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EXISTENCE OF SOLUTIONS AND HYERS-ULAM STABILITY FOR A COUPLED SYSTEM OF NONLINEAR LANGEVIN EQUATIONS INVOLVING ϕ-RIEMANN-LIOUVILLE AND ψ-CAPUTO FRACTIONAL DERIVATIVES SUBJECT TO MIXED BOUNDARY CONDITIONS

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Abstract. This paper investigates the existence, uniqueness, and Hyers-Ulam stability of solutions to a class of coupled system of nonlinear fractional Langevin equations involving mixed generalized fractional derivatives. The system under consideration is governed by fractional differential equations that involve generalized fractional derivatives, subject to mixed boundary conditions. After presenting the necessary preliminaries and definitions within the framework of generalized fractional calculus, we employ fixed point theorems to establish sufficient conditions for the existence and uniqueness result of solutions. Additionally, Hyers-Ulam stability of the considered system is investigated. Two pertinent examples are provided to verify the validity of the reported theoretical results. Our results not only provide a new contributions to the field but also expand upon several previously reported findings in the literature.

Keywords. Coupled system of Langevin fractional problems, fixed-point theorems , generalized fractional derivatives, Hyers-Ulam Stability
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1. Introduction

In recent years, fractional differential equations have become an increasingly important area of study, largely due to their ability to describe complex systems with memory and hereditary effects. These equations have found applications across a wide range of fields, including biomedical engineering [16], finance economics [10], physics [12], modeling in dynamics and chaos [5], control systems [21], viscoelasticity [17, 8] and the references therein. Foundational texts on fractional calculus offer a detailed account of its classical theory and mathematical tools in [19, 13].

In particular, coupled systems of fractional differential equations play a significant role in a wide range of applied problems, with notable applications in the biosciences. The qualitative properties of such systems specifically the existence,

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uniqueness, and stability of solutions have been the focus of extensive research, often employing fixed point theorems and related analytical techniques [9, 18].

On the other hand, a significant advancement in this field was introduced by Almeida [1], who proposed a new type of fractional derivative known as the ψ-Caputo derivative. This formulation generalizes the well-known Caputo derivative by incorporating a flexible kernel function, and builds upon the work of earlier researchers. Since its introduction, this new operator has attracted substantial interest, with many researchers exploring its properties and potential applications [2, 20].

Alternatively, the Langevin equation plays a fundamental role in the stochastic modeling of physical systems subject to both deterministic and random forces. Originally introduced by Paul Langevin in 1908 to describe the motion of a Brownian particle in a fluid [15]. Mathematically, the Langevin equation is closely linked to the Fokker-Planck equation, which governs the evolution of the probability density function associated with the stochastic process $x(t)$. It has also been generalized to incorporate memory effects and anomalous diffusion through the use of fractional derivatives, leading to fractional Langevin equations (FLEs), which are increasingly studied in the context of viscoelastic media, biological transport, and subdiffusive systems. More recently, there has been growing interest in investigating the existence, uniqueness, and various types of stability notably Ulam-Hyers stability for solutions to nonlinear fractional Langevin equation involving different types of fractional derivatives. Researchers have applied a diverse analytical techniques to study these problems, such as classical fixed point theorems, Leray-Schauder degree theory [7, 24].

At the same time, extensive research has been devoted to the theoretical analysis of the problem of fractional differential equations using different types of fractional derivatives. Topics such as the existence and uniqueness of solutions, as well as Ulam-type stability of fractional differential systems (FDS) including Langevin-type systems, employing a range of analytical techniques have been extensively studied by numerous researchers [14, 20, 23, 24]. Recently, Abbas [3] investigated a coupled system of fractional differential equations with four-point boundary conditions involving ψ-Caputo fractional derivatives. The author employed the same fixed-point theories as that used in the present study to establish the existence and uniqueness results of solutions, without investigating the Hyers-Ulam stability analysis.

