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HOMOGENIZATION OF A QUASILINEAR ELLIPTIC PROBLEM IN POROUS MEDIA WITH ROBIN-TYPE CONDITIONS AND L^1 DATA

CHAFIA KAREK[1], TAHA KHALIL ABASSI[2], AMAR OULD-HAMMOUDA[3*] and FARES YAZID[4]

ABSTRACT. We consider a class of quasilinear elliptic problems with L^1 data, set in a domain periodically perforated by two kinds of holes. A Robin-type condition depending on a parameter γ is imposed on the boundary of some holes, while a Dirichlet condition is prescribed on the boundary of the remaining holes as well as on the external boundary of the domain. Our goal is to describe the asymptotic behavior of the renormalized solution of this problem, given that the solution does not belong to H^1 and to derive the corresponding limit problem.

Keywords. Renormalized solution, L^1 data, periodic unfolding, small holes.
2020 Mathematics Subject Classification. 35D99; 35A01; 74Q05.

1. Introduction

In this work, we study the asymptotic behavior of a class of quasilinear elliptic problems with L^1 data, with a Robin-type condition set in a perforated domain $\Omega_{\varepsilon\delta}$ of $\mathbb{R}^N$, $N > 2$ where the holes of size $r_1(\varepsilon\delta_1) = \varepsilon\delta_1 = o(\varepsilon)$ and $r_2(\varepsilon\delta_2) = \varepsilon\delta_2 = o(\varepsilon)$, are $\varepsilon-$periodically distributed. Our primary tool is the unfolding method, introduced by Cioranescu et al. (2002). We consider here the case where $\delta = \delta(\varepsilon)$ is such that $\delta \to 0$ as $\varepsilon \to 0$. These types of holes are referred to in the literature as "small holes"(see e.g.[9]), or "tiny holes" (see e.g. ([1],[19]). Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 2$ and $|\partial\Omega| = 0$. The reference (periodicity) cell Y is given by $Y = \left]-\dfrac{1}{2}, \dfrac{1}{2}\right[^N$.

We consider two disjoint compact subsets T and B of Y. We suppose that both B and T are with Lipschitz boundaries. There exists $\bar{\delta} > 0$ such that $\bar{\delta}\big(\overline{T} \cup \overline{B}\big) \subset Y$. The parameters $\varepsilon, \delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ are small and satisfy $\big(\delta_1(\varepsilon), \delta_2(\varepsilon)\big) \in (0, \bar{\delta}]^2$.

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* Corresponding author

The perforated domain $\Omega_{\varepsilon\delta}$ is constructed by removing the following sets of holes:

$$B_{\varepsilon\delta_1} = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_1\overline{B}+\varepsilon\xi)\Big), \qquad T_{\varepsilon\delta_2} = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\delta_2\overline{T}+\varepsilon\xi)\Big). \tag{1.1}$$

Thus, the perforated domain is defined as $\Omega_{\varepsilon\delta} = \Omega \setminus (\overline{B}_{\varepsilon\delta_1} \cup \overline{T}_{\varepsilon\delta_2})$, where $\Xi_\varepsilon = \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega\}$.

We consider the following quasi-linear elliptic problem:

$$(\mathcal{P}_{\varepsilon\delta})\begin{cases} -div\,(A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}) = f & \text{in } \Omega_{\varepsilon\delta},\\ u_{\varepsilon\delta} = 0 & \text{on } \partial\Omega^0_{\varepsilon\delta_2},\\ A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}.\eta + \varepsilon^\gamma\tau^\varepsilon(x)h(u_{\varepsilon\delta}) = g_\varepsilon & \text{on } \partial T_{\varepsilon\delta_1}.\end{cases} \tag{1.2}$$

The constant $\gamma \geq 0$ is given and the outward normal to $T_{\varepsilon\delta_1}$ is denoted by η.

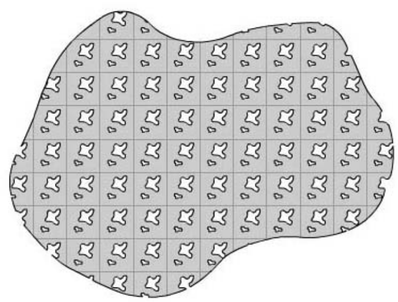

Fig.1. The domain $\Omega_{\varepsilon\delta}$

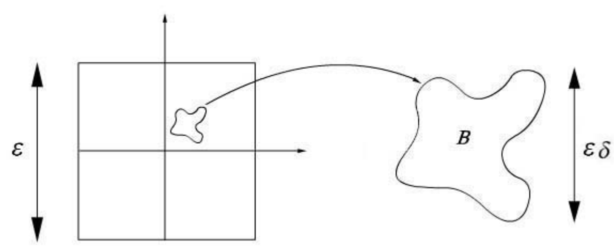

Fig.2. The cell εY_δ.

Problems of type (1.2) arise for instance, in the modeling of chemically reactive flows around domain containing reactive solid grains(or reactive obstacles) that are periodically distributed. For an in-depth study of the chemical processes considered in this model, we refer the reader to Antontsev et al. [2] and to Bear [3]. The study of the asymptotic behavior of such models has been carried out within the framework of classical homogenization(i.e.,for $\delta_1 = 1$) and under the assumption that $B_{\varepsilon\delta_2} = \emptyset$. In this context, Conca, Diaz, Linan and Timofte [11] examined a particular case where $A^\varepsilon = I,\ \gamma = 0, g = 0$ and $for\ \ any\ \alpha,\beta > 0, h(v) = (\alpha v = /(1+\beta v)$; corresponding to the Langmuir kinetics model.

On the other hand, the motivation for considering L^1 data in problem (1.2) stems from thermoviscoelasticity models that account for nonlinear mechanical

dissipation. In such cases, the energy balance equation typically includes a right-hand side term belonging to L^1 (see[15]).
In this work, since the solution $u_{\varepsilon\delta}$ is not necessarily in $H^1(\Omega_{\varepsilon\delta})$, we work with renormalized solutions, a concept originally introduced by DiPerna-Lions in [13] for the Boltzmann equations. This framework allow us to prove existence and stability, and, with further assumptions, uniqueness as well.

There is an extensive literature on the homogenization in perforated domains in $\mathbb{R}^N$. In the case of "small" holes of size ε^α, $(\alpha > 0)$, Cioranescu and Murat [9], analysed the homogeneous Dirichlet problem for the Poisson equation. They proved that the critical hole size $\varepsilon^{N/N-2}$ $(N > 2)$ where the limit problem involves both the laplacian and an additional zero-order term, often referred to as a "strange term", which depends on the capacity of the holes. The nonhomogeneous Neumann problem for the Laplacian in the same geometric setting was studied by Conca and Donato [12]. In this case, the critical hole size is of order $\varepsilon^{N/N-1}$, and the contribution of the holes at the limit appears as an additional right-hand side integral term. The above-mentioned studies were conducted within a variational framework, where the data is set in Hilbert spaces, specifically L^2
In contrast, studies on the asymptotic behavior of problems with L^1 data are more recent. In 2001, M. Bencheikh Ali [4] treated the homogenization of renormalized solutions in perforated domains where Neumann boundary condition were prescribed on the hole boundaries. More recently, in 2017, P. Donato, O. Guibé, and A. Oropeza [14] investigated problem (1.2) in the classical homogenization setting with $B_{\varepsilon\delta_2} = \emptyset$.
Returning to problem $(\mathcal{P}_{\varepsilon\delta})$ we assume that the parameters ε, $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ satisfy the conditions

$$k_1 = \lim_{\varepsilon\to 0} \frac{\delta_1^{N-1}}{\varepsilon} \in [0, +\infty[\quad \text{and } k_2 = \lim_{\varepsilon\to 0} \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon} \in [0, +\infty[. \tag{1.3}$$

Under this assumption, and withrespect to γ, we establish that the solution of (1.2) converges in an appropriate function space to u which is characterized as the solution of a new limit problem (see Theorem 5.4 and corollary 5.6). These results reveal four distinct types of contributions in the limit system.
In particular, the term $k_2^2\Theta$ in equation (5.6) represents the contribution of the Dirichlet holes $B_{\varepsilon\delta_2}$. This contribution corresponds to the "strange term" introduced in Cioranescu and Murat [9] which depends on the capacity of the holes at the limit.
Our results are both novel and original, as no prior studies have examined this specific case within the framework of quasilinear elliptic problems with L^1 data.
The paper is structured as follows: Section 2 provides the necessary preliminaries and recalls essential results on time-dependent unfolding operators for "small holes."
Section 3 establishes a priori estimates for renormalized solutions to problem (1.2). Section 4 presents the unfolding homogenization results.
Section 5 demonstrates that the solution of $(\mathcal{P}_{\varepsilon\delta})$ converges, in an appropriate

function space, to the solution of the homogenized limit problem in its strong form.

2. Preliminaries

As in [6], for a.e. $z \in \mathbb{R}^N$, we denote by $[z] \in \mathbb{Z}^N$ the integer part of z and by $\{z\}$ the fractional part of z

$$\{z\} = z - [z] \in Y, \quad \text{for almost every } z \in \mathbb{R}^N.$$

Hence

$$x = \varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big] + \Big\{\frac{x}{\varepsilon}\Big\}\Big) for\ almost\ every; x \in \mathbb{R}^N.$$

Introduce now the following sets:

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \Big\}, \quad \Lambda_\varepsilon = \Omega \setminus \overline{\widehat{\Omega}}_\varepsilon. \tag{2.1}$$

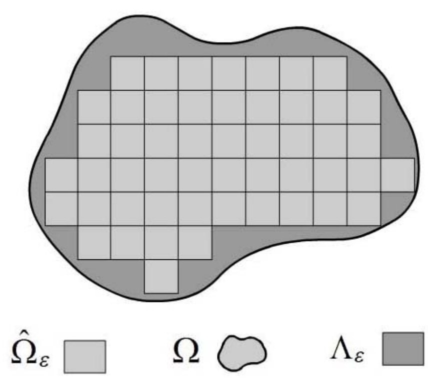

Fig.3. The domain Ω_ε

The set $\widehat{\Omega}_\varepsilon$ is defined as the largest finite union of $\varepsilon(\xi+\overline{Y})$ cells, $\xi \in \mathbb{Z}^N$, contained in Ω while Λ_ε is the subset of Ω containing parts of $\varepsilon(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$. Throughout this paper, given a functions v defined on $\Omega_{\varepsilon\delta}$, we denote by $\widetilde{v}$ its extension by zero outside Ω, furthermore T be a given positive number.

In the context of classical homogenization(i.e.,for $\delta_1 = 1$) and with $B_{\varepsilon\delta_2} = \emptyset$, the perforated domain is denoted Ω_ε.

In the sequel, we write $\mathcal{M}_{\partial T}(v) = \frac{1}{|\partial T|}\int_{\partial T} v(y) d\sigma y$, to refer to the mean value of any function $v \in L^1(\partial T)$ over ∂T

Let us do the following hypotheses. Let f, g, h and τ be functions satisfying these hypotheses:

(1) f belongs to $L^1(\Omega)$.
(2) $h : \mathbb{R} \to \mathbb{R}$ be continuous, increasing and satisfy $h(0) = 0$.
(3) τ is Y-periodic and positive function in $L^\infty(\partial T)$ and we have $\tau^\varepsilon(x) = \tau\left(\dfrac{x}{\varepsilon\delta_1}\right)$.

