49128
0.5	0.54097	0.54105	0.54099	0.54097
0.6	0.58161	0.58169	0.58163	0.58162
0.7	0.61436	0.61444	0.61438	0.61436
0.8	0.63992	0.64000	0.63994	0.63993
0.9	0.65881	0.65889	0.65883	0.65881
1	0.67139	0.67147	0.67141	0.67140

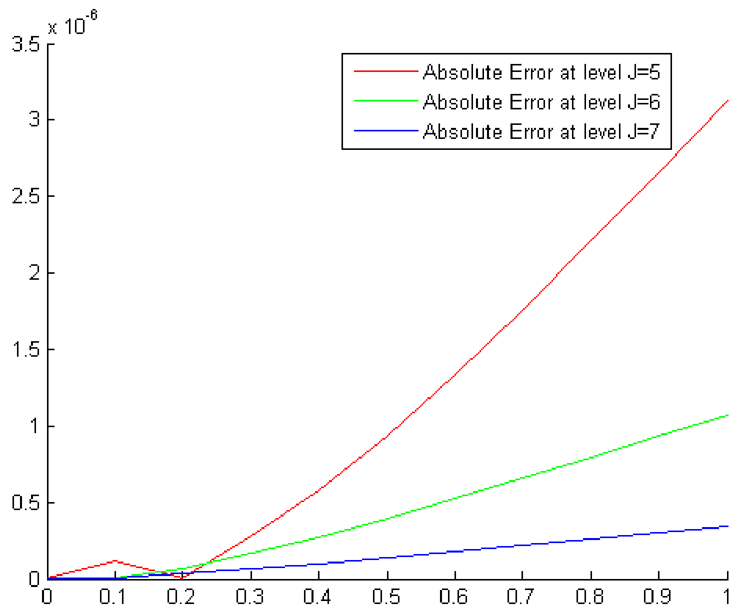

FIGURE 4. Absolute Error for the Bessel function $B_{\frac{1}{2}}$ at levels $J = 5, 6$ and 7

Obviously by comparing columns of Tables (5.1-5.4), one can easily see that our proposed approximations for Bessel functions $\eta = 0, 1, 2$ and $\eta = \frac{1}{2}$ are in excellent agreement with their tabulated and exact values. Where, the results of our approach are very significant and more accurate when the level J increases.

Conclusion In this paper, an effective approach for approximating Bessel functions of the first kind J_η has been presented. This approach is obtained by using Haar wavelets, and then it is very simple and efficient. By using this approach, a good agreement between the results obtained and the tabulated values is very significant.

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