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EXISTENCE OF EXTREMAL SOLUTIONS FOR MIXED NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

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ABSTRACT. In this study, we establish the existence of minimal and maximal solutions for a class of mixed fractional differential equations involving the p-Laplacian operator, under appropriate boundary conditions. The nonlinear term is assumed to be continuous. By employing the method of upper and lower solutions together with a monotone iterative approach, we investigate the existence of extremal solutions. To demonstrate the effectiveness and applicability of the theoretical results, an illustrative example is included.

Keywords. Fractional differential equations, mixed fractional derivatives, upper and lower solutions method, monotone iterative technique, extremal solution, p-Laplacian
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1. Introduction

Fractional differential equations FDEs have emerged as powerful tools for modeling complex dynamics across a wide range of scientific and engineering fields. Their applications span areas such as viscoelasticity, physics, biology, chemistry, biotechnology, polymer rheology, and electromagnetic theory in complex media, as well as population dynamics and economics, among others. For additional context, see [6, 7, 8, 15, 16, 22] and the references therein. The theoretical framework of FDEs is well established and has been thoroughly developed in various monographs, including, [10, 13, 14, 17, 18]. In recent years, FDEs have attracted considerable attention owing to their effectiveness in modeling processes that exhibit memory and hereditary characteristics, which commonly arise in areas such as physics, biology, and control theory. Of particular importance are problems that involve nonlinearities and nonstandard boundary conditions, which have attracted considerable research attention. Numerous existence results have been obtained through methods such as fixed point theory, topological degree theory, and the Leray-Schauder principle (see, [2, 5, 19, 20]).

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Despite this progress, the application of the upper and lower solutions technique to FDEs involving the p-Laplacian operator remains relatively under-explored. Furthermore, fractional problems with nonlinear and nonlocal boundary conditions especially when formulated through functionals introduce additional analytical challenges that warrant deeper investigation. In this work, we consider the existence of extremal solutions, specifically minimal and maximal solutions, for the following mixed fractional BVP:

$$\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}\Big(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(\kappa))\Big) = f(\kappa,\mathfrak{h}(\kappa)),\ \kappa\in(0,1)\,, \\ \mathfrak{h}(0)=\mathfrak{h}'(0)=0, \\ a_1\mathfrak{h}(1)+a_2\ {}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(1)=\mathcal{L}(\mathfrak{h}), \end{cases} \tag{1.1}$$

where $\varphi_p(\mathfrak{h}) = |\mathfrak{h}|^{p-2}.\mathfrak{h}$, such that $p>1$. ${}^{R}\mathfrak{D}^{\varrho}_{0^+}$ and ${}^{C}\mathfrak{D}^{\beta}_{0^+}$ denote the Riemann-Liouville and Caputo fractional derivatives respectively, with $1<\varrho\le 2$ and $0<\beta\le 1$. $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and the boundary condition involves a nonlinear functional $\mathcal{L}:\mathcal{C}([0,1])\to\mathbb{R}$ which is also continuous and increasing. The coefficients $a_1,a_2\in\mathbb{R}^+$.

Our methodology relies on the classical technique of upper and lower solutions. By imposing appropriate conditions on the nonlinear term f, we demonstrate the existence of extremal (minimal and maximal) solutions for problem (1.1). To support the theoretical findings, a concrete example is provided.

The method of upper and lower solutions is a well-established and versatile tool for proving the existence of solutions to various boundary value problems BVPs, particularly those with nonlinear, multi-point, or nonlocal boundary conditions. In this work, we extend and generalize several cited results by combining the Riemann-Liouville and Caputo fractional derivatives within the p-Laplacian framework. Our operator $\mathcal{L}$ is designed to incorporate a broad class of boundary conditions, including classical, multi-point, integral, and weighted integral conditions, thereby significantly extending the scope of existing studies on integer-order differential equations (see [3, 4, 11, 12, 21]).

The structure of the paper is as follows. Section 2 presents preliminary definitions and results from fractional calculus. In Section 3, we state and show the main existence results. Section 4 is devoted to an example that demonstrates the usefulness and effectiveness of our theoretical findings.

2. Preliminaries

In this section, we provide the foundational definitions and preliminary results necessary for the development of our main results. We first introduce fundamental notions from fractional calculus, then define the concepts of lower and upper solutions, and finally present a weak comparison principle relevant to our study.

Definition 2.1. [17] The Riemann-Liouville fractional integral operator of order $\varrho>0$ corresponding to a function $\vartheta:(0,+\infty)\longrightarrow\mathbb{R}$ is expressed as

$$\mathfrak{I}^{\varrho}_{0^+}\vartheta(\kappa) = \frac{1}{\Gamma(\varrho)}\int_0^{\kappa}(\kappa-s)^{\varrho-1}\vartheta(s)\,ds,$$

Definition 2.2. [17] The Riemann Liouville fractional derivative operator of order $\varrho > 0$ of a continuous function $\vartheta : (0, +\infty) \longrightarrow \mathbb{R}$ is formulated as

$$ {}^{R}\mathfrak{D}^{\varrho}_{0^+}\vartheta(\kappa) = \frac{1}{\Gamma(n-\varrho)}\Big(\frac{d}{d\kappa}\Big)^{n}\int_0^{\kappa}(\kappa - s)^{n-\varrho-1}\vartheta(s)\,ds, \quad n = [\varrho] + 1. $$

Lemma 2.3. [17] *Let $\varrho > 0$. Then*

$$ \mathfrak{I}^{\varrho}_{0^+}\,{}^{R}\mathfrak{D}^{\varrho}_{0^+}\vartheta(\kappa) = \vartheta(\kappa) + d_0\kappa^{\varrho-1} + d_1\kappa^{\varrho-2} + \cdots + d_{n-1}\kappa^{\varrho-n}, $$

where $d_j \in \mathbb{R}$; $j = 0, 1, ..., n-1$ with $n - 1 < \varrho \leq n$.