(4) $g_\varepsilon(x) = g\left(\dfrac{x}{\varepsilon\delta_1}\right)$, $g \in L^1(\partial T)$ be an Y-periodic function.

(5) The real matrix field $A : (y,t) \in Y \times \mathbb{R} \mapsto A(y,t) \in \mathbb{R}^{N^2}$ such that $A(.,t) = \{a_{ij}(.,t)\}_{i,j=1\dots N}$ is Y-periodic for any t and satisfies the following three conditions:

(6) the matrix A is a Caratheodory function, that is, for every fixed $t \in \mathbb{R}$ the mapping $y \mapsto A(y,t)$ is measurable, and almost every $y \in Y$, the mapping $t \mapsto A(y,t)$ is continuous.

(7) For some constant $\alpha > 0$, $A(y,t)\xi\xi \geq \alpha|\xi|^2$, for almost everywhere $y \in Y,\ for\, all\ t \in \mathbb{R}\, and\, for\, any\, \xi \in \mathbb{R}^N$,

(8) $A(y,t) \in L^\infty(Y \times (-k,k)^{N\times N}), \forall k > 0$ and one sets

$$A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right) \ for\ \ every\ (x,t) \in \Omega \times \mathbb{R}. \tag{2.2}$$

We introduce the following functional space

$$V_{\varepsilon\delta} = \{v \in H^1(\Omega_{\varepsilon\delta}) : v = 0 \ \ on\ \partial\Omega^0_{\varepsilon\delta_2}\}, \tag{2.3}$$

endowed with the scalar product

$$\langle\varphi, \psi\rangle = \int_{\Omega_{\varepsilon\delta}} \nabla\varphi.\nabla\psi\, dx = \sum_{j=1}^{N} \int_{\Omega_{\varepsilon\delta}} \frac{\partial\varphi}{\partial x_j}\frac{\partial\psi}{\partial x_j} dx,$$

is a Hilbert space.
Let $T_k : \mathbb{R} \mapsto \mathbb{R}$ be the truncation function at height $\underline{+}k$, with $k > 0$, defined as

$$T_k(t) = \min(k, \max(t, -k)), \tag{2.4}$$

for all $t \in \mathbb{R}$.

Remark 2.1. for any function v such that $T_k(v) \in V,\ \ for\ all\ k > 0$, we have

$$\nabla v \nabla T_k(v) = \nabla T_k(v) \nabla T_k(v). \tag{2.5}$$

The renormalized solution to problem (1.2) is defined as follows.

Definition 2.2. $u_{\varepsilon\delta}$ is a renormalized solution of (1.2) if

$$T_k(u_{\varepsilon\delta}) \in V \quad\ for\ \ all\ \ k > 0, \tag{2.6}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{x\in\Omega_{\varepsilon\delta}:|u_{\varepsilon\delta}|<n\}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta} dx = 0, \tag{2.7}$$

and for all $\psi \in W^{1,\infty}(\mathbb{R})$ with compact support, $u_{\varepsilon\delta}$ verifies

$$\int_{\Omega_{\varepsilon\delta}} \psi(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla v dx + \int_{\Omega_{\varepsilon\delta}} \psi'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta} v dx$$
$$+ \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma \tau^\varepsilon(x)\psi(u_{\varepsilon\delta})h(u_{\varepsilon\delta})v d\sigma x = \int_{\Omega_{\varepsilon\delta}} f\psi(u_{\varepsilon\delta})v dx + \int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi(u_{\varepsilon\delta})v d\sigma x, \tag{2.8}$$

for all $v \in V \cap L^\infty(\Omega_{\varepsilon\delta})$.

The existence and the uniqueness of renormalized solutions of problem (1.2) was proved in 2017 by O. Guibé and A. Oropeza (see [16] for detail).

Remark 2.3. From (1.3), one can see easily that k_1 corresponds to the critical size of Neumann small holes from [12], that is $C\varepsilon^{N/N-1}$, respectively k_2 corresponds to the critical size from [9], that is $C\varepsilon^{N/N-2}$. Observe also that, as $\varepsilon \to 0$, the holes in $B_{\varepsilon\delta_2}$ approach quicker zero, than those of $T_{\varepsilon\delta_1}$. Consequently, for ε small enough, the holes of $T_{\varepsilon\delta_1}$ do not intersect those of $B_{\varepsilon\delta_2}$ for each ε. It is why that one could make the hypothesis that from now on, in the limiting process $\varepsilon \to 0$, the holes $T_{\varepsilon\delta_1}$ and $B_{\varepsilon\delta_2}$ do not intersect.

2.1. **The periodic unfolding operator $\mathcal{T}_\varepsilon$ for fixed domains.** The definition and some properties of the periodic unfolding operator introduced in [7] and [6], are recalled here from [17] and [10].

Definition 2.4. For $\phi \in L^p(\Omega)$, $p \in [1, +\infty]$, the unfolding operator is defined as follows: $\mathcal{T}_\varepsilon : L^p(\Omega) \longrightarrow L^p(\Omega \times Y)$,

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big]_Y + \varepsilon y\Big) & a.e. \ \text{ for } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & a.e. \ \text{ for } (x,y) \in \ \Lambda_\varepsilon \times Y. \end{cases}$$

Definition 2.5. For any ϕ in $L^p(\Omega)$, $1 \le p < \infty$, The local average M_Y^ε : $L^p(\Omega) \longrightarrow L^p(\Omega)$ is given by

$$M_Y^\varepsilon(\phi)(x) = \int_Y \mathcal{T}_\varepsilon(\phi)(x,y)\,dy.$$

It is known that if $\{v_\varepsilon\}$ is a bounded sequence in $L^p(\Omega)$, such that $v_\varepsilon \to v$ strongly in $L^p(\Omega)$, then

$$M_Y^\varepsilon(v_\varepsilon) \to v \quad \text{strongly in } L^p(\Omega). \tag{2.9}$$

Proposition 2.6. [6, 8]

(1) *The operator $\mathcal{T}_\varepsilon$ is continuous and linear.*
(2) $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\phi), \forall \ \ \varphi, \psi \in L^p(\Omega)$
(3) *For every w belong to $L^p(\Omega)$, then $\mathcal{T}_\varepsilon(w) \to w$ strongly in $L^p(\Omega \times Y)$.*
(4) *One has*

$$\int_{\widehat{\Omega}_\varepsilon} \varphi(x)dx = \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\varphi)(x,y)dxdy, \ \ \forall\varphi \in L^1(\Omega).$$

(5) *For any $w \in L^p(\Omega)$, one has $\|\mathcal{T}_\varepsilon(w)\|_{L^p(\Omega\times Y)} \le |Y|^{\frac{1}{p}}\|w\|_{L^p(\Omega)}$.*
(6) *Let $w_\varepsilon \in L^p(\Omega)$ so that $\|w_\varepsilon\|_{L^p(\Omega)} \le C$, for some positive constant C. If*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w} \ \ \textit{weakly in } L^p(\Omega, L^p_{loc}(Y)),$$

then

$$\widetilde{w_\varepsilon}\mathcal{M}_Y \rightharpoonup (\widehat{w}) \ \ \textit{weakly in } \ L^p(\Omega).$$

(7) $\nabla_y \mathcal{T}_\varepsilon(\varphi)(x,y) = \varepsilon\mathcal{T}_\varepsilon(\nabla_x\varphi)(x,y)$, $\forall (x,y) \in \mathbb{R}^N \times Y$.

(8) *Let* $\varphi_\varepsilon \in L^p(\Omega)$ *satisfies*

$$\varphi_\varepsilon \to \varphi \quad \textit{strongly in } L^p(\Omega),$$

then

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \varphi \quad \textit{strongly} \quad \textit{in } L^p(\Omega \times Y).$$

Proposition 2.7. *Let* $r(x)$ *be continuous function , with* $r(0) = 0$*, then*

$$\mathcal{T}_\varepsilon(r(u_{\varepsilon\delta})) = r(\mathcal{T}_\varepsilon(u_{\varepsilon\delta})), \tag{2.10}$$

holds in $\Omega \times Y$*.*

Proof. for every $(x, y) \in \widehat{\Omega}_\varepsilon \times Y$ one has

$$\mathcal{T}_\varepsilon(r(u_{\varepsilon\delta}))(x, y) = r(u_{\varepsilon\delta})\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) = r\left(u_{\varepsilon\delta}\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)\right) = r\left(\mathcal{T}_\varepsilon(u_{\varepsilon\delta})\right),$$

while if $(x, y) \in \Lambda_\varepsilon \times Y$, one has

$$\mathcal{T}_\varepsilon(r(u_{\varepsilon\delta}))(x, y) = 0 \textit{ while } r(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))(x, y) = r(0),$$

then $r(0) = 0$ for (2.10) to hold in $\Omega \times Y$.
However, for any Lebesgue measurable function φ, we can write

$$\mathcal{T}_\varepsilon(r(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\varphi) = r(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\varphi), \tag{2.11}$$

even if $r(0) \neq 0$. □

Before concluding this subsection, we would like to highlight a few remarks regarding the use of the unfolding operator $\mathcal{T}_\varepsilon$, originally introduced for domains Ω_ε in the framework of classical homogenization (see [6]), specifically in the context of perforated domains $\Omega_{\varepsilon\delta}$ with small holes. Let $\mathbf{C}$ be a bounded domain with Lipschitz boundary. For sufficiently small δ we assume that $\delta\overline{\mathbf{C}} \subset Y$, and define $Y_\delta = Y \backslash \delta\overline{\mathbf{C}}$. As $\delta \to 0$, we observe that $Y_\delta \approx Y$ (this means that $|Y_\delta| \to |Y|$). This justifies applying the unfolding operator $\mathcal{T}_\varepsilon$ to functions defined on $\Omega_{\varepsilon\delta}$ in the following way:

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\Big(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\Big) & \text{for a.e.} \quad (x, y) \in \widehat{\Omega}_\varepsilon \times Y_\delta, \\ 0 & \qquad\qquad \text{otherwise.} \end{cases}$$

The main properties of the unfolding operator, as discussed previously; remain valid in this new setting. What changes is that the convergences become now only local with respect to the variable y. Theorem 3 from [18] for functions defined in $\Omega_{\varepsilon\delta}$ now reads as follows.

Proposition 2.8. *Let* $p \in]1, \infty]$*. Let* (w_ε) *be a sequence converging to some* w *in* $W^{1,p}(\Omega)$*. Up to a subsequence, there exists some* $\hat{w}$ *in* $L^p(\Omega; W^{1,p}_{per}(Y))$ *such that:*

$$\begin{aligned} &\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w \quad \textit{weakly in } L^p(\Omega; W^{1,p}_{loc}(Y)), \\ &\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} \quad \textit{weakly in } (L^p(\Omega, L^p_{loc} Y))^N. \end{aligned} \tag{2.12}$$

2.2. **The unfolding operator $\mathcal{T}_{\varepsilon\delta}$.** The geometry of domains with small holes requires a specific unfolding operator depending on both parameters ε and δ. The functions belonging to $H_0^1(\Omega_{\varepsilon\delta})$, it is natural to extend them by zero the whole of Ω. Clearly, these extensions belong to $H_0^1(\Omega)$. So, one may not distinguish the functions of $H_0^1(\Omega_{\varepsilon\delta})$ and their extensions in $H_0^1(\Omega)$.