In particular, Baitiche et al. [6] considered a class of nonlinear fractional Langevin differential equations involving two fractional derivative operators in the ψ-Caputo sense of the following form

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{\beta;\psi}\left({}^{c}\mathcal{D}_{a^+}^{\alpha;\psi}-\mu\right)x(t)=f(t,x(t)), & a<t<b,\\ x_{\psi}^{[k]}(a)=a_k, & 0\le k<l,\\ {}^{c}\mathcal{D}_{a^+}^{\alpha+k;\psi}x(a)=b_k, & 0\le k<n, \end{cases}$$

where ${}^{c}\mathcal{D}_{a^+}^{\alpha;\psi}$ and ${}^{c}\mathcal{D}_{a^+}^{\beta;\psi}$ denote the ψ-Caputo fractional derivatives such that $m-1<\alpha\leq m, n-1<\beta\leq n, l=\max\{m,n\}, n,m\in\mathbb{N}^*, \mu, a_k, b_k\in\mathbb{R}$ and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The existence, uniqueness and stability of solutions were established using the Banach contraction principle and the Schauder fixed-point theorem and the Hyers-Ulam stability concept.

Zibar et al. [25] investigated a coupled system of nonlinear fractional differential problems involving mixed generalized fractional derivatives of the following form

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\varrho_1,\phi_1}\left({}^{C}\mathcal{D}_{0^+}^{\varrho_2,\psi_1}x\right)(t)=G\big(t,x(t),y(t)\big), & (t\in\mathcal{O}:=[0,1]),\\ {}^{C}\mathcal{D}_{0^+}^{\varrho_3,\phi_2}\left({}^{RL}\mathcal{D}_{0^+}^{\varrho_4,\psi_2}y\right)(t)=H\big(t,x(t),y(t)\big), & (t\in\mathcal{O}:=[0,1]),\\ \gamma x(0)=\delta x(1)=\alpha^{C}\mathcal{D}_{0^+}^{\varrho_2,\psi_1}x(0)=\beta^{C}\mathcal{D}_{0^+}^{\varrho_2,\psi_1}x(1)=0,\\ \gamma^* y(0)=\delta^* y(1)=\alpha^{*RL}\mathcal{D}_{0^+}^{\varrho_4,\psi_2}y(0)=\beta^{*RL}\mathcal{D}_{0^+}^{\varrho_4,\psi_2}y(1)=0, \end{cases}$$

where ${}^{RL}\mathcal{D}_{0^+}^{\varrho;\phi}$ and ${}^{C}\mathcal{D}_{0^+}^{\varrho;\psi}$are the standard ϕ-Riemann-Liouville and ψ-Caputo fractional derivatives of order ϱ, respectively. Also $1<\varrho_i\leq 2$ for $i\in\{1,2,3,4\}$, and $\gamma,\delta,\alpha,\beta,\gamma^*,\delta^*,\alpha^*,\beta^*\in\mathbb{R}_+,\ \phi,\psi:[0,1]\to[0,\infty[$ are a bounded for $j\in\{1,2\}$; $G,H\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$; ${}^{\mathrm{RL}}\mathcal{I}_{0^+}^{\varrho;\psi}$ is the ψ-Riemann-Liouville fracional integral of order $\varrho>0$.

The existence, uniqueness, and stability of solutions were obtained using the Banach contraction principle, the Leray-Schauder alternative fixed point theorem, and the Hyers-Ulam stability approach.

However, to the best of our knowledge, there are few studies on the existence, uniqueness and Hyers-Ulam stability of solutions for coupled system of nonlinear Langevin equations involving mixed generalized fractional derivatives and subject to mixed boundary conditions.

Motivated by the above-mentioned research [25, 6], in this paper, we introduce and investigate a novel coupled system of nonlinear fractional Langevin equations involving the ϕ-Riemann-Liouville and ψ-Caputo fractional derivatives, subject to mixed boundary conditions of the following form:

$$\begin{cases} {}^{RL}\mathcal{D}_{b^+}^{\alpha_1;\phi}\left({}^{C}\mathcal{D}_{b^+}^{\alpha_2;\psi}-\mu_1\right)x(t)=\mathfrak{L}(t,x(t),y(t)), & b<t<\ell_0,\\ {}^{C}\mathcal{D}_{b^+}^{\alpha_3;\tilde{\phi}}\left({}^{RL}\mathcal{D}_{b^+}^{\alpha_4;\tilde{\psi}}-\mu_2\right)y(t)=\mathfrak{R}(t,x(t),y(t)), & b<t<\ell_0,\\ x_{\phi}^{[k]}(b)=b_k,\, {}^{C}\mathcal{D}_{b^+}^{\alpha_2+k;\phi}x(b)=\ell_k,\, x_{\psi}^{[k]}(b)=b_k^*, & 0\leq k<n,\\ y_{\tilde{\phi}}^{[j]}(b)=\tilde{b}_j,\, {}^{RL}\mathcal{D}_{b^+}^{\alpha+j;\tilde{\phi}}y(b)=\tilde{\ell}_j,\, y_{\tilde{\psi}}^{[j]}(b)=\tilde{b}_j^*, & 0\leq j<n, \end{cases} \tag{1.1}$$

where ${}^{RL}\mathcal{D}_{b^+}^{\alpha;\phi}$ and ${}^{C}\mathcal{D}_{b^+}^{\alpha;\psi}$ denote the ϕ-Riemann-Liouville and ψ-Caputo fractional derivatives of order α such that $n-1<\alpha_i\leq n$ for $i\in\{1,2,3,4\}, n\in\mathbb{N}^*$, and $\mu_1,\mu_2,b_k,\ell_k,b_k^*,\tilde{b}_j,\tilde{\ell}_j,\tilde{b}_j^*\in\mathbb{R},\ \phi,\psi,\tilde{\phi},\tilde{\psi}:\Omega_{\mathrm{b}}\to\mathbb{R}$ are a bounded and $\mathfrak{L},\mathfrak{R}:\Omega_{\mathrm{b}}\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

The existence, uniqueness, and stability of solutions for problem (1.1) are established using the Banach fixed-point theorem, the Leray-Schauder alternative fixed-point theorem, and the Hyers-Ulam stability concept.

In comparison with the aforementioned studies, the innovative contributions of this paper can be summarized as follows:
First, the current study investigates a novel coupled system of nonlinear fractional Langevin equations characterized by mixed generalized fractional derivatives involving four distinct fractional orders and subjected to mixed boundary conditions, whereas the work in [6] studied a nonlinear Langevin differential equations with two fractional orders formulated within the framework of the generalized Caputo derivative associated with initial conditions. Accordingly, The findings presented here generalize the results established in [6].

Second, it is clear that for $\mu_1 = \mu_2 = 0$ and $n = 2$, the problem considered in this paper reduces to the same problem studied in [25] but under different boundary conditions. Consequently, the results derived in this paper generalize those of [25, 6], which are obtained as particular cases under certain conditions.

The structure of this paper is as follows: Section 2 introduces the fundamental concepts, including key definitions and auxiliary propositions which will be used throughout the paper. Section 3 discusses the main results related to both existence and uniqueness of the considered problem using Banach fixed-point theorem as well as existence results employing Leray-Schauder alternative. In Section 4, we investigate the Hyers-Ulam stability. In Section 5, we present two illustrative examples to clarify and support our theoretical results. Finally, we conclude the paper in Section 6 by summarizing the main theoretical contributions and results established in this study and by outlining possible future research directions.

2. Preliminaries

In this section, we establish the notational framework and recall essential definitions and preliminary results of ψ-fractional calculus that will serve as foundational tools throughout the paper.

Let $\Omega_{\mathrm{b}}(0 < b < \infty)$, be a finite interval and $\psi : \Omega_{\mathrm{b}} \to \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$, for all $t \in \Omega_{\mathrm{b}}$

Let $\mathbb{C} = C(\Omega_{\mathrm{b}}, \mathbb{R})$ denotes the Banach space of all continuous functions from Ω_{b} to $\mathbb{R}$. Let us introduce the space $\mathcal{X} = \{x(t) : x(t) \in C(\Omega_{\mathrm{b}}, \mathbb{R})\}$, endowed with the norm $\|x\| = \max\{|x(t)| : t \in \Omega_{\mathrm{b}}\}$. Obviously, $(\mathcal{X}, \|.\|)$ is a Banach space. Also let $\mathcal{Y} = \{y(t) : y(t) \in C(\Omega_{\mathrm{b}}, \mathbb{R})\}$ be endowed with the norm $\|y\| = \max\{|y(t)| : t \in \Omega_{\mathrm{b}}\}$. Obviously the product space $(\mathcal{X} \times \mathcal{Y}, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