Definition 2.4. [10] The Caputo's fractional derivative operator of order $\varrho > 0$ of a continuous function $\vartheta : (0, +\infty) \longrightarrow \mathbb{R}$ is formulated as

$$ {}^{C}\mathfrak{D}^{\varrho}_{0^+}\vartheta(\kappa) = \frac{1}{\Gamma(n-\varrho)}\int_0^{\kappa}(\kappa - s)^{n-\varrho-1}\vartheta^{(n)}(s)\,ds. $$

Lemma 2.5. [10] *Let $\varrho > 0$, then*

$$ \mathfrak{I}^{\varrho}_{0^+}\,{}^{C}\mathfrak{D}^{\varrho}_{0^+}\vartheta(\kappa) = \vartheta(\kappa) - \vartheta(0) - \vartheta'(0)\kappa - \frac{\vartheta^{(2)}(0)}{2!}\kappa^2 - \cdots - \frac{\vartheta^{(n-1)}(0)}{(n-1)!}\kappa^{n-1}, $$

where $n - 1 < \varrho \leq n$.

- We now consider the following BVP:

$$ \begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(\kappa))) = -\mathcal{W}(\kappa, \mathfrak{h}(\kappa)),\ \kappa \in (0,1)\,, \\ \mathfrak{h}(0) = \mathfrak{h}'(0) = 0, \\ \mathfrak{h}(1) + a\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(1) = \lambda, \end{cases} \tag{2.1} $$

where $\mathcal{W} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is assumed to be both continuous and strictly increasing in its second argument, $a > 0$, and $\lambda \in \mathbb{R}$.

Lemma 2.6. *(Weak comparison principle)*
Let $\mathfrak{h}_1,\mathfrak{h}_2 \in \mathcal{C}^1([0,1])$, $\varphi_p(\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_j) \in \mathcal{C}^1([0,1])$, $j = 1, 2$. Suppose that:

$$ \begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p(\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_1(\kappa))) + \mathcal{W}(\kappa, \mathfrak{h}_1(\kappa)) \leq -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_2(\kappa))) + \mathcal{W}(\kappa, \mathfrak{h}_2(\kappa)), \\ \mathfrak{h}_1(0) \leq \mathfrak{h}_2(0), \\ \mathfrak{h}_1'(0) \leq \mathfrak{h}_2'(0), \\ \mathfrak{h}_1(1) + a\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_1(1) \leq \mathfrak{h}_2(1) + a\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_2(1). \end{cases} \tag{2.2} $$

Therefore, it follows that $\mathfrak{h}_1(\kappa) \leq \mathfrak{h}_2(\kappa)$, $\forall\kappa \in [0,1]$.

Proof. Assume by contradiction that there exists $\kappa_0 \in [0,1]$ such that

$$ \mathfrak{R}(\kappa_0) := \mathfrak{h}_2(\kappa_0) - \mathfrak{h}_1(\kappa_0) = \min_{0\leq\kappa\leq\kappa_0}\mathfrak{R}(\kappa) < 0, \tag{2.3} $$

and that

$$ \mathfrak{R}(\kappa) > \mathfrak{R}(\kappa_0),\ \text{for all}\ \kappa \in (\kappa_0, 1]. \tag{2.4} $$

We distinguish the following three cases:
Case 1:
If $\kappa_0 = 0$, we obtain a contradiction

$$ 0 > \mathfrak{h}_2(0) - \mathfrak{h}_1(0) \geq 0. $$

Case 2: If $\kappa_0 = 1$, then

$$\begin{aligned} 0 > \mathfrak{h}_2(1) - \mathfrak{h}_1(1) &\geq -a\left({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(1) - {}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(1)\right) \\ &= \frac{-a}{\Gamma(1-\beta)} \int_0^1 (1-s)^{-\beta}\left[\mathfrak{h}_2'(s) - \mathfrak{h}_1'(s)\right] ds \\ &= \frac{-a}{\Gamma(1-\beta)} \int_0^1 (1-s)^{-\beta}\mathfrak{R}'(s)\, ds \geq 0. \end{aligned}$$

The final result is obtained by using the fact that $\mathfrak{R}$ is decreasing, this leads to another contradiction.

Case 3: If $\kappa_0 \in (0,1)$, we distinguish the following subcases:

(i) If ${}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{R}(\kappa_0) > 0$, then $\mathfrak{I}^{\beta}_{0+}\,{}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{R}(\kappa_0) > 0$, therefore by the use of lemma 2.5, we obtain: $\mathfrak{R}(\kappa_0) > \mathfrak{R}(0) \geq 0$, which contradicts the definition of κ_0 in (2.3).

(ii) If ${}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{R}(\kappa_0) < 0$, so $\mathfrak{I}^{\beta}_{0+}\,{}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{R}(\kappa_0) < 0$, thus $\mathfrak{R}(\kappa_0) < \mathfrak{R}(0)$ for all $\kappa_0 \in (0,1)$. Consequently $\dfrac{\mathfrak{R}(\kappa_0) - \mathfrak{R}(0)}{\kappa_0} < 0 \iff \mathfrak{R}'(0) < 0$, again contradicting the condition $\mathfrak{R}'(0) \geq 0$.

(iii) If ${}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{R}(\kappa_0) = 0$, so ${}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0) = {}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0)$ and then $\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0)) = \varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0))$. The strict monotonicity of φ_p leads to

$$ {}^R\mathfrak{D}^{\varrho}_{0+}[-\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0)) + \varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0))] = 0. $$

Here, we observe that

$$ -{}^R\mathfrak{D}^{\varrho}_{0+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0))) + \mathcal{W}(\kappa_0, \mathfrak{h}_2(\kappa)) + {}^R\mathfrak{D}^{\varrho}_{0+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0))) - \mathcal{W}(\kappa_0, \mathfrak{h}_1(\kappa)) \geq 0. $$

This indicates that

$$ -{}^R\mathfrak{D}^{\varrho}_{0+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0))) + {}^R\mathfrak{D}^{\varrho}_{0+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0))) \geq \mathcal{W}(\kappa_0, \mathfrak{h}_1(\kappa)) - \mathcal{W}(\kappa_0, \mathfrak{h}_2(\kappa)). $$