Definition 2.9. Let $p \in [1, +\infty]$ and for any $\phi \in L^p(\Omega)$, the unfolding operator $\mathcal{T}_{\varepsilon\delta} : L^p(\Omega) \longrightarrow L^p(\Omega \times \mathbb{R}^n)$, is defined by

$$\mathcal{T}_{\varepsilon\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x, \delta z) & \text{a.e. for} \quad (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta} Y, \\ 0 & \text{otherwise.} \end{cases}$$

This operator is linear and continuous.

Proposition 2.10. (1) *For any $u \in L^2(\Omega)$, $\|\mathcal{T}_{\varepsilon\delta}(u)\|^2_{L^2(\Omega\times\mathbb{R}^N)} \le \frac{1}{\delta^N}\|u\|^2_{L^2(\Omega)}$.*
(2) *Let $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ such that $\int_{\Lambda_\varepsilon} |w_\varepsilon|\, dx \to 0$, then*

$$\int_\Omega w_\varepsilon \, dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon\delta}(w_\varepsilon)\, dxdz \to 0.$$

Proposition 2.11. *Denote by 2^* the Sobolev exponent $\frac{2N}{N-2}$ associated to 2 for all $N \ge 3$. Let $\omega \subset \mathbb{R}^N$ be an open and bounded domain. Then the following estimates hold:*

$$\|\nabla_z\big(\mathcal{T}_{\varepsilon\delta}(u)\big)\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \le \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.13}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u - M_Y^\varepsilon(u)\big)\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \le \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.14}$$

$$\|\mathcal{T}_{\varepsilon\delta}\big(u)\|^2_{L^2(\Omega\times\omega)} \le \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{\frac{2}{N}}\|\nabla u\|^2_{L^2(\Omega)} + 2|\,\omega\,|\;\|u\|^2_{L^2(\Omega)}, \tag{2.15}$$

here, C denotes the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$ and M_Y^ε is actually given by Definition 2.5.

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domains with Lipschitz boundary. Suppose that w_ε in $W_0^{1,p}(\Omega_{\varepsilon\delta})$ verifies*

$$\|\nabla w_\varepsilon\|_{L^p(\Omega_{\varepsilon\delta})} \le C,$$

then there exist w in $W_0^{1,p}(\Omega)$ and $\widehat{w}$ in $L^p(\Omega; W^{1,p}_{per}(\frac{1}{\delta}Y))$ with $M_{Y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,

$$\mathcal{T}_{\varepsilon\delta}(w_\varepsilon) \to w \qquad \textit{strongly in} \quad L^p(\Omega; W^{1,p}(\frac{1}{\delta}Y)).$$

Proof. The proof is obtained in a manner similar to that of Theorem 2.13 in [8] by applying the change of variable $y = \delta z$ the result for $\mathcal{T}_{\varepsilon\delta}$ follows. $\square$

2.3. The boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$.

Definition 2.13. Let $p \in [1, +\infty]$ and ϕ in $L^p(\partial T_{\varepsilon\delta})$, we define the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta} : L^p(\partial T_{\varepsilon\delta}) \mapsto L^p(\mathbb{R}^N \times \partial T)$ by:

$$\mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon\delta z\Big), \quad x \in \mathbb{R}^N,\ z \in \partial T. \tag{2.16}$$

Remark 2.14. Observe that if $\delta = 1$, $\mathcal{T}^b_{\varepsilon 1}$ is none other than the boundary unfolding operator $\mathcal{T}^b_{\varepsilon}$ defined in [8].

Proposition 2.15. *The linear and continuous boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta}$ is satisfies the following formula:*

$$\int_{\partial T_{\varepsilon\delta}} \phi(x)\, d\sigma(x) = \frac{\delta^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T} \mathcal{T}^b_{\varepsilon\delta}(\phi)(x,z)\, dx d\sigma(z), \quad \forall\phi \in L^1(\partial T_{\varepsilon\delta}). \tag{2.17}$$

Proposition 2.16. [18] *Let φ_ε in $W^{1,p}(\Omega_{\varepsilon\delta})$ for any ε, and suppose that $\varphi \in W^{1,p}(\frac{1}{\delta}Y))$ such that*

$$\mathcal{T}_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \textit{weakly in } L^p(\Omega; W^{1,p}(\frac{1}{\delta}Y)).$$

then we have the following convergence:

$$\mathcal{T}^b_{\varepsilon\delta}(\varphi_\varepsilon) \rightharpoonup \varphi \quad \textit{weakly in } L^p(\Omega; W^{1-\frac{1}{p},p}(\partial T)),$$

Lemma 2.17. *For any function φ is a function in $L^\infty(\partial\Omega_{\varepsilon\delta} \cap \partial T_{\varepsilon\delta_1})$ and Suppose g is a Y-periodic function in $L^1(\partial T)$, then*

$$\left|\int_{\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta}} g\left(\frac{x}{\varepsilon\delta}\right)\varphi(x) d\sigma x\right| \le C\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^1(\partial T)}\|\varphi\|_{L^\infty(\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta})}, \tag{2.18}$$

where C a constant independent of ε

Proof. We consider a Y-periodic function g belonging to $L^1(\partial T)$ and $\varphi \in L^\infty(\partial\Omega_{\varepsilon\delta} \cap \partial T_{\varepsilon\delta})$; using the integration formula (2.17), one has,

$$\left|\int_{\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta}} g\left(\frac{x}{\varepsilon\delta}\right)\varphi(x) d\sigma x\right| = \frac{\delta^{N-1}}{\varepsilon}\left|\int_{\mathbb{R}^N\times\partial T} \mathcal{T}^b_{\varepsilon\delta}\left(g(\frac{x}{\varepsilon\delta})\varphi\right)(x,z) dx d\sigma z\right|$$

$$\le C\frac{\delta^{N-1}}{\varepsilon}\|g\|_{L^1(\partial T)}\|\varphi\|_{L^\infty(\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta})}.$$

□

Proposition 2.18. *Suppose that g in $L^1(\partial T)$ is a Y-periodic function, consequently, there exists C a constant does not depend on ε such that*

$$\left|\int_{\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta}} g_\varepsilon(x)\,\varphi(x) d\sigma x\right| \le C\|\varphi\|_{L^\infty(\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta})}. \tag{2.19}$$

Proposition 2.19. *Suppose g is a Y-periodic function in $L^1(\partial T)$ and g_ε is the function defined by $g_\varepsilon = g\left(\frac{x}{\varepsilon\delta}\right)$, then*

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta}} g_\varepsilon(x)\varphi(x) d\sigma x = k_1|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega \varphi(x) dx, \tag{2.20}$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Using proposition 2.15 we obtain

$$\lim_{\varepsilon \to 0} \int_{\partial\Omega_{\varepsilon\delta} \cap \partial T_{\varepsilon\delta}} g_\varepsilon(x)\varphi(x) d\sigma x = \lim_{\varepsilon \to 0} \frac{\delta^{N-1}}{\varepsilon} \int_{\mathbb{R}^N \times \partial T} g(z) \mathcal{T}^b_{\varepsilon\delta}(\varphi)(x,z) dx d\sigma z.$$

From property (4) of proposition 16 in [18], one has $\mathcal{T}^b_{\varepsilon\delta}(\varphi) \to \varphi$ strongly in $L^2(\mathbb{R}^N \times \partial T)$. As $\varphi \in L^\infty(\Omega)$ *and the operator* $\mathcal{T}^b_{\varepsilon\delta}(\varphi)$ is bounded in $L^\infty(\mathbb{R}^N \times \partial T)$; hence,

$$\mathcal{T}^b_{\varepsilon\delta}(\varphi) \to \varphi \ \ weakly^* \ \ in \ L^\infty(\mathbb{R}^N \times \partial T).$$

From the assumption $g \in L^1(\partial T)$, we deduce that,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\partial\Omega_{\varepsilon\delta} \cap \partial T_{\varepsilon\delta}} g_\varepsilon(x)\varphi(x) d\sigma x &= \lim_{\varepsilon \to 0} \frac{\delta^{N-1}}{\varepsilon} \int_{\mathbb{R}^N \times \partial T} g(z)\varphi(x,z) dx d\sigma z \\ &= k_1 |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega \varphi(x) dx. \end{aligned}$$

□

3. A priori estimates

This section is devoted to presenting some a priori estimates for a renormalized solution of problem (1.2).

We, first, define, the function ψ_n, $\ \ for \ n \in \mathbb{N}^*$, as

$$\psi_n(t) = \begin{cases} 0 & , \ t \le -2n \\ \dfrac{t}{n} + 2 & , \ -2n \le t \le -n \\ 1 & , \ -n \le t \le n \\ -\dfrac{t}{n} + 2 & , \ n \le t \le 2n \\ 0 & , \ t \ge 2n \end{cases} \tag{3.1}$$

One remarks easily that by construction, ψ_n is Lipschiz-continuous function and satisfies

$$0 \le \psi_n \le 1, \tag{3.2}$$

with support given by

$$supp\psi_n = [-2n, 2n]. \tag{3.3}$$

We deduce that

$$\lim_{n \to \infty} \psi_n = 1, \qquad almost\ everywhere \ \ in \ \mathbb{R}. \tag{3.4}$$

In addition, its derivative satisfies

$$|\psi_n'(s)| \le \frac{1}{n} \ \ for \ \ |s| \le 2n, almost\ everywhere \ \ in \ \mathbb{R}. \tag{3.5}$$

Proposition 3.1. *Let the assumptions (1) to (8) in the preliminaries on f, g, h, τ and A and (2.2) hold. Let $u_{\varepsilon\delta}$ denote a renormalized solution of problem (1.2). Then there exist a constant $C > 0$ both independent of ε and k, such that for each $k \in \mathbb{N}$ and $\varepsilon > 0$*

$$\|T_k(u_{\varepsilon\delta})\|^2_{H^1(\Omega_{\varepsilon\delta})} \leq Ck. \tag{3.6}$$

Moreover, for $\gamma \geq 0$, one can find a positive constant C independent of ε, such that

$$\int_{\partial\Omega_{\varepsilon\delta}\cap\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma \tau^\varepsilon(x)|h(u_{\varepsilon\delta})|d\sigma x \leq C. \tag{3.7}$$

Proof. We consider $u_{\varepsilon\delta}$ be a renormalized solution of problem (1.2). In the renormalized formulation (2.8) we replace ψ *by* ψ_n and from (2.6) we can take $v = T_k(u_{\varepsilon\delta})$ as test function in (2.8), to get

$$\begin{aligned}
&\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla T_k(u_{\varepsilon\delta})dx + \int_{\Omega_{\varepsilon\delta}} \psi_n'(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}T_k(u_{\varepsilon\delta})dx\\
&+ \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x\\
&= \int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})dx + \int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x.
\end{aligned} \tag{3.8}$$