The left sided ψ-RL fractional integral and derivative of order $\alpha > 0$ with respect to an increasing differentiable function $\psi : \Omega_{\mathrm{b}} \to \mathbb{R}$, $\psi'(t) \neq 0$, for a given real valued function $x : \Omega_{\mathrm{b}} \to \mathbb{R}$ are defined as,

$$^{\mathrm{RL}}\mathcal{I}^{\alpha;\psi}_{\mathrm{b}^+} x(t) = \int_{\mathrm{b}}^{t} \tfrac{\psi'(s)}{\Gamma(\alpha)} \left(\psi_s(t)\right)^{\alpha-1} x(s)\,\mathrm{d}s, \qquad \psi_s(t) := \psi(t) - \psi(s), \tag{2.1}$$

and

$$^{\mathrm{RL}}\mathcal{D}^{\alpha;\psi}_{\mathrm{b}^+} x(t) = x^{[n]}_{\psi}(t)\, {}^{\mathrm{RL}}\mathcal{I}^{n-\alpha;\psi}_{\mathrm{b}^+} x(t) = x^{[n]}_{\psi}(t) \int_{\mathrm{b}}^{t} \tfrac{\psi'(s)}{\Gamma(n-\alpha)} \left(\psi_s(t)\right)^{n-\alpha-1} x(s)\,\mathrm{d}s, \tag{2.2}$$

respectively, where $x_{\psi}^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n x(t)$, $\Gamma(\alpha) = \int_0^{+\infty} e^{-s}s^{\alpha-1}\,\mathrm{d}s$ is the Euler's Gamma function, $\psi, x \in C^n(\Omega_\mathrm{b})$, $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$ [1, 13]. Under the mentioned assumptions for the functions ψ and x, the left-sided fractional derivative of ψ-Caputo type of order α for x is expressed as

$$ {}^{\mathrm{C}}\mathcal{D}_{\mathrm{b}^+}^{\alpha;\psi} x(t) = {}^{\mathrm{RL}}\mathcal{I}_{\mathrm{b}^+}^{n-\alpha;\psi}\, x_{\psi}^{[n]}(t), \tag{2.3}$$

where $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$ [1]. Thus

$$ {}^{\mathrm{C}}\mathcal{D}_{\mathrm{b}^+}^{\alpha;\psi} x(t) = \begin{cases} \displaystyle\int_{\mathrm{b}}^{t} \frac{\psi'(s)(\psi_s(t))^{n-\alpha-1}}{\Gamma(n-\alpha)} x_{\psi}^{[n]}(s)\mathrm{d}s, & \alpha \notin \mathbb{N} \\ x_{\psi}^{[n]}(t), & \alpha \in \mathbb{N}. \end{cases} \tag{2.4}$$

This generalization in (2.4) implies the Caputo and the Caputo-Hadamard fractional derivative operators for $\psi(t) = t$ and $\psi(t) = \ln t$, respectively and additionally,

$$ {}^{\mathrm{C}}\mathcal{D}_{\mathrm{b}^+}^{\alpha;\psi} x(t) = {}^{\mathrm{RL}}\mathcal{D}_{\mathrm{b}^+}^{\alpha;\psi}\left[x(t) - \sum_{k=0}^{n-1} \frac{x_{\psi}^{[n]}(\mathrm{b})}{k!}\left(\psi_\mathrm{b}(t)\right)^k\right]. \tag{2.5}$$

Lemma 2.1 ([2])**.** *: Let $n-1 < \alpha < n$ and $x \in C^n(\Omega_\mathrm{b}, \mathbb{R})$. Then the following holds:*

$$ {}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi C}\mathcal{D}_{b^+}^{\alpha;\psi} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x_{\psi}^{[k]}(b)}{k!}[\psi_b(t)]^k $$

for all $t \in \Omega_\mathrm{b}$. Moreover, if $m \in \mathbb{N}$ be an integer and $x \in C^{n+m}(\Omega_\mathrm{b}, \mathbb{R})$ a function. Then, the following holds:

$$ \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m {}^{C}\mathcal{D}_{b^+}^{\alpha;\psi} x(t) = {}^{C}\mathcal{D}_{b^+}^{\alpha+m;\psi} x(t) + \sum_{k=0}^{m-1} \frac{(\psi_b(t))^{k+n-\alpha-m}}{\Gamma(k+n-\alpha-m+1)} x_{\psi}^{[k+n]}(b). \tag{2.6}$$