Using the monotonicity of $\mathcal{W}$ in its second parameter and the fact that $\mathfrak{h}_2(\kappa_0) < \mathfrak{h}_1(\kappa_0)$, we reach a contradiction as follows:

$$ -{}^R\mathfrak{D}^{\varrho}_{0+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_2(\kappa_0))) + {}^R\mathfrak{D}^{\varrho}_{a}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathfrak{h}_1(\kappa_0))) > 0. $$

□

Definition 2.7. A function $\mathcal{X}$ is called a lower solution of problem (2.1) if

(i) $\mathcal{X} \in \mathcal{C}^1([0,1])$, $\varphi_p({}^C\mathfrak{D}^{\beta}_{0+}\mathcal{X}) \in \mathcal{C}^1([0,1])$.

$$(ii)\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathcal{X}(\kappa))) \le -\mathcal{W}(\kappa,\mathcal{X}(\kappa)),\ \kappa\in(0,1)\,, \\ \mathcal{X}(0)\le 0, \\ \mathcal{X}'(0)\le 0, \\ \mathcal{X}(1)+a\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathcal{X}(1)\le\lambda. \end{cases}$$

Definition 2.8. A function $\mathcal{Z}$ is said to be an upper solution of problem (2.1) if

(i) $\mathcal{Z}\in\mathcal{C}^1([0,1])$, $\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathcal{Z})\in\mathcal{C}^1([0,1])$.

$$(ii)\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathcal{Z}(\kappa))) \ge -\mathcal{W}(\kappa,\mathcal{Z}(\kappa)),\ \kappa\in(0,1)\,, \\ \mathcal{Z}(0)\ge 0, \\ \mathcal{Z}'(0)\ge 0, \\ \mathcal{Z}(1)+a\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathcal{Z}(1)\ge\lambda. \end{cases}$$

Theorem 2.9. *Assume that $\mathcal{X}$ and $\mathcal{Z}$ are lower and upper solutions of (2.1), respectively, and that $\mathcal{X}(\kappa)\le\mathcal{Z}(\kappa)$, $\forall\kappa\in[0,1]$. Then problem (2.1) admits a unique solution $\mathfrak{h}\in\mathcal{C}^1([0,1])$ with $\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h})\in\mathcal{C}^1([0,1])$ such that*

$$\mathcal{X}(\kappa)\le\mathfrak{h}(\kappa)\le\mathcal{Z}(\kappa),\ \forall\kappa\in[0,1]\,.$$

Proof. The proof of Theorem 3.1 in [1] is based on constructing a modified boundary value problem and defining a corresponding integral operator whose fixed points represent solutions. The operator is shown to be well-defined, bounded, and continuous in a Banach space, allowing the application of Brouwer's fixed point theorem. A contradiction argument is then used to ensure the solution remains within the prescribed bounds. We follow a similar construction by defining an appropriate compact operator in a function space and proving that it satisfies the conditions for the Schauder fixed point theorem. Uniqueness follows from a contraction argument as established in Lemma 2.6. These steps mirror the approach in [1] and justify the statement in the original proof.

□

3. Main results

The purpose of this section is to formulate and rigorously show our key results.

Definition 3.1. A function $\underline{\mathfrak{h}}$ is said to be a lower solution of problem (1.1) if:

(i) $\underline{\mathfrak{h}}\in\mathcal{C}^1([0,1])$, $\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}})\in\mathcal{C}^1([0,1])$.

$$(ii)\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}(\kappa))) \le f(\kappa,\underline{\mathfrak{h}}(\kappa)),\ \kappa\in(0,1)\,, \\ \underline{\mathfrak{h}}(0)\le 0, \\ \underline{\mathfrak{h}}'(0)\le 0, \\ a_1\underline{\mathfrak{h}}(1)+a_2\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}(1)\le\mathcal{L}(\underline{\mathfrak{h}}). \end{cases}$$

Definition 3.2. A function $\overline{\mathfrak{h}}$ is said to be an upper solution of problem (1.1) if:

(i) $\overline{\mathfrak{h}}\in\mathcal{C}^1([0,1])$, $\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}})\in\mathcal{C}^1([0,1])$.

$$(ii)\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}(\kappa))) \ge f(\kappa,\overline{\mathfrak{h}}(\kappa)),\ \kappa\in(0,1)\,, \\ \overline{\mathfrak{h}}(0)\ge 0, \\ \overline{\mathfrak{h}}'(0)\ge 0, \\ a_1\overline{\mathfrak{h}}(1)+a_2\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}(1)\ge\mathcal{L}(\overline{\mathfrak{h}}). \end{cases}$$

• Due to the nonlinearity of f, we require the ensuing hypothesis to hold:
($\mathcal{G}$) There exists a continuous and strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that for all $\kappa \in [0,1]$ the mapping $(\kappa, \mathfrak{h}(\kappa)) \longmapsto f(\kappa, \mathfrak{h}(\kappa)) + g(\mathfrak{h}(\kappa))$ is increasing.
The principal finding of this study is encapsulated in the following theorem.

Theorem 3.3. *Assume that hypothesis ($\mathcal{G}$) is fulfilled and $\underline{\mathfrak{h}}$ and $\overline{\mathfrak{h}}$ are respectively lower and upper solutions of problem (1.1) such that $\underline{\mathfrak{h}} \leq \overline{\mathfrak{h}}$ on $[0,1]$. Hence problem (1.1) admits a minimal solution $\mathfrak{h}_{\min}$ and maximal solution $\mathfrak{h}_{\max}$ such that for any solution $\mathfrak{h}$ of (1.1) satisfying $\underline{\mathfrak{h}} \leq \mathfrak{h} \leq \overline{\mathfrak{h}}$ on $[0,1]$, the following expression holds:*

$$\underline{\mathfrak{h}} \leq \mathfrak{h}_{\min} \leq \mathfrak{h} \leq \mathfrak{h}_{\max} \leq \overline{\mathfrak{h}} \quad on \quad [0,1]\,.$$