Due to (2.5), we fix $k > 0$, which allows usto write

$$\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla T_k(u_{\varepsilon\delta})dx = \int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})dx \tag{3.9}$$

moreover, (3.3) together with (3.5) we have, $\forall n \in \mathbb{N}$ where $2n > k$,

$$\begin{aligned}
&\left|\int_{\Omega_{\varepsilon\delta}} \psi_n'(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}T_k(u_{\varepsilon\delta})dx\right|\\
&\leq \frac{k}{n}\int_{\Omega_{\varepsilon\delta}\cap\{|u_{\varepsilon\delta}|<2n\}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}dx = k\rho(n),
\end{aligned} \tag{3.10}$$

by (2.7) $\rho(n)$ go to zero as $\varepsilon \to 0$.
Using the Hölder's inequality and (3.2) for the forth integral of (3.8) we get

$$\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})dx \leq \|f\|_{L^1(\Omega_{\varepsilon\delta})}\|\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})\|_{L^\infty(\Omega_{\varepsilon\delta})} \leq k\|f\|_{L^1(\Omega)}. \tag{3.11}$$

In the last integral of (3.8), using (3.2) and since $T_k(u_{\varepsilon\delta}) \in L^\infty(\partial T_{\varepsilon\delta_1})$ we apply proposition 2.18 to obtain

$$\left|\int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x\right| \leq C_1\|\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})\|_{L^\infty(\partial T_{\varepsilon\delta_1})} \leq C_1k, \tag{3.12}$$

Combining (3.9), (3.10), (3.11) and (3.12) we deduce from (3.8) that

$$\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla T_k(u_{\varepsilon\delta})dx + \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x$$
$$\leq\ k(\rho(n)+\|f\|_{L^1(\Omega)}+C_1),$$

such that $\rho(n)\to 0$ when $n\ \to\infty$. Using (3.4) and by Fatou's Lemma, we get

$$\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla T_k(u_{\varepsilon\delta})dx + \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma(x) \leq C_2 k, \tag{3.13}$$

since both h *and* T_k are nondecreasing functions and vanish at zero; ze obtain that $h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta}) \geq 0$. Moreover, because of τ is a Y-periodic function and positive, then $\varepsilon^\gamma\tau^\varepsilon(x)\geq 0$, so

$$\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma(x) \geq 0, \tag{3.14}$$

using (3.14), from (3.13) it yields

$$\alpha\int_{\Omega_{\varepsilon\delta}} |\nabla T_k(u_{\varepsilon\delta})|^2\,dx \leq C_3 k,$$

then

$$\|T_k(u_{\varepsilon\delta})\|_{V_{\varepsilon\delta}} \leq Ck,$$

where $C=\dfrac{C_3}{\alpha}$ does not depend on ε *and* k.
In order to establish the boundary integral estimate, we observe that, due to the ellipticity of the matrix A, from (3.13), one obtains

$$\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma(x) \leq C_2 k,$$

for any k, with a constant C_2 independent of ε *and* k. In the case $k=1$, one has:

$$\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)h(u_{\varepsilon\delta})T_1(u_{\varepsilon\delta})d\sigma x \leq C_2.$$

Let us observe that $|h(u_{\varepsilon\delta})| \leq h(u_{\varepsilon\delta_1\delta_2\delta})T_1(u_{\varepsilon\delta})+C$ where $C>0$ is a constant independent of ε. Therefore, applying the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$, to the previous inequality, we deduce:

$$\begin{aligned}\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon|h(u_{\varepsilon\delta})| &\leq \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon h(u_{\varepsilon\delta})T_1(u_{\varepsilon\delta})d\sigma x + \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)c d\sigma x\\ &\leq C_2 + C\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T} \varepsilon^\gamma\tau(z)dxd\sigma z\\ &= C_2 + C\frac{\delta_1^{N-1}}{\varepsilon}|\partial T|\mathcal{M}_{\partial T}(\tau)\varepsilon^\gamma,\end{aligned}$$

which gives (3.7) if $\gamma\geq 0$.

□

Corollary 3.2. [14] *Suppose that assumptions of proposition 3.1 hold. Let $u_{\varepsilon\delta}$ be a renormalized solution of (1.2). Then, $\forall\, k \in \mathbb{N}$, $\exists\, u_k^0 \in H_0^1(\Omega)$ and $\widehat{u_k} \in L^2(\Omega, H^1_{per}(Y))$ with $\mathcal{M}_Y(\widehat{u_k}) = 0$ such that, up to subsequence, as $\varepsilon \to 0$,*

$$\begin{cases} (i). \quad \mathcal{T}_\varepsilon(T_k(u_{\varepsilon\delta})) \to u_k^0 & \textit{strongly in } L^2(\Omega, H^1(Y)), \\ (ii). \quad \mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta})) \rightharpoonup \nabla u_k^0 + \nabla_y \widehat{u_k} & \textit{weakly in } L^2(\Omega \times Y), \\ (iii). \quad T_k(\widetilde{u_{\varepsilon\delta}}) = \widetilde{T_k(u_{\varepsilon\delta})} \rightharpoonup u_k^0 & \textit{weakly in } L^2(\Omega) \textit{ and weakly}^* \textit{ in } L^\infty(\Omega), \\ (iv). \quad \|T_k(u_{\varepsilon\delta}) - u_k^0\|_{L^2(\Omega_{\varepsilon\delta})} \to 0. \end{cases} \tag{3.15}$$

4. Main results

Proposition 4.1. *Let $u_{\varepsilon\delta}$ denote a renormalized solution to (1.2) under assumptions of f, g, τ and the matrix A. then up to a subsequence, there exists a function $u^0 : \Omega \to \mathbb{R}$ measurable and finite almost everywhere, such that*

$$\mathcal{T}_\varepsilon(u_{\varepsilon\delta}) \to u^0 \quad \textit{almost everywhere in } \Omega \times Y, \tag{4.1}$$

and

$$\mathcal{T}^b_{\varepsilon\delta}(u_{\varepsilon\delta}) \to u^0 \ \textit{almost everywhere } \ \textit{in } \Omega \times \partial T, \tag{4.2}$$

with

$$T_k(u_0) = u_k^0 \quad \textit{for all } k > 0, \tag{4.3}$$

and

$$\mathcal{T}_{\varepsilon\delta}(A(x, u_{\varepsilon\delta}(x))) \to A(z, u^0) \quad \textit{a.e} \ \ \textit{in } \Omega \times \frac{1}{\delta}Y. \tag{4.4}$$

Furthermore, if $\gamma = 0$, we have

$$h(u^0) \in L^1(\Omega). \tag{4.5}$$

Proof. For all natural nmbers k and for every $\varepsilon > 0$ we observe that:

$$\mathcal{T}_\varepsilon(T_k(u_{\varepsilon\delta})) = T_k(\mathcal{T}_\varepsilon(u_{\varepsilon\delta})), \tag{4.6}$$

since $T_k(0) = 0$ and

$$\mathcal{T}_\varepsilon(T_k(u_{\varepsilon\delta}))(x,y) = T_k(u_{\varepsilon\delta})\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) = T_k\left(u_{\varepsilon\delta}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right)\right) = T_k(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))(x,y).$$

Thus, from (i) of corollary 3.2 and (4.6)

$$T_k(\mathcal{T}_\varepsilon(u_{\varepsilon\delta})) \to u_k^0 \textit{ strongly in } L^2(\Omega, H^1(Y)). \tag{4.7}$$

Applying the trace theorem, we obtain:

$$\begin{aligned} \left\|\mathcal{T}^b_{\varepsilon\delta}(T_k(u_{\varepsilon\delta})) - u_k^0\right\|^2_{L^2(\mathbb{R}^N \times \partial T)} &= \int_{\mathbb{R}^N} \|\mathcal{T}^b_{\varepsilon\delta}(T_k(u_{\varepsilon\delta})) - u_k^0\|_{L^2(\partial T)} dx \\ &\leq C \int_\Omega \|\mathcal{T}_\varepsilon(T_k(u_{\varepsilon\delta})) - u_k^0\|_{H^1(Y)} dx. \end{aligned}$$

According to corollary 3.2 this term go to zero. From (4.6) we get

$$T_k(\mathcal{T}^b_{\varepsilon\delta}(u_{\varepsilon\delta})) \to u_k^0 \textit{ strongly } \ \textit{in } L^2(\mathbb{R}^N \times \partial T). \tag{4.8}$$

Now by property (5) of proposition 2.6 we have

$$k^2 meas\{|\mathcal{T}_\varepsilon(u_{\varepsilon\delta})| \geq k\} = \int_{\{|\mathcal{T}_\varepsilon(u_{\varepsilon\delta})|\geq k\}} T_k^2(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))dxdy \leq \|T_k(u_{\varepsilon\delta})\|^2_{L^2(\Omega_{\varepsilon\delta})} \leq Ck,$$

where C constant independent of k and ε. This imply that

$$meas\{|\mathcal{T}_\varepsilon(u_{\varepsilon\delta})| \geq k\} \leq \frac{C}{k}. \tag{4.9}$$

In the space V, the Poincare inequality hold with a constant that does not depend on ε, we use the same method used in [16], we can show that the sequence $\{\mathcal{T}_\varepsilon(u_{\varepsilon\delta})\}$ is Cauchy in measure. Hence, there exists a subsequence, $\mathcal{T}_\varepsilon(u_{\varepsilon\delta})$ converges almost everywhere.
Then, there exists a measurable function $u_0 : \Omega \to \mathbb{R}$ finite almost everywhere, such that (4.1) holds.
Now, combining (4.1) and (4.7) yields

$$T_k(u^0) = u_k^0 \in H_0^1(\Omega), \quad for \;\; all \;\; k > 0.$$

Due to (4.8), we have

$$T_k(\mathcal{T}^b_{\varepsilon\delta}(u_{\varepsilon\delta})) \to T_k(u^0) \quad a.e \;\; in \; L^2(\Omega \times \partial T).$$

Using the same argument as above, and after extracting a subsequence, we show that (4.2) holds up to a subsequence.
From definition 2.9 we have

$$\begin{aligned}\mathcal{T}_{\varepsilon\delta}(A(x, u_{\varepsilon\delta}(x))) &= A\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\delta z, u_{\varepsilon\delta}\left(\left[\frac{x}{\varepsilon}\right]_Y + \delta z\right)\right)\\ &= A\left(z, \mathcal{T}_{\varepsilon\delta}\left(u_{\varepsilon\delta}\right)(x, \delta z)\right) \; a.e \; in \; \Omega_{\varepsilon\delta} \times \frac{1}{\delta}Y.\end{aligned}$$

On the other hand, using (4.1), we can extract a subsequence such that $\mathcal{T}_\varepsilon(u_{\varepsilon\delta})$ converge to u^0 in $\Omega \times \frac{1}{\delta}Y$. This together with A be a Caratheodory function, yields

$$\mathcal{T}_{\varepsilon\delta}(A(x, u_{\varepsilon\delta}(x))) \to A(z, u^0) \quad almost \; everywhere \;\; in \; \Omega \times \frac{1}{\delta}Y.$$

Finally, from (3.7) when $\gamma = 0$ we have

$$\int_{\partial T} \tau^\varepsilon(x)|h(u_{\varepsilon\delta})|d\sigma x \leq C.$$

Using the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$, we get

$$\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T} \tau(z)|h(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))|dxd\sigma z \leq C,$$