We know that from Equation (2.6) if $x_{\psi}^{[k]}(b) = 0$, for all $k = n, n+1, \ldots, n+m-1$ we can get the following relation

$$ \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^m {}^{C}\mathcal{D}_{b^+}^{\alpha;\psi} x(t) = {}^{C}\mathcal{D}_{b^+}^{\alpha+m;\psi} x(t), \quad t \in \Omega_\mathrm{b} \tag{2.7}$$

Lemma 2.2 ([13, 2])**.** *: Let $\alpha, \beta > 0$, and $x \in (\Omega_\mathrm{b}, \mathbb{R})$. Then for each $t \in \mathrm{J}$ we have*

(1) ${}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi\mathrm{RL}}\mathcal{I}_{b^+}^{\beta;\psi} x(t) = {}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha+\beta;\psi} x(t)$,

(2) ${}^{C}\mathcal{D}_{b^+}^{\alpha;\psi\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi} x(t) = x(t)$,

(3) ${}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi}(\psi_b(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi_b(t))^{\beta+\alpha-1}$,

(4) ${}^{C}\mathcal{D}_{b^+}^{\alpha;\psi}(\psi_b(t))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi_b(t))^{\beta-\alpha-1}$,

(5) ${}^{C}\mathcal{D}_{b^+}^{\alpha;\psi}(\psi_b(t))^k = 0$, *for all $k \in \{0, \ldots, n-1\}, n \in \mathbb{N}$.*

Lemma 2.3 ([1]). *Let $x \in C^n(\Omega_b, \mathbb{R})$ and $n-1<\alpha<n$. Then we have*
(1) ${}^{RL}\mathcal{D}_{b^+}^{\alpha;\psi}\,{}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi}x(t)=x(t)$
(2) ${}^{\mathrm{RL}}\mathcal{I}_{b^+}^{\alpha;\psi}\,{}^{RL}\mathcal{D}_{b^+}^{\alpha;\psi}x(t)=x(t)-\displaystyle\sum_{k=1}^{n}\frac{x_\psi^{[k-1]}(b)}{\Gamma(k-\alpha)}[\psi_b(t)]^{k-\alpha}$

We next recall the well known fixed point theorems that will be utilized in subsequent sections.

Theorem 2.4 (Banach [22]). . *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space X and $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ be a contraction operator. Then there is a unique $x\in\mathcal{C}$ with $\mathcal{T}x=x$.*

Theorem 2.5. *([11]) Let $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ be a completely continuous operator . Let*

$$\Theta(\mathcal{T})=\{x\in\mathcal{C}: x=\varrho\mathcal{T}(x)\ for\ some\ 0<\varrho<1\}.$$

Then either the set $\Theta(\mathcal{T})$ is unbounded or $\mathcal{T}$ has at least one fixed point.

3. Main results

Lemma 3.1. *Assume $\mathfrak{g},\mathfrak{h}$ are continuous functions from Ω_b to $\mathbb{R}$. Then the solution for the linear system of sequential fractional differential equations:*

$$\begin{cases} {}^{RL}\mathcal{D}_{b^+}^{\alpha_1;\phi}\left({}^{C}\mathcal{D}_{b^+}^{\alpha_2;\psi}-\mu_1\right)x(t)=\mathfrak{g}(t), & b<t<\ell_0,\\ {}^{C}\mathcal{D}_{b^+}^{\alpha_3;\tilde\phi}\left({}^{RL}\mathcal{D}_{b^+}^{\alpha_4;\tilde\psi}-\mu_2\right)y(t)=\mathfrak{h}(t), & b<t<\ell_0,\\ x_\phi^{[k]}(b)=b_k,\ {}^{C}\mathcal{D}_{b^+}^{\alpha_2+k;\phi}x(b)=\ell_k,\ x_\psi^{[k]}(b)=b_k^*, & 0\le k<n,\\ y_{\tilde\phi}^{[j]}(b)=\tilde b_j,\ {}^{RL}\mathcal{D}_{b^+}^{\alpha+j;\tilde\phi}y(b)=\tilde\ell_j,\ y_{\tilde\psi}^{[j]}(b)=\tilde b_j^*, & 0\le j<n,\end{cases} \tag{3.1}$$