To prove this theorem, we begin by establishing an auxiliary result in the form of the following lemma.
Let $\underline{\mathfrak{R}}$ and $\overline{\mathfrak{R}} \in \mathcal{C}^1([0,1])$ be two functions such that
(*i*) $\varphi_p\left({}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{R}}\right)$, $\varphi_p\left({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{R}}\right) \in \mathcal{C}^1([0,1])$,
(*ii*) $\underline{\mathfrak{h}} \leq \underline{\mathfrak{R}} \leq \overline{\mathfrak{R}} \leq \overline{\mathfrak{h}}$ on $[0,1]$.
The focus is on the following class of problems:

$$\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(\kappa))) + h(\mathfrak{h}(\kappa)) = f(\kappa, \overline{\mathfrak{R}}(\kappa)) + g(\overline{\mathfrak{R}}(\kappa)),\ \kappa \in (0,1)\,, \\ \mathfrak{h}(0) = 0, \\ \mathfrak{h}'(0) = 0, \\ a_1\mathfrak{h}(1) + a_2{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(1) = \mathcal{L}(\overline{\mathfrak{R}}), \end{cases} \tag{3.1}$$

and

$$\begin{cases} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(\kappa))) + h(\mathfrak{h}(\kappa)) = f(\kappa, \underline{\mathfrak{R}}(\kappa)) + g(\underline{\mathfrak{R}}(\kappa)),\ \kappa \in (0,1)\,, \\ \mathfrak{h}(0) = 0, \\ \mathfrak{h}'(0) = 0, \\ a_1\mathfrak{h}(1) + a_2{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}(1) = \mathcal{L}(\underline{\mathfrak{R}}). \end{cases} \tag{3.2}$$

Lemma 3.4. *Let $\underline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}$ be a lower and upper solutions respectively of problem (1.1), and assume that ($\mathcal{G}$) holds. Then, the auxiliary problems (3.1) and (3.2) admit unique solution $\widehat{\mathfrak{h}}$ and $\widetilde{\mathfrak{h}}$, respectively, such that*

$$\underline{\mathfrak{h}} \leq \underline{\mathfrak{R}} \leq \widetilde{\mathfrak{h}} \leq \widehat{\mathfrak{h}} \leq \overline{\mathfrak{R}} \leq \overline{\mathfrak{h}} \quad on \quad [0,1]\,.$$

Proof. The demonstration is carried out step by step for clarity.
First Step: $\underline{\mathfrak{R}}$ is a lower solution of (3.1).
Let $\kappa \in (0,1)$, we have

$$\begin{aligned} -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{R}}(\kappa))) + g(\underline{\mathfrak{R}}(\kappa)) \quad &\leq \quad f(\kappa, \underline{\mathfrak{R}}(\kappa)) + g(\underline{\mathfrak{R}}(\kappa)), \\ &\leq \quad f(\kappa, \overline{\mathfrak{R}}(\kappa)) + g(\overline{\mathfrak{R}}(\kappa)). \end{aligned}$$

It follows that

$$\forall \kappa \in (0,1)\colon -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{R}}(\kappa))) + g(\underline{\mathfrak{R}}(\kappa)) \leq f(\kappa, \overline{\mathfrak{R}}(\kappa)) + g(\overline{\mathfrak{R}}(\kappa)). \quad (3.3)$$

Furthermore

$$\underline{\mathfrak{R}}(0) \ \leq \ 0, \quad (3.4)$$

with

$$\underline{\mathfrak{R}}'(0) \ \leq \ 0, \quad (3.5)$$

and we have

$$a_1\underline{\mathfrak{R}}(1) + a_2\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{R}}(1) \leq \mathcal{L}(\overline{\mathfrak{R}}). \quad (3.6)$$

By (3.3), (3.4), (3.5) and (3.6), we conclude that $\underline{\mathfrak{R}}$ is a lower solution of (3.1).
Second Step: $\overline{\mathfrak{R}}$ is an upper solution of problem (3.1).
Let $\kappa \in (0,1)$, we have

$$-{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{R}}(\kappa))) + g(\overline{\mathfrak{R}}(\kappa)) \geq f(\kappa, \overline{\mathfrak{R}}(\kappa)) + g(\overline{\mathfrak{R}}(\kappa)).$$

We obtain

$$\forall \kappa \in (0,1) : -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{R}}(\kappa))) + g(\overline{\mathfrak{R}}(\kappa)) \geq f(\kappa, \overline{\mathfrak{R}}(\kappa)) + g(\overline{\mathfrak{R}}(\kappa)). \quad (3.7)$$

Moreover, we have

$$\overline{\mathfrak{R}}(0) \geq 0, \quad (3.8)$$

with

$$\underline{\mathfrak{R}}'(0) \ \geq \ 0, \quad (3.9)$$

and

$$a_1\overline{\mathfrak{R}}(1) + a_2\,{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{R}}(1) \geq \mathcal{L}(\overline{\mathfrak{R}}). \quad (3.10)$$

Then, by (3.7), (3.8), (3.9) and (3.10), it follows that $\overline{\mathfrak{R}}$ is an upper solution of (3.1).

From **First Step** and **Second Step**, and given that the mapping $(\kappa, \mathfrak{h}(\kappa)) \longmapsto f(\kappa, \mathfrak{h}(\kappa)) + g(\mathfrak{h}(\kappa))$ is continuous, bounded, and strictly increasing, we apply Theorem 2.9, to conclude that problem (3.1) admits a unique solution $\widehat{\mathfrak{h}}$ satisfying

$$\underline{\mathfrak{R}} \leq \widehat{\mathfrak{h}} \leq \overline{\mathfrak{R}}.$$

Following the same approach, one may deduce the existence and uniqueness of a solution to problem (3.2), denoted by $\widetilde{\mathfrak{h}}$ with

$$\underline{\mathfrak{R}} \leq \widetilde{\mathfrak{h}} \leq \overline{\mathfrak{R}}.$$

Ultimately, the result follows by employing a proof technique analogous to that of Lemma 2.6, We infer that $\widetilde{\mathfrak{h}} \leq \widehat{\mathfrak{h}}$ on $[0,1]$. $\square$

Proof. We establish Theorem 3.3 by means of the following successive steps.