Since $\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}) \to u^0$ almost everywhere in $\mathbb{R}^N \times \partial T$, Fatou's lemma yields $\mathcal{T}^b_{\varepsilon\delta_1}$, we get

$$k_1 \int_{\mathbb{R}^N\times\partial T} \tau(z)|h((u_0))|dxd\sigma z \leq C,$$

then

$$k_1 \int_{\mathbb{R}^N \times \partial T} \tau(z)|h((u_0))|dxd\sigma z = k_1|\partial T|\mathcal{M}_{\partial T} \int_\Omega |h(u^0)|dx,$$

this implies

$$k_1|\partial T|\mathcal{M}_{\partial T}(\tau) \int_\Omega |h(u^0)|dx \leq C.$$

Since $\tau > 0$, it follows that $\mathcal{M}_{\partial T}(\tau) > 0$. Thus, (4.5) follows. □

Proposition 4.2. [14] *Given any $k > 0$, suppose $\widehat{u_k} \in L^2(\Omega, H^1_{per}(Y^*))$ satisfies $\mathcal{M}_{Y^*}(\widehat{u_k}) = 0$. Then there exists a unique measurable function $\widehat{u} : \Omega \times Y^* \to \mathbb{R}$ such that for all function $R \in C^1(\mathbb{R})$ with compact support such that supp$R \subset [-n, n]$ for some $n \in \mathbb{N}$, the following relation holds:*

$$R(u^0)\widehat{u_k} = R(u^0)\widehat{u} \qquad a.e.\ in\ \Omega \times Y^*, \quad \forall k \geq n. \tag{4.10}$$

where the function u^0 is defined by Proposition 4.1

5. Homogenization results

Proposition 5.1. *Suppose that assumptions of f, g, h, τ and the matrix A. Consider $u_{\varepsilon\delta}$ as the subsequence of the renormalized solution to problem (1.2), provided by Corollary 3.2. Then for any $k \in \mathbb{N}$ and $\varepsilon > 0$ we obtain:*

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{k} \int_{|u_{\varepsilon\delta}<k|} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta} dx = 0. \tag{5.1}$$

Proof. We consider $u_{\varepsilon\delta}$ be a renormalized solution of problem (1.2). In the renormalized formulation (2.8) we replace ψ *by* ψ_n and we choose $\frac{1}{k}T_k(u_{\varepsilon\delta}) \in V$ as test function, we obtain:

$$\frac{1}{k}\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla T_k(u_{\varepsilon\delta})dx + \frac{1}{k}\int_{\Omega_{\varepsilon\delta}} \psi_n'(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}T_k(u_{\varepsilon\delta})dx$$
$$+ \frac{1}{k}\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma \tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x$$
$$= \frac{1}{k}\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})dx + \frac{1}{k}\int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x,$$

we note the previous equation as

$$I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon = I_4^\varepsilon + I_5^\varepsilon. \tag{5.2}$$

For the first integral we use (2.5) to get

$$I_1^\varepsilon = \frac{1}{k}\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x, u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})dx. \tag{5.3}$$

Using the definition of T_k and (3.5) for the second integral, we get

$$|I_2^\varepsilon| \leq \frac{1}{n}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x, u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta} dx,$$

Now by (2.7), we have

$$\lim_{n\to\infty} |I_2^\varepsilon| = 0. \tag{5.4}$$

For the third integral, by properties of h *and* T_k, and for fixed k *and* ε. we get $h(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta}) \geq 0$. Also from assumption on τ and (3.2), we have $\varepsilon^\gamma \tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta}) \geq 0$. We deduce

$$I_3^\varepsilon \geq 0. \tag{5.5}$$

For the forth integral, we use (3.2) to get $|f\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})| \leq k|f| \in L^1(\Omega)$, using the fact that $\psi_n \to 1 \;\; as\; n \to \infty$ together with the Lebesgue dominated convergence theorem, yields:

$$\lim_{n\to\infty} I_4^\varepsilon = \lim_{n\to\infty} \frac{1}{k}\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})dx = \frac{1}{k}\int_{\Omega_{\varepsilon\delta}} fT_k(u_{\varepsilon\delta})dx.$$

Further, by (iii) of corollary 3.2 and (4.3) lead to

$$\limsup_{\varepsilon\to 0}\lim_{n\to\infty} I_4^\varepsilon = \frac{1}{k}\int_\Omega f u_k^0 dx = \int_\Omega f\frac{T_k(u^0)}{k}.$$

Since u^0 is finite almost everywhere, we note that as $k\to\infty$

$$\frac{T_k(u^0)}{k} \to 0 \quad\;\; a.e \;\; in\,\Omega. \tag{5.6}$$

Moreover, we have

$$\left| f\frac{T_k(u^0)}{k}\right| \leq |f| \in L^1(\Omega),$$

thanks the Lebesgue dominated convergence theorem we deduce that

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\lim_{n\to\infty} I_4^\varepsilon = 0. \tag{5.7}$$

For the fifth integral, using (3.2) we have $|g_\varepsilon\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})| \leq k|g_\varepsilon| \in L^1(\partial T)$, applying the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty} I_5^\varepsilon = \lim_{n\to\infty}\int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon\psi_n(u_{\varepsilon\delta})T_k(u_{\varepsilon\delta})d\sigma x = \frac{1}{k}\int_{\partial T_{\varepsilon\delta_1}} g_\varepsilon T_k(u_{\varepsilon\delta})d\sigma x.$$

Applying the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$ and using (4.6), we get

$$\lim_{n\to\infty} I_5^\varepsilon = \frac{1}{k}\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T} g(z)T_k(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))dxd\sigma z,$$

using (4.2) gives

$$\limsup_{\varepsilon\to 0}\lim_{n\to\infty} I_5^\varepsilon = \frac{1}{k}k_1\int_{\mathbb{R}^N\times\partial T} g(z)T_k(u^0)dxd\sigma z.$$

Indeed $\left|g\frac{T_k(u^0)}{k}\right| \leq |g| \quad \in L^1(\partial T)$, due to (5.6), using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\lim_{n\to\infty} I_5^\varepsilon = 0. \tag{5.8}$$

Finally, since $\lim_{n\to\infty} \psi_n = 1$, together (5.2), (5.3), (5.4), (5.5) and Fatou's lemma yield

$$\frac{1}{k}\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta}) = \lim_{n\to\infty} I_1^{\varepsilon} \le \lim_{n\to\infty} I_4^{\varepsilon} + \lim_{n\to\infty} I_5^{\varepsilon},$$

using (5.7) and (5.8), we deduce that

$$\lim_{k\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{k}\int_{\Omega_{\varepsilon\delta}} A^{\varepsilon}(x,u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta}) = 0.$$

Hence, the proof of proposition is established. □

Other tools that will be needed below are some results from [7]. To formulate them, define the space K_B as

$$K_B = \{\Phi \in L^{2^*}(\mathbb{R}^N) \mid \nabla\Phi \in L^2(\mathbb{R}^N),\ \Phi \text{ constant on } B\}.$$

Lemma 5.2. *([7]) Let $N \ge 3$. Then, for every $\delta_0 > 0$, the set*

$$\bigcup_{0<\delta_2<\delta_0} \{\phi \in H^1_{per}(Y) \big|\ \phi = 0 \ \text{on}\ \delta_2 B\},$$

is dense in $H^1_{per}(Y)$.

Lemma 5.3. *([7]) Let $v \in D(\mathbb{R}^N) \cap K_B$ and*

$$w_{\varepsilon\delta_2}(x) = v(B) - v\left(\frac{1}{\delta_2}\left\{\frac{x}{\varepsilon}\right\}\right) \quad \text{for} \quad x \in \mathbb{R}^N.$$

Then

$$w_{\varepsilon\delta_2} \rightharpoonup v(B) \quad \text{weakly in } H^1(\Omega). \tag{5.9}$$

Now, as in [7], let the function χ be the solution of the following cell problem:

$$\begin{cases} \chi \in L^{\infty}(\Omega; K_B), \\ \displaystyle\int_{\mathbb{R}^N\setminus B} \nabla_z\chi(x,z)\,\nabla_z\Psi(z)\,dz = 0 \quad \text{for a.e.} x \in \Omega,\ \ \forall \Psi \in K_B \text{ with } \Psi(B) = 0, \end{cases} \tag{5.10}$$

and introduce the function Θ given by

$$\Theta(x) = \int_{\mathbb{R}^N\setminus B} \nabla_z\chi(x,z)\,\nabla_z\chi(x,z)\,dz. \tag{5.11}$$

Notice that Θ can be interpreted as a local capacity of the set B since it is positive by definition. Assume now that there exist k_1 with $0 \le k_1 < +\infty$, and k_2 with $0 \le k_2 < +\infty$, such that

$$k_1 = \lim_{\varepsilon\to 0}\frac{\delta_1(\varepsilon)^{N-1}}{\varepsilon} \quad \text{and} \quad k_2 = \lim_{\varepsilon\to 0}\frac{\delta_2(\varepsilon)^{\frac{N}{2}-1}}{\varepsilon}. \tag{5.12}$$

We now present the main homogenization result associated with the problem (1.2).

Theorem 5.4. *Under the assumptions of $f, g, h, \tau,$ and the matrix A, we consider $u_{\varepsilon\delta}$ to be a renormalized solution of (1.2). We define the function $J(\gamma)$ for every $\gamma \geq 0$ as follows:*

$$J(\gamma)\begin{cases}|\partial T| & if\ \gamma = 0,\\ 0 & if\ \gamma > 0.\end{cases} \tag{5.13}$$

suppose that as $\varepsilon \longrightarrow 0$ there exist two matrices fields A and A^0 such that, respectively:

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \longrightarrow A(x,y)\ a.e.\ in\quad \Omega \times Y, \tag{5.14}$$

$$\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon)(x,z) \longrightarrow A^0(x,z)\ a.e.\ in\quad \Omega \times (\mathbb{R}^N \setminus B). \tag{5.15}$$

Let $u_{\varepsilon\delta}$ be the solution of (1.2). Then there exists a subsequence such that there exist a measurable function $u^0 : \Omega \to \mathbb{R}$, and finite almost everywhere, and for any $k \in \mathbb{N}$, $\widehat{u_k} \in L^2(\Omega, H^1_{per}(Y^*))$ with $\mathcal{M}_y(\widehat{u_k}) = 0$ satisfying

$$\begin{cases}(i). & \mathcal{T}_\varepsilon(T_k(u_{\varepsilon\delta})) \to T_k(u^0) & strongly\ in\ L^2(\Omega, H^1(Y)),\\ (ii). & \mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta})) \rightharpoonup \nabla T_k(u^0) + \nabla_y \widehat{u_k} & weakly\ in\ L^2(\Omega \times Y),\\ (iii). & T_k(\widetilde{u_\varepsilon}) = \widetilde{T_k(u_\varepsilon)} \rightharpoonup T_k(u_0) & weakly\ in\ L^2(\Omega.),\\ (iv). & \|T_k(u_{\varepsilon\delta}) - u^0_k\|_{L^2(\Omega_{\varepsilon\delta})} \to 0.\end{cases} \tag{5.16}$$