is

$$\begin{aligned} x(t)=&\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s\frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}\mathfrak{g}(r)dr\right)ds\\ &+\mu_1\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}x(s)\right)ds+\varpi_1(t),\end{aligned} \tag{3.2}$$

where

$$\varpi_1(t)=\sum_{k=0}^{n-1}\frac{\ell_k-\mu_1b_k}{\Gamma(k+1-\alpha_1)}\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}[\phi_b(s)]^{k+1-\alpha_1}\right)ds+\sum_{k=0}^{n-1}\frac{b_k^*}{\Gamma(k+1)}[\psi_b(t)]^k,$$

and

$$\begin{aligned} y(t)=&\int_b^t\left(\frac{\tilde\psi'(s)}{\Gamma(\alpha_4)}\left(\tilde\psi_s(t)\right)^{\alpha_4-1}\int_b^s\frac{\tilde\phi'(r)}{\Gamma(\alpha_3)}\left(\tilde\phi_r(s)\right)^{\alpha_3-1}\mathfrak{h}(r)dr\right)ds\\ &+\mu_2\int_b^t\left(\frac{\tilde\psi'(s)}{\Gamma(\alpha_4)}\left(\tilde\psi_s(t)\right)^{\alpha_4-1}y(s)\right)ds+\varpi_2(t),\end{aligned} \tag{3.3}$$

where

$$\varpi_2(t)=\sum_{j=0}^{n-1}\frac{\tilde{\ell}_j-\mu_2\tilde{b}_j}{\Gamma(j+1)}\int_b^t\left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)}\left(\tilde{\psi}_s(t)\right)^{\alpha_4-1}\left[\tilde{\phi}_b(s)\right]^j\right)ds+\sum_{j=0}^{n-1}\frac{\tilde{b}_j^*}{\Gamma(j+1-\alpha_4)}\left[\tilde{\psi}_b(t)\right]^{j+1-\alpha_4}$$

Proof. By applying the operators ${}^{RL}\mathcal{I}_{b^+}^{\alpha_1;\psi}$ and ${}^{RL}\mathcal{I}_{b^+}^{\alpha_3;\psi}$ to both sides of the equations in (3.1) respectively, and then using Lemmas 2.1, 2.2 and 2.3, we obtain

$$ {}^{C}\mathcal{D}_{b^+}^{\alpha_2;\psi}x(t)={}^{RL}\mathcal{I}_{b^+}^{\alpha_1;\phi}\mathfrak{g}(t)+\mu_1x(t)+\sum_{k=0}^{n-1}\frac{{}^{C}\mathcal{D}_{b^+}^{\alpha_2+k;\psi}x(b)-\mu_1x_{\phi}^{[k]}(b)}{\Gamma(k+1-\alpha_1)}[\phi_b(t)]^{k+1-\alpha_1} \tag{3.4}$$

and

$$ {}^{RL}\mathcal{D}_{b^+}^{\alpha_4;\tilde{\psi}}y(t)={}^{RL}\mathcal{I}_{b^+}^{\alpha_3;\tilde{\phi}}\mathfrak{h}(t)+\mu_2y(t)+\sum_{i=0}^{n-1}\frac{{}^{RL}\mathcal{D}_{b^+}^{\alpha_4+i;\tilde{\psi}}y(b)-\mu_2y_{\tilde{\phi}}^{[i]}(b)}{\Gamma(i+1)}[\tilde{\phi}_b(t)]^{i} \tag{3.5}$$

from the initial condition (3.1), it follows that

$$ {}^{C}\mathcal{D}_{b^+}^{\alpha_2;\psi}x(t)={}^{RL}\mathcal{I}_{b^+}^{\alpha_1;\phi}\mathfrak{g}(t)+\mu_1x(t)+\sum_{k=0}^{n-1}\frac{\ell_k-\mu_1b_k}{\Gamma(k+1-\alpha_1)}[\phi_b(t)]^{k+1-\alpha_1} \tag{3.6}$$

and

$$ {}^{RL}\mathcal{D}_{b^+}^{\alpha_4;\tilde{\psi}}y(t)={}^{RL}\mathcal{I}_{b^+}^{\alpha_3;\tilde{\phi}}\mathfrak{h}(t)+\mu_2y(t)+\sum_{i=0}^{n-1}\frac{\tilde{\ell}_i-\mu_2\tilde{b}_i}{\Gamma(i+1)}[\tilde{\phi}_b(t)]^{i} \tag{3.7}$$