We have $\underline{\mathfrak{h}}_0 = \underline{\mathfrak{h}}$, $\overline{\mathfrak{h}}_0 = \overline{\mathfrak{h}}$ and let's introduce the sequences $(\underline{\mathfrak{h}}_n)_{n\in\mathbb{N}^*}$ and $(\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}^*}$ by

$$\begin{cases} -{}^R\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}_{n+1}(\kappa))) + g(\underline{\mathfrak{h}}_{n+1}(\kappa)) = f(\kappa, \underline{\mathfrak{h}}_n(\kappa)) + g(\underline{\mathfrak{h}}_n(\kappa)),\ \kappa \in (0,1)\,, \\ \underline{\mathfrak{h}}_{n+1}(0) = 0, \\ \underline{\mathfrak{h}}'_{n+1}(0) = 0, \\ a_1\underline{\mathfrak{h}}_{n+1}(1) + a_2{}^C\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}_{n+1}(1) = \mathcal{L}(\underline{\mathfrak{h}}_n), \end{cases} \tag{3.11}$$

and

$$\begin{cases} -{}^R\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa))) + g(\overline{\mathfrak{h}}_{n+1}(\kappa)) = f(\kappa, \overline{\mathfrak{h}}_n(\kappa)) + g(\overline{\mathfrak{h}}_n(\kappa)),\ \kappa \in (0,1)\,, \\ \overline{\mathfrak{h}}_{n+1}(0) = 0, \\ \overline{\mathfrak{h}}'_{n+1}(0) = 0, \\ a_1\overline{\mathfrak{h}}_{n+1}(1) + a_2{}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(1) = \mathcal{L}(\overline{\mathfrak{h}}_n). \end{cases} \tag{3.12}$$

Step 1. We begin by showing that

$$\forall n \in \mathbb{N} : \underline{\mathfrak{h}} \le \underline{\mathfrak{h}}_n \le \underline{\mathfrak{h}}_{n+1} \le \overline{\mathfrak{h}}_{n+1} \le \overline{\mathfrak{h}}_n \le \overline{\mathfrak{h}},\ \text{on}\ \ [0,1]\,.$$

(i) Firstly, for $n = 0$:

$$\begin{cases} -{}^R\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}_1(\kappa))) + g(\underline{\mathfrak{h}}_1(\kappa)) = f(\kappa, \underline{\mathfrak{h}}_0(\kappa)) + g(\underline{\mathfrak{h}}_0(\kappa)), \kappa \in (0,1)\,, \\ \underline{\mathfrak{h}}_1(0) = 0, \\ \underline{\mathfrak{h}}'_1(0) = 0, \\ a_1\underline{\mathfrak{h}}_1(1) + a_2{}^C\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}_1(1) = \mathcal{L}(\underline{\mathfrak{h}}_0), \end{cases} \tag{3.13}$$

and

$$\begin{cases} -{}^R\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_1(\kappa))) + g(\overline{\mathfrak{h}}_1(\kappa)) = f(\kappa, \overline{\mathfrak{h}}_0(\kappa)) + g(\overline{\mathfrak{h}}_0(\kappa)),\ \kappa \in (0,1)\,, \\ \overline{\mathfrak{h}}_1(0) = 0, \\ \overline{\mathfrak{h}}'_1(0) = 0, \\ a_1\overline{\mathfrak{h}}_1(1) + a_2{}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_1(1) = \mathcal{L}(\overline{\mathfrak{h}}_0). \end{cases} \tag{3.14}$$

Since $\underline{\mathfrak{h}}$, $\overline{\mathfrak{h}}$ are a lower and upper solutions for problem (1.1), respectively, an application of Lemma 3.4 yields

$$\underline{\mathfrak{h}} \le \underline{\mathfrak{h}}_0 \le \underline{\mathfrak{h}}_1 \le \overline{\mathfrak{h}}_1 \le \overline{\mathfrak{h}}_0 \le \overline{\mathfrak{h}},\ \text{on}\ \ [0,1]\,.$$

(ii) Assume $n > 1$ fixed, then

$$\underline{\mathfrak{h}} \le \underline{\mathfrak{h}}_{n-1} \le \underline{\mathfrak{h}}_n \le \overline{\mathfrak{h}}_n \le \overline{\mathfrak{h}}_{n-1} \le \overline{\mathfrak{h}},\ \text{on}\ \ [0,1]\,,$$

and we demonstrate that

$$\underline{\mathfrak{h}} \le \underline{\mathfrak{h}}_n \le \underline{\mathfrak{h}}_{n+1} \le \overline{\mathfrak{h}}_{n+1} \le \overline{\mathfrak{h}}_n \le \overline{\mathfrak{h}},\ \text{on}\ \ [0,1]\,.$$

Let $\kappa \in (0,1)$, we have

$$-\,{}^R\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa))) + g(\overline{\mathfrak{h}}_{n+1}(\kappa)) = f(\kappa, \overline{\mathfrak{h}}_n(\kappa)) + g(\overline{\mathfrak{h}}_n(\kappa)), \tag{3.15}$$

such as $\overline{\mathfrak{h}}_{n-1} \geq \overline{\mathfrak{h}}_n$, and by employing the hypothesis $(\mathcal{G})$, we get

$$f(\kappa, \overline{\mathfrak{h}}_{n-1}(\kappa)) + g(\overline{\mathfrak{h}}_{n-1}(\kappa)) \geq f(\kappa, \overline{\mathfrak{h}}_n(\kappa)) + g(\overline{\mathfrak{h}}_n(\kappa)). \tag{3.16}$$

Therefore, combining (3.15) and (3.16), we deduce that

$$\forall \kappa \in (0,1): \; -{}^{R}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa))) \geq f(\kappa, \overline{\mathfrak{h}}_n(\kappa)).$$

In addition, we observe that

$$\begin{array}{rcl} \overline{\mathfrak{h}}_n(0) & = & 0, \\ & \geq & 0, \end{array} \tag{3.17}$$

with

$$\begin{array}{rcl} \overline{\mathfrak{h}}'_n(0) & = & 0, \\ & \geq & 0, \end{array} \tag{3.18}$$

and

$$\begin{array}{rcl} a_1\overline{\mathfrak{h}}_n(1) + a_2{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(1) & = & \mathcal{L}(\overline{\mathfrak{h}}_{n-1}), \\ & \geq & \mathcal{L}(\overline{\mathfrak{h}}_n). \end{array} \tag{3.19}$$