Moreover, there exists $\widehat{u} : \Omega \times Y \to \mathbb{R}$ a measurable function such that the triple $(u^0, \widehat{u}, U)$ satisfies the following limit problem

$$\begin{cases}\displaystyle\int_{\Omega\times Y} A(y,u^0)(\nabla u^0 + \nabla_y \widehat{u})(\nabla(S(u^0)\eta_0) + S_1(u^0)\nabla_y \Psi(x,y))dxdy\\ \displaystyle + k_1 J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_\Omega h(u^0)S(u^0)\eta_0 dx\\ \displaystyle - k_2 \int_{\Omega\times\partial B} A^0(z,u^0)\nabla_z U(x,z,t)\nu_B S(u^0)\eta_0 dxd\sigma z\\ \displaystyle = \int_\Omega f S(u^0)\eta_0 dx + k_1|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega S(u^0)\eta_0 dx\\ for\ every\ \eta_0 \in H^1_0(\Omega) \cap L^\infty(\Omega)\ and\ for\ every\ \Psi \in L^2(\Omega, H^1_{per}(Y)),\end{cases} \tag{5.17}$$

for every function S and S_1 in $C^1(\mathbb{R})$ with compact supports.
While the function U obeys

$$\int_{\mathbb{R}^N\setminus B} A^0(z,u^0)\nabla_z U(x,z)\nabla_z v(z)dz = 0, \tag{5.18}$$

for a.e. x in Ω, and for any $v \in K_B$, with $v(B) = 0$.
Further, we have the convergence

$$\lim_{k\to\infty}\frac{1}{k}\int_{\Omega\times Y} A(y,u^0)(\nabla T_k(u^0) + \nabla_y \widehat{u_k})(\nabla T_k(u^0) + \nabla_y \widehat{u_k})dxdy = 0. \tag{5.19}$$

Proof. Let $u_{\varepsilon\delta}$ be a renormalized solution of (1.2). We use $\varphi w_{\varepsilon\delta_2}$ as test function in (2.8) where $\varphi \in \mathcal{D}(\Omega)$ and $w_{\varepsilon\delta_2}$ defined by Lemma 5.3. We also replace ψ *by* ψ_n

where ψ_n is given by (3.1). We get

$$\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla(\varphi w_{\varepsilon\delta_2})dx + \int_{\Omega_{\varepsilon\delta}} \psi_n'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}(\varphi w_{\varepsilon\delta_2})dx$$
$$+\int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}d\sigma x$$
$$=\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})(\varphi w_{\varepsilon\delta_2})dx + \int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi_n(u_{\varepsilon\delta})(\varphi w_{\varepsilon\delta_2})d\sigma x,$$

then

$$\int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla\varphi w_{\varepsilon\delta_2}dx + \int_{\Omega_{\varepsilon\delta}} \psi_n(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\varphi\nabla w_{\varepsilon\delta_2}dx$$
$$+\int_{\Omega_{\varepsilon\delta}} \psi_n'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}(\varphi w_{\varepsilon\delta_2})dx + \int_{\partial T_{\varepsilon\delta_1}} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}d\sigma x$$
$$=\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})(\varphi w_{\varepsilon\delta_2})dx + \int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon\psi_n(u_{\varepsilon\delta})(\varphi w_{\varepsilon\delta_2})d\sigma x,$$

which reads as

$$J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 + J_\varepsilon^4 = J_\varepsilon^5 + J_\varepsilon^6. \tag{5.20}$$

In order to pass to the limit in each term with n fixed, we make use the unfolding method. Given that φ has a compact support in Ω, using Proposition 2.6, Remark 2.7 and (3.3), we can write

$$\lim_{\varepsilon\to 0} J_\varepsilon^1 = \lim_{\varepsilon\to 0}\int_{\Omega\times Y} \psi_n(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta})))\mathcal{T}_\varepsilon(\nabla T_{2n}(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla\varphi)\mathcal{T}_\varepsilon(w_{\varepsilon\delta_2})dxdy. \tag{5.21}$$

Since (4.1) holds

$$\mathcal{T}_\varepsilon(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta}))) = A^\varepsilon(y,T_{2n}(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))) \to A(y,T_{2n}(u^0)) \quad a.e \;\; in\ \Omega\times Y. \tag{5.22}$$

Thanks to assumption of A, $\mathcal{T}_\varepsilon(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta})))$ is bounded in $L^\infty(\Omega\times Y)$ so that

$$A^\varepsilon(y,\mathcal{T}_\varepsilon(T_{2n}(u_{\varepsilon\delta}))) \to A(y,T_{2n}(u^0)) \;\; weakly^* \;\; in\ L^\infty(\Omega\times Y).$$

And from (4.1) and the continuity of ψ_n, for every n, we have

$$\psi_n(\mathcal{T}_\varepsilon(u_{\varepsilon\delta})) \to \psi_n(u^0) \quad almost \;\; everywhere \;\; in\ \Omega\times Y. \tag{5.23}$$

Due to hypotheses on A, property (3) of proposition 2.6, (3.1), (5.22) and (5.23), in view the Lebesgue dominated convergence theorem we get

$$\psi_n(\mathcal{T}_\varepsilon(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(A(x,T_{2n}(u_{\varepsilon\delta})))\mathcal{T}_\varepsilon(\nabla\varphi)$$
$$\to \psi_n(u^0)A(y,T_{2n}(u^0))\nabla\varphi \quad strongly\ in\ (L^2(\Omega\times Y))^N.$$

Now, passing to the limit in (5.21), and using (ii) of corollary 3.2, (3.3), (4.3) and (5.9) we get

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon^1 &= v(B)\int_{\Omega\times Y}\psi_n(u_0)A(y,T_{2n}(u^0))\nabla\varphi(\nabla u_{2n}^0+\nabla_y\widehat{u_{2n}})dxdy\\ &= v(B)\int_{\Omega\times Y}\psi_n(u_0)A(y,u^0)\nabla\varphi(\nabla u^0+\nabla_y\widehat{u_{2n}})dxdy.\end{aligned} \tag{5.24}$$

Unfolding the second integral by $\mathcal{T}_{\varepsilon\delta_2}$ and using lemma 5.3, we get

$$\lim_{\varepsilon\to 0} J_\varepsilon^2 = \frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\int_{\Omega\times\mathbb{R}^N}\psi_n(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta})))\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla T_{2n}(u_{\varepsilon\delta})))\mathcal{T}_{\varepsilon\delta_2}(\varphi)(-\nabla_z v)dxdz. \tag{5.25}$$

Thanks to estimates (3.6) , we can apply Proposition 2.11. Thus, there exists some U_{2n} *in* $L^2(\Omega, L^2_{loc}(\mathbb{R}^N))$ such that, up to a subsequence

up to a subsequence,

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon\delta_2}(T_{2n}(u_{\varepsilon\delta}))\rightharpoonup U_{2n} \text{ weakly in } L^2(\Omega;L^2_{loc}(\mathbb{R}^N)). \tag{5.26}$$

Using (2.9), one also has the convergence

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}_Y^\varepsilon(T_{2n}(u_{\varepsilon\delta}))\to k_2\, u_{2n}^0 \text{ strongly in } L^2(\Omega;L^2_{loc}(\mathbb{R}^N)). \tag{5.27}$$

However, according to Proposition 2.11, there exists W_{2n} in $L^2(\Omega;L^{2^*}(\mathbb{R}^N))$ with $\nabla_z W_{2n}$ in $L^2(\Omega\times\mathbb{R}^N)$, such that

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon\delta_2}(T_{2n}(u_{\varepsilon\delta}))-\mathcal{M}_Y^\varepsilon(T_{2n}(u_{\varepsilon\delta})))\rightharpoonup W_{2n} \text{ weakly in } L^2(\Omega;L^{2^*}(\mathbb{R}^N)). \tag{5.28}$$

From (5.26),(5.27) and (5.28), one conclude that

$$U_{2n}=W_{2n}+k_2u_{2n}^0 \quad\text{and}\quad \nabla_z U_{2n}=\nabla_z W_{2n},$$

by Proposition 2.11 again, one also has convergence

$$\frac{\delta_2^{\frac{N}{2}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon\delta_2}(T_{2n}(u_{\varepsilon\delta}))=\delta_2^{\frac{N}{2}}\mathcal{T}_{\varepsilon\delta_2}(\nabla T_{2n}(u_{\varepsilon\delta}))\rightharpoonup\nabla_z U_{2n} \text{ weakly in } L^2(\Omega\times\mathbb{R}^N). \tag{5.29}$$

On the other hand, since (4.1) holds and using (4.4) we have

$$\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta})))=A^\varepsilon(z,T_{2n}(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta})))\to A^0(z,T_{2n}(u^0)) \quad a.e \;\; in\; \Omega\times(\mathbb{R}^N\setminus B). \tag{5.30}$$

Due to assumption of A, $\mathcal{T}_{\varepsilon\delta_2}(A^\varepsilon(x,T_{2n}(u_{\varepsilon\delta})))$ is bounded in $L^\infty(\Omega\times(\mathbb{R}^N\setminus B))$ so that

$$A^\varepsilon(y,\mathcal{T}_{\varepsilon\delta_2}(T_{2n}(u_{\varepsilon\delta})))\to A(z,T_{2n}(u^0)) \;\; weakly^* \;\; in\; L^\infty(\Omega\times(\mathbb{R}^N\setminus B)).$$

And from (4.1) and the continuity of ψ_n, for every n, we have

$$\psi_n(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta}))\to\psi_n(u^0) \quad a.e \;\; in\; \Omega\times\mathbb{R}^N. \tag{5.31}$$

Thanks to assumptions on A, (3.1), (5.30) and (5.31), the Lebesgue dominated convergence theorem can be applied, and we obtain the following result

$$\psi_n(\mathcal{T}_{\varepsilon\delta_2}(u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon\delta_2}(A(x,T_{2n}(u_{\varepsilon\delta})))\mathcal{T}_{\varepsilon\delta_2}(\nabla\varphi) \\ \to \psi_n(u^0)A^0(z,T_{2n}(u^0))\nabla\varphi \quad \textit{strongly in } (L^2(\Omega\times(\mathbb{R}^N\setminus B))).$$

We then compute the limit in (5.25) using (5.29), (1.3) and (3.3) and get

$$\lim_{\varepsilon\to 0} J_\varepsilon^2 = -k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} \psi_n(u^0)A^0(z,u^0)\nabla_z U\nabla_z v(z)\varphi dxdz \tag{5.32}$$

For the third integral of (5.20), using (3.3) and (3.5), we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}|J_\varepsilon^3| &= \limsup_{\varepsilon\to 0}\left|\int_{\Omega_{\varepsilon\delta}}\psi_n'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}(\varphi w_{\varepsilon\delta_2})dx\right| \\ &\le 2v(B)\|\varphi\|_{L^\infty(\Omega)}\limsup_{\varepsilon\to 0}\frac{1}{2n}\int_{|u_{\varepsilon\delta}|<2n}A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}dx.\end{aligned} \tag{5.33}$$

Hence, as consequence of proposition 5.1

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0} J_\varepsilon^3 = 0. \tag{5.34}$$

Now for the forth integral J_ε^4, we have two cases:
The first case if $\gamma=0$, under the assumption on τ, using the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$, (3.3) and (1.3) we get

$$\begin{aligned}&\int_{\partial T_{\varepsilon\delta_1}}\tau^\varepsilon(x)\psi(u_{\varepsilon\delta})h(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}d\sigma x \\ &= \frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T}\tau(z)\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(\varphi w_{\varepsilon\delta_2})dxd\sigma z,\end{aligned} \tag{5.35}$$

using (4.2) and the continuity of ψ_n, we get

$$\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\to\psi_n(u^0)\quad \textit{a.e. in } \mathbb{R}^N\times\partial T. \tag{5.36}$$