We apply the operators again ${}^{RL}\mathcal{I}_{b^+}^{\alpha_2;\psi}$ and ${}^{RL}\mathcal{I}_{b^+}^{\alpha_4;\psi}$ to both sides of the equations (3.6) and (3.7) respectively, and then using Lemmas 2.1 , 2.2 and 2.3,we have

$$\begin{aligned}
x(t)=&{}^{RL}\mathcal{I}_{b^+}^{\alpha_2;\psi}{}^{RL}\mathcal{I}_{b^+}^{\alpha_1;\phi}\mathfrak{g}(t)+\sum_{k=0}^{n-1}\frac{\ell_k-\mu_1b_k}{\Gamma(k+1-\alpha_1)}{}^{RL}\mathcal{I}_{b^+}^{\alpha_2;\psi}\left[\phi_b(t)\right]^{k+1-\alpha_1}\\
&+\mu_1{}^{RL}\mathcal{I}_{b^+}^{\alpha_2;\psi}x(t)+\sum_{k=0}^{n-1}\frac{x_{\psi}^{[k]}(b)}{\Gamma(k+1)}\left[\psi_b(t)\right]^k\\
=&\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}\left(\psi_s(t)\right)^{\alpha_2-1}\int_b^s\frac{\phi'(r)}{\Gamma(\alpha_1)}\left(\phi_r(s)\right)^{\alpha_1-1}\mathfrak{g}(r)dr\right)ds\\
&+\mu_1\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}\left(\psi_s(t)\right)^{\alpha_2-1}x(s)\right)ds+\sum_{k=0}^{n-1}\frac{b_k^*}{\Gamma(k+1)}\left[\psi_b(t)\right]^k\\
&+\sum_{k=0}^{n-1}\frac{\ell_k-\mu_1b_k}{\Gamma(k+1-\alpha_1)}\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}\left(\psi_s(t)\right)^{\alpha_2-1}\left[\phi_b(s)\right]^{k+1-\alpha_1}\right)ds,
\end{aligned}$$

and

$$
\begin{aligned}
y(t) =& {}^{RL}\mathcal{I}_{b^+}^{\alpha_4;\tilde{\psi}}\,{}^{RL}\mathcal{I}_{b^+}^{\alpha_3;\tilde{\phi}}\mathfrak{h}(t) + \sum_{j=0}^{n-1} \frac{\tilde{\ell}_j - \mu_2 \tilde{b}_j}{\Gamma(j+1)}\,{}^{RL}\mathcal{I}_{b^+}^{\alpha_4;\tilde{\psi}} \left[\tilde{\phi}_b(t)\right]^j \\
&+ \mu_2\,{}^{RL}\mathcal{I}_{b^+}^{\alpha_4;\tilde{\psi}} y(t) + \sum_{j=0}^{n-1} \frac{y_{\tilde{\psi}}^{[j]}(b)}{\Gamma(j+1-\alpha_4)} \left[\tilde{\psi}_b(t)\right]^{j+1-\alpha_4} \\
=& \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)} \left(\tilde{\psi}_s(t)\right)^{\alpha_4-1} \int_b^s \frac{\tilde{\phi}'(r)}{\Gamma(\alpha_3)} \left(\tilde{\phi}_r(s)\right)^{\alpha_3-1} \mathfrak{h}(r)dr \right) ds \\
&+ \mu_2 \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)} \left(\tilde{\psi}_s(t)\right)^{\alpha_4-1} y(s) \right) ds + \sum_{j=0}^{n-1} \frac{\tilde{b}_j^*}{\Gamma(j+1-\alpha_4)} \left[\tilde{\psi}_b(t)\right]^{j+1-\alpha_4} \\
&+ \sum_{j=0}^{n-1} \frac{\tilde{\ell}_j - \mu_2 \tilde{b}_j}{\Gamma(j+1)} \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)} \left(\tilde{\psi}_s(t)\right)^{\alpha_4-1} \left[\tilde{\phi}_b(s)\right]^j \right) ds
\end{aligned}
$$