From (3.16), (3.17), (3.18) and (3.19), it follows that $\overline{\mathfrak{h}}_n$ is an upper solution of problem (1.1). Similarly, using the same arguments, one can verify that $\underline{\mathfrak{h}}_n$ is a lower solution of (1.1). Therefore, by invoking Lemma 3.4, we conclude that there exist unique solutions $\underline{\mathfrak{h}}_{n+1}$ and $\overline{\mathfrak{h}}_{n+1}$ to problems (3.11) and (3.12) respectively, such that

$$\underline{\mathfrak{h}} \leq \underline{\mathfrak{h}}_n \leq \underline{\mathfrak{h}}_{n+1} \leq \overline{\mathfrak{h}}_{n+1} \leq \overline{\mathfrak{h}}_n \leq \overline{\mathfrak{h}}, \text{ on } [0,1].$$

Consequently, we obtain

$$\forall n \in \mathbb{N}: \; \underline{\mathfrak{h}} \leq \underline{\mathfrak{h}}_n \leq \underline{\mathfrak{h}}_{n+1} \leq \overline{\mathfrak{h}}_{n+1} \leq \overline{\mathfrak{h}}_n \leq \overline{\mathfrak{h}}, \text{ on } [0,1].$$

Step 2. Let $K_1 > 0$ be a constant independent of the index $n \in \mathbb{N}$, such that:

$$\|{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n\|_\infty \leq K_1, \forall n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ and $\kappa \in [0,1]$, by the lemma 2.3, we have:

$$-\mathfrak{I}^{\varrho}_{0^+}\mathfrak{D}^{\varrho}_{0^+}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa))) = \mathfrak{I}^{\varrho}_0\Big[f(\kappa, \overline{\mathfrak{h}}_n(\kappa)) + g(\overline{\mathfrak{h}}_n(\kappa)) - g(\overline{\mathfrak{h}}_{n+1}(\kappa))\Big],$$

so

$$-\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa)) = d_1\kappa^{\varrho-1} + d_2\kappa^{\varrho-2} + \frac{1}{\Gamma(\varrho)}\int_0^\kappa (\kappa-s)^{\varrho-1}\Big[f(s, \overline{\mathfrak{h}}_n(s)) + g(\overline{\mathfrak{h}}_n(s)) - g(\overline{\mathfrak{h}}_{n+1}(s))\Big]ds.$$

Since ${}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(0) = 0$ implies that $d_2 = 0$.

Therefore

$$\begin{aligned}\left|\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa))\right| &= \left|d_1\kappa^{\varrho-1}+\frac{1}{\Gamma(\varrho)}\int\limits_0^{\kappa}(\kappa-s)^{\varrho-1}\Big[f(s,\overline{\mathfrak{h}}_n(s))+g(\overline{\mathfrak{h}}_n(s))-g(\overline{\mathfrak{h}}_{n+1}(s))\Big]ds\right| \\ &\le |d_1|+\frac{\mathfrak{M}_1(f)+2\mathfrak{M}_2(g)}{\Gamma(\varrho)}\left|\int\limits_0^{\kappa}(\kappa-s)^{\varrho-1}ds\right| \\ &\le |d_1|+\frac{\mathfrak{M}_1(f)+2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)},\end{aligned}$$

where

$$\mathfrak{M}_1(f)=\max\left\{|f(\kappa,\mathfrak{h})|:\ \kappa\in[0,1]\,,\ \underline{\mathfrak{h}}\le\mathfrak{h}\le\overline{\mathfrak{h}}\right\},$$

and

$$\mathfrak{M}_2(g)=\max\left\{|g(\mathfrak{h})|:\ \underline{\mathfrak{h}}\le\mathfrak{h}\le\overline{\mathfrak{h}}\right\}.$$

Accordingly, by setting

$$K_1^{p-1}=|d_1|+\frac{\mathfrak{M}_1(f)+2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)},$$

we obtain

$$\|{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}\|_\infty\le K_1.$$

In addition, we get

$$\begin{aligned}\|\overline{\mathfrak{h}}_{n+1}\|_\infty &= \|\mathfrak{I}^{\beta}_{0^+}{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}\|_\infty \\ &\le \frac{\|{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}\|_\infty}{\Gamma(\beta+1)} \\ &\le \frac{K_1}{\Gamma(\beta+1)}.\end{aligned}$$

The first result follows directly from Lemma 2.5 with condition $\overline{\mathfrak{h}}_{n+1}(0)=0$, and the second result follows from Lemma 2.1 part (a) in [10]. Consequently, the two sequences $({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ and $(\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ are uniformly bounded.

Step 3. Let $K_2>0$ be a constant independent of the index $n\in\mathbb{N}$, with:

$$\|{}^{C}\mathfrak{D}^{\beta}_{0^+}\underline{\mathfrak{h}}_n\|_\infty\le K_2,\forall n\in\mathbb{N}.$$

Since the proof follows the same arguments as in **Step 2**, it is omitted here.

Step 4. $({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

For this, let $\epsilon>0$ such that

$$\eta=\frac{1}{2}\min\left\{\left(\frac{\epsilon}{2|d_1|}\right)^{\frac{1}{\varrho-1}},\left(\frac{\epsilon\Gamma(\varrho+1)}{2\left[\mathfrak{M}_1(f)+2\mathfrak{M}_2(g)\right]}\right)^{\frac{1}{\varrho}}\right\}.$$