Furthermor, using the continuity of h and the fact that $h(0)=0$, together with (4.2) we obtain,

$$\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta})) = h(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\to h(u^0)\quad \textit{almost everywhere in } \mathbb{R}^N\times\partial T. \tag{5.37}$$

From the above convegrnces (5.36) and (5.37), for every fixed n, we get

$$\tau(z)\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta}))\to\tau(z)\psi_n(u^0)h(u^0)\quad \textit{a.e. in } \mathbb{R}^N\times\partial T. \tag{5.38}$$

By (3.3) we deduce that

$$\left|\tau(z)\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta}))\right|^2\le 4\|\tau\|_{L^\infty(\Omega)}n^2,$$

applying the Lebesgue dominated convergence theorem we get

$$\tau(z)\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta}))\to\tau(z)\psi_n(u^0)h(u^0)\quad \textit{strongly in } L^2(\mathbb{R}^N\times\partial T). \tag{5.39}$$

Thus, (5.9), (1.3) and (5.39) we can pass to the limit in expression (5.35) we get

$$\lim_{\varepsilon\to 0}\int_{\partial T_{\varepsilon\delta_1}}\tau^{\varepsilon}(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}d\sigma x = k_1 v(B)|\partial T|\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)\varphi dx. \tag{5.40}$$

The second cases, if $\gamma > 0$, we have from (5.40)

$$\lim_{\varepsilon\to 0}\varepsilon^{\gamma}\int_{\partial T_{\varepsilon\delta_1}}\tau^{\varepsilon}(x)\psi_n(u_{\varepsilon\delta})h(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}d\sigma x = 0. \tag{5.41}$$

Combining (5.40) and (5.41), we have

$$\lim_{\varepsilon\to 0} J_{\varepsilon}^3 = K_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)\varphi dx, \tag{5.42}$$

where $J(\gamma)$ is defined by (5.13).
For the fifth term, by the unfolding operator $\mathcal{T}_{\varepsilon}$, remark 2.1, and obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} J_{\varepsilon}^5 &= \lim_{\varepsilon\to 0}\int_{\Omega_{\varepsilon\delta}} f\psi_n(u_{\varepsilon\delta})\varphi w_{\varepsilon\delta_2}dx\\ &= \lim_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_{\varepsilon}(f)\psi_n(\mathcal{T}_{\varepsilon}(u_{\varepsilon\delta}))\mathcal{T}_{\varepsilon}(\varphi)\mathcal{T}_{\varepsilon}(w_{\varepsilon\delta_2})dxdy.\end{aligned}$$

Now, by (5.23), we pass to the limit, using (1.3) and (5.9) and get

$$\lim_{\varepsilon\to 0} J_{\varepsilon}^5 = v(B)\int_{\Omega\times Y} f\psi_n(u^0)\varphi dxdy = v(B)\int_{\Omega} f\psi_n(u^0)\varphi dx. \tag{5.43}$$

Using the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$ on the last integral J_{ε}^5 and by remark 2.1 we obtain

$$\begin{aligned} J_{\varepsilon}^6 &= \int_{\partial T} g\left(\frac{x}{\varepsilon\delta_1}\right)\psi_n(u_{\varepsilon\delta})\varphi(x)w_{\varepsilon\delta_2}d\sigma x\\ &= \frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T} g(z)\psi_n(\mathcal{T}^b_{\varepsilon\delta_1}(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(\varphi)\mathcal{T}^b_{\varepsilon\delta_1}(w_{\varepsilon\delta_2})dxd\sigma z.\end{aligned}$$

Due to (3.2), (5.9), (1.3) and (5.36), taking the limit as $\varepsilon\to 0$ for n fixed and using the Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0} J_{\varepsilon}^6 = v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_{\Omega}\psi_n(u^0)\varphi dx. \tag{5.44}$$

Finally, by (5.24), (5.32), (5.34), (5.42), (5.43) and (5.44) we are able to pass to the limit in expression (5.20), which yields the following limit equation:

$$\begin{aligned} &v(B)\int_{\Omega\times Y}\psi_n(u_0)A(y,u^0)\nabla\varphi(\nabla u^0 + \nabla_y\widehat{u_{2n}})dxdy - k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)}\psi_n(u^0)A^0(z,u^0)\nabla_z U\nabla_z v(z)\varphi dxdz\\ &+ v(B)w(n,\varphi) + k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)\varphi dx\\ &= v(B)\int_{\Omega} f\psi_n(u^0)\varphi dx + v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_{\Omega}\psi_n(u^0)\varphi dx,\end{aligned}$$

for every $\varphi \in \mathcal{D}(\Omega)$, and where

$$|w(n,\varphi)| \leq \|\varphi\|_{L^\infty(\Omega)} w(n), \quad with\ w(n) \to 0\ as\ n \to \infty, \tag{5.45}$$

in view of (5.33) and (5.34).
by proposition 4.2, we take $R = \psi_n$, there exists $\widehat{u} : \Omega \times Y \to \mathbb{R}$ a measurable function, such that

$$\psi_n(u^0)\widehat{u_{2n}} = \psi_n(u^0)\widehat{u} \ \ for\ every\ n. \tag{5.46}$$

Then we can write the integral equation above as

$$\begin{aligned} & v(B)\int_{\Omega\times Y} \psi_n(u_0)A(y,u^0)\nabla\varphi(\nabla u^0 + \nabla_y\widehat{u})dxdy \\ & - k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} \psi_n(u^0)A^0(z,u^0)\nabla_z U\nabla_z v(z)\varphi dxdz \\ & + v(B)w(n,\varphi) + k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)\varphi dx \\ & = v(B)\int_\Omega f\psi_n(u^0)\varphi dx + v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega \psi_n(u^0)\varphi dx. \end{aligned} \tag{5.47}$$

By density and due to (5.45) and (5.47) the result is valid for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.
Now, let $S\ and\ S_1\ be\ in\ C^1(\mathbb{R})$ with compact supports. Take $k > 0$ such that

$$suppS \subset [-k,k] \quad and \ \ suppS_1 \subset [-k,k]. \tag{5.48}$$

We set $\varphi = S(u^0)\eta_0$ in (5.47) where $\eta_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ we get

$$\begin{aligned} & v(B)\int_{\Omega\times Y} \psi_n(u_0)A(y,u^0)(\nabla u^0 + \nabla_y\widehat{u})S(u^0)\nabla\eta_0 dxdy \\ & + v(B)\int_{\Omega\times Y} \psi_n(u_0)A(y,u^0)(\nabla u^0 + \nabla_y\widehat{u})S'(u^0)\eta_0 dxdy \\ & - k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} \psi_n(u^0)A^0(z,u^0)\nabla_z U\nabla_z v(z)S(u^0)\eta_0 dxdz \\ & + v(B)w(n,S(u^0)\eta_0) + k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)S(u^0)\eta_0 dx \\ & = v(B)\int_\Omega f\psi_n(u^0)S(u^0)\eta_0 dx + v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega \psi_n(u^0)S(u^0)\eta_0 dx. \end{aligned} \tag{5.49}$$

For $n \geq k$, from (3.1) and (5.48), we have

$$\psi_n(u^0)S(u^0) = S(u^0) \ \ and\ \psi_n(u^0)S'(u^0) = S'(u^0),$$

this implies that

$$
\begin{aligned}
& v(B)\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})S(u^0)\nabla\eta_0 dxdy\\
&+v(B)\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})S'(u^0)\eta_0 dxdy\\
&-k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} A^0(z,u^0)\nabla_z U\nabla_z v(z)S(u^0)\eta_0 dxdz+v(B)w(n,S(u^0)\eta_0)\\
&+k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)S(u^0)\eta_0 dx\\
&=v(B)\int_\Omega f\psi_n(u^0)S(u^0)\eta_0 dx+v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega S(u^0)\eta_0 dx.
\end{aligned}
\tag{5.50}
$$

Taking to the limit as $n\to\infty$ in (5.50) and using (5.45) we obtain

$$
\begin{aligned}
& v(B)\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})\nabla(S(u^0)\eta_0) dxdy\\
&-k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} A^0(z,u^0)\nabla_z U\nabla_z v(z)S(u^0)\eta_0 dxdz\\
&+k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)S(u^0)\eta_0 dx\\
&=v(B)\int_\Omega f\psi_n(u^0)S(u^0)\eta_0 dx+v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega S(u^0)\eta_0 dx.
\end{aligned}
\tag{5.51}
$$

If $v(B)=0$ we get the equation (5.18).
Now, we choose $\psi=S_1$ and $v=v_\varepsilon$ as a test functions in (2.8), with v_ε defined by

$$
v_\varepsilon=\varepsilon\varphi(x)\xi\left(\frac{x}{\varepsilon}\right)\quad \textit{such that}\ \ \varphi\in\mathcal{D}(\Omega)\quad \textit{and}\ \ \xi\in H^1_{per}(Y)\cap L^\infty(Y),
$$

then, we have

$$
\begin{aligned}
&\int_{\Omega_{\varepsilon\delta}} S_1(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla v_\varepsilon dx+\int_{\Omega_{\varepsilon\delta}} S_1'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}v_\varepsilon dx\\
&+\int_{\partial T_{\varepsilon\delta_1}}\varepsilon^\gamma\tau^\varepsilon(x)S_1(u_{\varepsilon\delta})h(u_{\varepsilon\delta})v_\varepsilon d\sigma x=\int_{\Omega_{\varepsilon\delta}} fS_1(u_{\varepsilon\delta})v_\varepsilon dx+\int_{\partial T_{\varepsilon\delta_1}} g^\varepsilon S_1(u_{\varepsilon\delta})v_\varepsilon d\sigma x,
\end{aligned}
$$

which can written as

$$
L^1_\varepsilon+L^2_\varepsilon+L^3_\varepsilon=L^4_\varepsilon+L^5_\varepsilon. \tag{5.52}
$$

By proposition 2.6, we have

$$
\mathcal{T}_\varepsilon(v_\varepsilon)=\varepsilon\mathcal{T}_\varepsilon(\varphi)\xi,\quad \mathcal{T}_\varepsilon(\nabla v_\varepsilon)=\varepsilon\mathcal{T}_\varepsilon(\nabla\varphi)\xi\left(\frac{\cdot}{\varepsilon}\right)+\mathcal{T}_\varepsilon(\varphi)\nabla_y\xi\left(\frac{\cdot}{\varepsilon}\right), \tag{5.53}
$$

and

$$
\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\to\varphi\nabla_y\xi\quad \textit{strongly in}\ L^2(\Omega\times Y)\ \ \textit{as}\ \varepsilon\to 0. \tag{5.54}
$$

According, by same method used to show (5.24), combining proposition 2.6, (5.22), (5.23), corollary 3.2, (5.46) and (5.54) yields

$$\begin{aligned}\lim_{\varepsilon\to 0} L^1_\varepsilon &= \lim_{\varepsilon\to 0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(S_1(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(A^\varepsilon(x,T_k(u_{\varepsilon\delta})))\mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla v_\varepsilon)dxdy\\ &= \int_{\Omega\times Y} S_1(u^0)(A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})\varphi(x)\nabla_y\xi(y)dxdy.\end{aligned} \tag{5.55}$$