Then, for any $\kappa_1,\kappa_2\in[0,1]$ satisfying $\kappa_1<\kappa_2$ with $\kappa_2-\kappa_1<\eta$, it follows that

$$\begin{aligned}
&\left|\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_2)) - \varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_1))\right| \\
= \;& \Bigg| d_1\kappa_2^{\varrho-1} - d_1\kappa_1^{\varrho-1} + \frac{1}{\Gamma(\varrho)}\int_0^{\kappa_2}(\kappa_2-s)^{\varrho-1}\left[f(s,\overline{\mathfrak{h}}_n(s)) + g(\overline{\mathfrak{h}}_n(s)) - g(\overline{\mathfrak{h}}_{n+1}(s))\right]\, ds \\
& -\frac{1}{\Gamma(\varrho)}\int_0^{\kappa_1}(\kappa_1-s)^{\varrho-1}\left[f(s,\overline{\mathfrak{h}}_n(s)) + g(\overline{\mathfrak{h}}_n(s)) - g(\overline{\mathfrak{h}}_{n+1}(s))\right]\, ds\Bigg| \\
\leq \;& |d_1|(\kappa_2^{\varrho-1} - \kappa_1^{\varrho-1}) + \Bigg|\int_0^{\kappa_1}\frac{\left[(\kappa_2-s)^{\varrho-1} - (\kappa_1-s)^{\varrho-1}\right]}{\Gamma(\varrho)}\left[f(s,\overline{\mathfrak{h}}_n(s)) + g(\overline{\mathfrak{h}}_n(s)) - g(\overline{\mathfrak{h}}_{n+1}(s))\right]\, ds \\
& +\frac{1}{\Gamma(\varrho)}\int_{\kappa_1}^{\kappa_2}(\kappa_2-s)^{\varrho-1}\left[f(s,\overline{\mathfrak{h}}_n(s)) + g(\overline{\mathfrak{h}}_n(s)) - g(\overline{\mathfrak{h}}_{n+1}(s))\right]\, ds\Bigg| \\
\leq \;& |d_1|(\kappa_2^{\varrho-1} - \kappa_1^{\varrho-1}) + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho)}\int_0^{\kappa_1}\left[(\kappa_2-s)^{\varrho-1} - (\kappa_1-s)^{\varrho-1}\right]\, ds \\
& +\frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho)}\int_{\kappa_1}^{\kappa_2}(\kappa_2-s)^{\varrho-1}\, ds \\
\leq \;& |d_1|(\kappa_2^{\varrho-1} - \kappa_1^{\varrho-1}) + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)}(\kappa_2^{\varrho} - \kappa_1^{\varrho}).
\end{aligned}$$

To estimate the expressions $\kappa_2^{\varrho-1} - \kappa_1^{\varrho-1}$ and $\kappa_2^{\varrho} - \kappa_1^{\varrho}$, we adopt an approach analogous to that presented in [2]. We have

Case 1. $\eta \leq \kappa_1 < \kappa_2 < 1$:

$$\begin{aligned}
\left|\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_2)) - \varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_1))\right| &\leq |d_1|\frac{\varrho-1}{\eta^{2-\varrho}}(\kappa_2-\kappa_1) + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)}\varrho(\kappa_2-\kappa_1), \\
&\leq |d_1|(\varrho-1)\eta^{\varrho-1} + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho)}\eta \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Case 2. $0 \leq \kappa_1 < \eta$, $\kappa_2 < 2\eta$:

$$\begin{aligned}
\left|\varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_2)) - \varphi_p({}^C\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1}(\kappa_1))\right| &\leq |d_1|\kappa_2^{\varrho-1} + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)}\kappa_2^{\varrho} \\
&< |d_1|(2\eta)^{\varrho-1} + \frac{\mathfrak{M}_1(f) + 2\mathfrak{M}_2(g)}{\Gamma(\varrho+1)}(2\eta)^{\varrho} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, the sequence $({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$. Such us φ_p^{-1} is the homeomorphisms from $\mathbb{R}$ into $\mathbb{R}$, we deduce that

$$\left|{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_2)-{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_1)\right|=\left|\varphi_p^{-1}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_2)))-\varphi_p^{-1}(\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_1)))\right|,$$

and, we have also

$$\left|\overline{\mathfrak{h}}_n(\kappa_2)-\overline{\mathfrak{h}}_n(\kappa_1)\right|=\left|\mathfrak{I}^{\beta}_{0^+}{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_2)-\mathfrak{I}^{\beta}_{0^+}{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n(\kappa_1)\right|.$$

Consequently $({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ and $(\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ are equicontinuous on $[0,1]$.

In view of the Arzelà-Ascoli theorem, one can extract a convergent subsequence $(\overline{\mathfrak{h}}_{n_j})_{n_j\in\mathbb{N}}$ in the space $\mathcal{C}^1([0,1])$.

Set

$$\mathfrak{h}:=\lim_{n_j\to+\infty}\overline{\mathfrak{h}}_{n_j},$$

therefore

$${}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}=\lim_{n_j\to+\infty}{}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n_j}.$$

By using **Step 1**, the sequence $(\overline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below. Hence, it converges pointwise to a limit, denoted by $\mathfrak{h}_{\max}$.

Hence, $\mathfrak{h}=\mathfrak{h}_{\max}$ and the whole sequence converges to $\mathfrak{h}_{\max}$ in $\mathcal{C}^1([0,1])$.

Let $\kappa\in(0,1)$, we have

$$-\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\overline{\mathfrak{h}}_{n+1})(\kappa)=d_1\kappa^{\varrho-1}+\frac{1}{\Gamma(\varrho)}\int_0^{\kappa}(\kappa-s)^{\varrho-1}(f(s,\overline{\mathfrak{h}}_n(s))+g(\overline{\mathfrak{h}}_n(s))-g(\overline{\mathfrak{h}}_{n+1}(s)))ds,$$

Letting $n\longrightarrow+\infty$, we deduce that

$$f(s,\overline{\mathfrak{h}}_n(s))+g(\overline{\mathfrak{h}}_n(s))-g(\overline{\mathfrak{h}}_{n+1}(s))\to f(s,\mathfrak{h}_{\max}(s)).$$

Also, we have

$$\exists Q>0;\forall n\in\mathbb{N};\forall s\in[0,1]:\left|f(s,\overline{\mathfrak{h}}_n(s))+g(\overline{\mathfrak{h}}_n(s))-g(\overline{\mathfrak{h}}_{n+1}(s))\right|\le Q.$$

It follows from the dominated convergence theorem of Lebesgue that

$$-\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_{\max}(\kappa))=d_1\kappa^{\varrho-1}+\frac{1}{\Gamma(\varrho)}\int_0^{\kappa}(\kappa-s)^{\varrho-1}f(s,\mathfrak{h}_{\max}(s))ds.$$

So, we get $\forall\kappa\in(0,1)$

$$-\varphi_p({}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_{\max}(\kappa))=d_1\kappa^{\varrho-1}+\mathfrak{I}^{\varrho}_{0^+}f(\kappa,\mathfrak{h}_{\max}(\kappa)),\tag{3.20}$$

Also, we have

$$\mathfrak{h}_{\max}(0)=0,\tag{3.21}$$

$${\mathfrak{h}_{\max}}'(0)=0,\tag{3.22}$$

and

$$a_1\mathfrak{h}_{\max}(1)+a_2{}^{C}\mathfrak{D}^{\beta}_{0^+}\mathfrak{h}_{\max}(1)=\mathcal{L}(\mathfrak{h}_{\max}).\tag{3.23}$$

From (3.20), (3.21), (3.22) and (3.23), it follows that $\mathfrak{h}_{\max}$ is a solution of the following problem

$$\begin{cases} -{}^{R}\mathfrak{D}_{0^+}^{\varrho}(\varphi_p({}^{C}\mathfrak{D}_{0^+}^{\beta}\mathfrak{h}_{\max})) = f(\kappa, \mathfrak{h}_{\max}),\ \kappa \in [0,1]\,, \\ \mathfrak{h}_{\max}(0) = 0, \\ {\mathfrak{h}_{\max}}'(0) = 0, \\ a_1\mathfrak{h}_{\max}(1) + a_2{}^{C}\mathfrak{D}_{0^+}^{\beta}\mathfrak{h}_{\max}(1) = \mathcal{L}(\mathfrak{h}_{\max}), \end{cases} \tag{3.24}$$

We now prove that if $\mathfrak{h}$ is the another solution of (1.1) verifying $\underline{\mathfrak{h}} \leq \mathfrak{h} \leq \overline{\mathfrak{h}}$ on $[0,1]$, then it follows that $\mathfrak{h} \leq \mathfrak{h}_{\max}$ on $[0,1]$.

Since $\mathfrak{h}$ is a lower solution of (1.1), then with **Step 1,** we get

$$\forall n \in \mathbb{N}\colon \mathfrak{h} \leq \overline{\mathfrak{h}}_n,$$

Taking the limit as $n \to +\infty$, we obtain

$$\mathfrak{h} \leq \lim_{n\to+\infty} \overline{\mathfrak{h}}_n = \mathfrak{h}_{\max},$$

which confirms that $\mathfrak{h}_{\max}$ is the maximal solution of (1.1).

Step 5. Similarly, one can show that the sequence $(\underline{\mathfrak{h}}_n)_{n\in\mathbb{N}}$ converges to a minimal solution $\mathfrak{h}_{\min}$ of (1.1). This concludes the proof. □

4. Application

Now, we employ the previous result to establish the problem described below.

$$\begin{cases} -{}^{R}\mathfrak{D}_{0^+}^{3/2}\Big(\varphi_p({}^{C}\mathfrak{D}_{0^+}^{1/2}\mathfrak{h}(\kappa))\Big) = \sqrt{\sigma\,|\mathfrak{h}(\kappa)|} - \mathfrak{h}(\kappa) + \sigma,\ \kappa \in (0,1)\,, \\ \mathfrak{h}(0) = \mathfrak{h}'(0) = 0, \\ \mathfrak{h}(1) + {}^{C}\mathfrak{D}_{0^+}^{1/2}\mathfrak{h}(1) = \int_0^1 \varpi(s)\mathfrak{h}(s)ds. \end{cases} \tag{4.1}$$

σ is a positive real number, and ϖ is real positive function.

Theorem 4.1. *Assume that* $\int_0^1 \varpi(s)\, ds \leq 1$, *then problem* (4.1) *admits a maximal solution* $\mathfrak{h}_{\max}$ *and minimal solution* $\mathfrak{h}_{\min}$.

Proof. Set $(\underline{\mathfrak{h}}, \overline{\mathfrak{h}}) = (0, \sigma)$. Firstly, for $\kappa \in [0,1]$, we have

$$\begin{cases} 0 = -{}^{R}\mathfrak{D}_{0^+}^{3/2}(\varphi_p({}^{C}\mathfrak{D}_{0^+}^{1/2}\underline{\mathfrak{h}}(\kappa))) \leq \sigma, \quad \kappa \in (0,1)\,, \\ \underline{\mathfrak{h}}(0) = 0 \leq 0, \\ \underline{\mathfrak{h}}'(0) = 0 \leq 0, \\ 0 = \underline{\mathfrak{h}}(1) + a{}^{C}\mathfrak{D}_{0^+}^{1/2}\underline{\mathfrak{h}}(1) \leq \int_0^1 0\, ds. \end{cases}$$

Secondly, since $\int_0^1 \varpi(s)ds \leq 1$ we have

$$\begin{cases} 0 = -{}^R\mathfrak{D}_{0^+}^{3/2}(\varphi_p({}^C\mathfrak{D}_{0^+}^{1/2}\overline{\mathfrak{h}}(\kappa))) \geq -\sigma, \quad \kappa \in (0,1), \\ \overline{\mathfrak{h}}(0) = \sigma \geq 0, \\ \overline{\mathfrak{h}}'(0) = 0 \geq 0, \\ \sigma = \overline{\mathfrak{h}}(1) + a\,{}^C\mathfrak{D}_{0^+}^{1/2}\overline{\mathfrak{h}}(1) \geq \int_0^1 \sigma\varpi(s)\, ds. \end{cases}$$

Therefore $\underline{\mathfrak{h}}$ and $\overline{\mathfrak{h}}$ are respectively a lower and an upper solution of problem (4.1).

Then, by Theorem 3.3, we conclude that problem (4.1) admits both a maximal solution $\mathfrak{h}_{\max}$ and a minimal solution $\mathfrak{h}_{\min}$, such that

$$0 \leq \mathfrak{h}_{\min}(\kappa) \leq \mathfrak{h}_{\max}(\kappa) \leq \sigma \quad \text{for all} \quad \kappa \in [0,1].$$

□

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