For the second term of (5.52), since S_1 has compact support and by assumption on A and (3.1), the term $\int_{\{|u_{\varepsilon\delta}|<k\}} S_1'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}dx$ is bounded, so that we have

$$|L^2_\varepsilon| = \left|\int_{\Omega_{\varepsilon\delta}} S_1'(u_{\varepsilon\delta})A^\varepsilon(x,u_{\varepsilon\delta})\nabla u_{\varepsilon\delta}\nabla u_{\varepsilon\delta}v_\varepsilon dx\right| \le C\varepsilon\|\varphi\|_{L^\infty(\Omega)}\|\xi\|_{L^\infty(Y)},$$

consequently

$$\limsup_{\varepsilon\to 0} L^2_\varepsilon = 0. \tag{5.56}$$

By the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$ for the third term and using (1.3), (5.36), (5.37) and (5.53), we get for $\gamma\ge 0$

$$\lim_{\varepsilon\to 0} L^3_\varepsilon = \lim_{\varepsilon\to 0}\varepsilon^\gamma\frac{\delta_1^{N-1}}{\varepsilon}\int_{\mathbb{R}^N\times\partial T}\tau(z)\mathcal{T}^b_{\varepsilon\delta_1}(S_1(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(h(u_{\varepsilon\delta}))\mathcal{T}^b_{\varepsilon\delta_1}(v_\varepsilon)dxd\sigma z = 0. \tag{5.57}$$

On the other hand, for the forth integral L^4_ε we use $\mathcal{T}_\varepsilon$, by proposition 2.6 together with (5.23) we can pass to the limit and get

$$\lim_{\varepsilon\to 0} L^4_\varepsilon = 0. \tag{5.58}$$

For the last surface integral L^5_ε using the boundary unfolding operator $\mathcal{T}^b_{\varepsilon\delta_1}$ and by (1.3) together with (5.36) we can pass to the limit and get

$$\lim_{\varepsilon\to 0} L^5_\varepsilon = 0. \tag{5.59}$$

We are now ready to pass to the limit in (5.52) as $\varepsilon\to 0$. Thanks to (5.55), (5.56), (5.58), (5) and (5.59), we deduce

$$\int_{\Omega\times Y} S_1(u^0)A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})\varphi(x)\nabla_y\xi(y)dxdy = 0, \tag{5.60}$$

for every $\varphi\in\mathcal{D}(\Omega)$ and for every $\xi\in H^1_{per}(Y)\cap L^\infty(Y)$.
Due to the compact support of S_1 , we obtain $S_1(u^0)A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})\in(L^2(\Omega\times Y))^N$. By density, (5.60) is equivalent to

$$\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})S_1(u^0)\nabla_y(\eta_1(x)\varphi(x)\xi(y))dxdy = 0,$$

for every $\eta_1\in L^2(\Omega)$ *and* $\xi\in H^1_{per}(Y)$.
We observe that $\eta_1(x)\xi(y)\in L^2(\Omega)\otimes H^1_{per}(Y)$ moreover, the density of $L^2(\Omega)\otimes H^1_{per}(Y)$ in $L^2(\Omega;H^1_{per}(Y))$ allows us to deduce that

$$\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})S_1(u^0)\nabla_y\Psi(x,y)dxdy=0, \tag{5.61}$$

for all $\Psi(x,y)\in L^2(\Omega;H^1_{per}(Y))$ Adding (5.51) and (5.61) we get

$$\begin{aligned}
&v(B)\int_{\Omega\times Y} A(y,u^0)(\nabla u^0+\nabla_y\widehat{u})(\nabla(S(u^0)\eta_0)+S_1(u^0)\nabla_y\Psi(x,y))dxdy\\
&-k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} A^0(z,u^0)\nabla_z U\nabla_z v(z)S(u^0)\eta_0 dxdz\\
&+k_1 v(B)J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_{\mathbb{R}^N}\psi_n(u^0)h(u^0)S(u^0)\eta_0 dx\\
&=v(B)\int_\Omega f\psi_n(u^0)S(u^0)\eta_0 dx+v(B)|\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega S(u^0)\eta_0 dx,
\end{aligned} \tag{5.62}$$

integrating by parts in second term, it is easily seen that

$$\begin{aligned}
&k_2\int_{\Omega\times(\mathbb{R}^{\mathbb{N}}\setminus B)} A^0(z,u^0)\nabla_z U\nabla_z v(z)S(u^0)\eta_0 dxdz\\
&\qquad=k_2 v(B)\int_{\Omega\times\partial B} A^0(z,u^0).\nu_B S(u^0)\eta_0 dxd\sigma z.
\end{aligned}$$

We replace this formula in (5.62) we obtain (5.17).
For the convergence behaviour of the truncated energy, we set

$$\mathcal{I}^0_k=\int_{\Omega\times Y} A(y,u^0)(\nabla T_k(u^0)+\nabla_y\widehat{u_k})(\nabla T_k(u^0)+\nabla_y\widehat{u_k})dxdy.$$

For any $\varepsilon>0$ let us define

$$D_\varepsilon=\mathcal{T}_\varepsilon(A^\varepsilon(x,u_{\varepsilon\delta})),\quad \zeta_\varepsilon=\mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta})).$$

By (5.16)(ii) and (5.22) we have,

$$\begin{aligned}
D_\varepsilon&\to A(y,u^0)\ \ \textit{a.e. in }\Omega\times Y,\\
\zeta_\varepsilon&\rightharpoonup\nabla T_k(u^0)+\nabla_y\widehat{u_k}\ \ \textit{weakly in }L^2(\Omega\times Y).
\end{aligned}$$

According to proposition 2.6 we have

$$\begin{aligned}
\mathcal{I}^0_k&\le\liminf_{\varepsilon\to0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon(x,u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta}))\mathcal{T}_\varepsilon(\nabla T_k(u_{\varepsilon\delta}))dxdy\\
&\le\limsup_{\varepsilon\to0}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x,u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})dx.
\end{aligned}$$

Usibg Proposition 5.1 we have

$$\lim_{k\to\infty}\frac{1}{k}\mathcal{I}^0_k\le\lim_{k\to\infty}\limsup_{\varepsilon\to0}\frac{1}{k}\int_{\Omega_{\varepsilon\delta}} A^\varepsilon(x,u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})\nabla T_k(u_{\varepsilon\delta})dx=0,$$

that is

$$\lim_{k\to\infty}\frac{1}{k}\int_{\Omega\times Y} A(y,u^0)(\nabla T_k(u^0)+\nabla_y\widehat{u_k})(\nabla T_k(u^0)+\nabla_y\widehat{u_k})dxdy=0,$$

which implies (5.19). $\square$

Theorem 5.5. [14] *According to the setting of Theorem 5.4, the function $\widehat{u}$ is given by*

$$\widehat{u}(y,x) = -\sum_{j=1}^{N} \widehat{\chi_{e_j}}(y, u^0(x)) \frac{\partial u^0}{\partial x_j}(x),$$

where $(e_j)_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$ and $\widehat{\chi_{e_j}}(.,t)$ is the solution of

$$\begin{cases} -div(A(.,t)\nabla_y \widehat{\chi_\lambda}(.,t)) = -div(A(.,t)\lambda) & in\ Y, \\ A(.,t)(\lambda - \nabla_y \widehat{\chi_\lambda}(.,t)).n = 0 & on\ \partial T, \\ \widehat{\chi_\lambda}(.,t) \qquad Y - periodic, \\ \mathcal{M}_Y(\widehat{\chi_\lambda}(.,t)) = 0. \end{cases}$$

for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$.

Corollary 5.6. *Let us consider the function u^0 defined by theorem 5.4. Then u^0 is a renormalized solution of the problem*

$$\begin{cases} -div(A^{hom}(u^0)\nabla u^0) + k_1 J(\gamma)|\partial T|\mathcal{M}_{\partial T}(\tau)h(u^0) + k_2^2 \Theta u^0 = f + k_1|\partial T|\mathcal{M}_{\partial T}(g) & on\ \ \Omega, \\ u^0 = 0 & on\ \ \partial\Omega, \end{cases} \tag{5.63}$$

that is, u^0 satisfies

$$T_k(u^0) \in H_0^1(\Omega) \quad for\ any\ k > 0,$$

$$\lim_{n\to 0} \frac{1}{n} \int_{\{x\in\Omega:|u^0|<n\}} A^{hom}(u^0)\nabla u^0 \nabla u^0 dx = 0,$$

where the function Θ giving rise to a "strange term", is given by (5.25)

The homogenized matrix A^{hom} is defined [5], for all fixed $t \in \mathbb{R}$, as

$$A^{hom}(t)\lambda = \int_{Y\setminus B} A(y,t)\nabla_y \widehat{w_\lambda}(y,t)dy \qquad \forall \lambda \in \mathbb{R}^N,$$

in which

$$\widehat{w_\lambda}(y,t) = \lambda_y - \widehat{\chi_\lambda}(y,t),$$

Remark 5.7.
(1) If $\delta_1 = 1$ and $k_2 = 0$, we are in the case of the classical homogenization, and we get the results established by P. Donato and al. [14]

(2) If $k_1 = 0$ *and* $k_2 = 0$, it means that the "small holes" have no effect on the considered problem, we get so,

$$\begin{cases} -div(A^{hom}(u^0)\nabla u^0) = f & in\ \ \Omega, \\ u^0 = 0 & on\ \ \partial\Omega. \end{cases}$$

(3) If $k_1 \neq 0$ *and* $k_2 = 0$, the "Robin" holes have an effect on the considered problem while the Dirichlet holes have no effect and (5.63) is reduced to the following,

$$\begin{cases} -div(A^{hom}(u^0)\nabla u^0) + k_1 J(\gamma)|\partial T|\mathcal{M}_{\partial T}(\tau)h(u^0) = f + k_1|\partial T|\mathcal{M}_{\partial T}(g) & on\ \ \Omega, \\ u^0 = 0 & on\ \ \partial\Omega. \end{cases}$$

(4) If $k_1 = 0$ *and* $k_2 \neq 0$, the "Dirichlet" holes have an effect on the considered problem while the "Robin" holes have no effect, and (5.63) is reduced to the following,

$$\begin{cases} -div(A^{hom}(u^0)\nabla u^0) + k_2^2 \Theta u^0 = f & on \quad \Omega, \\ u^0 = 0 & on \quad \partial\Omega. \end{cases}$$

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[1] Mathematics Department, Faculty of Sciences, August 20th 1955 University, Skikda, Algeria.

Email address: karekchafia@gmail.com

[2] Mathematics Department, Laboratory of Physics Mathematics and Applications, ENS,P. O. Box 92, 16050 Kouba, Algiers, Amar Telidji University , Laghouat, Algeria.

Email address: t.abassi@lagh-univ.dz

[3] National School of Nanoscience and nanotechnology, Sidi Abdallah & Laboratory of Physics Mathematics and Applications, ENS,P. O. Box 92, 16050 Kouba, Algiers, Algeria.

Email address: a.ouldhammouda@nsnn.edu.dz

[4] Mathematics Department, Laboratory of Pure and Applied Mathematics, Amar Telidji University, Laghouat 03000, Algeria.

Email address: f.yazid@lagh-univ.dz