and the converse follows easily. Hence, the proof is completed. □

To proceed with presenting the main results, we first set forth the following assumptions, which serve as the foundation for the subsequent analysis:
$(\mathbb{A}_1)$ The functions $\mathfrak{L}, \mathfrak{R} : \Omega_{\mathrm{b}} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist constants $L_{\mathfrak{L}}, L_{\mathfrak{R}} > 0$ where

$$
\begin{aligned}
&\left|\mathfrak{L}(t, x_1, y_1) - \mathfrak{L}(t, x_2, y_2)\right| \le L_{\mathfrak{L}} \left(|x_1 - x_2| + |y_1 - y_2|\right). \\
&\left|\mathfrak{R}(t, x_1, y_1) - \mathfrak{R}(t, x_2, y_2)\right| \le L_{\mathfrak{R}} \left(|x_1 - x_2| + |y_1 - y_2|\right).
\end{aligned} \tag{3.8}
$$

for $t \in \Omega_{\mathrm{b}}$ and $x_i, y_i \in \mathbb{R}, i = 1, 2$.
$(\mathbb{A}_2)$ For some positive numbers $c_{\mathfrak{L}}, d_{\mathfrak{L}}, m_{\mathfrak{L}}, c_{\mathfrak{R}}, d_{\mathfrak{R}}$ and $m_{\mathfrak{R}}$,

$$
\left|\mathfrak{L}(t, x(t), y(t))\right| \le c_{\mathfrak{L}} |x| + d_{\mathfrak{L}} |y| + m_{\mathfrak{L}} \tag{3.9}
$$

and

$$
\left|\mathfrak{R}(t, x(t), y(t))\right| \le c_{\mathfrak{R}} |x| + d_{\mathfrak{R}} |y| + m_{\mathfrak{R}}. \tag{3.10}
$$

for $t \in \Omega_{\mathrm{b}}$ and $x, y \in \mathbb{R}, i = 1, 2$.

In view of Lemma 3.1, we define an operator $\mathcal{T} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ by

$$
\mathcal{T}(x, y)(t) = \left(\mathcal{T}_1(x, y)(t), \mathcal{T}_2(x, y)(t)\right) \tag{3.11}
$$

where

$$
\begin{aligned}
\mathcal{T}_1(x, y)(t) =& \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} \left(\psi_s(t)\right)^{\alpha_2-1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} \left(\phi_r(s)\right)^{\alpha_1-1} \mathfrak{L}(r, x(r), y(r))dr \right) ds \\
&+ \mu_1 \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} \left(\psi_s(t)\right)^{\alpha_2-1} x(s) \right) ds + \varpi_1(t),
\end{aligned} \tag{3.12}
$$

and

$$\mathcal{T}_2(x,y)(t) = \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)} \left(\tilde{\psi}_s(t)\right)^{\alpha_4-1} \int_b^s \frac{\tilde{\phi}'(r)}{\Gamma(\alpha_3)} \left(\tilde{\phi}_r(s)\right)^{\alpha_3-1} \mathfrak{R}(r,x(r),y(r))dr \right) ds$$
$$+ \mu_2 \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)} \left(\tilde{\psi}_s(t)\right)^{\alpha_4-1} y(s) \right) ds + \varpi_2(t), \tag{3.13}$$

To facilitate the presentation of subsequent results, we set

$$\begin{aligned}
\mathfrak{L}_1 &= L_{\mathfrak{L}} \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + |\mu_1| \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
\mathfrak{L}_2 &= \eta_1 \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \varpi_1^* \\
\mathfrak{R}_1 &= L_{\mathfrak{R}} \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mu_2| \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
\mathfrak{R}_2 &= \eta_2 \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \varpi_2^*
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\mathring{\mho}_1 &= c_{\mathfrak{L}} \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + c_{\mathfrak{R}} \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mu_1| \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \\
\mathring{\mho}_2 &= d_{\mathfrak{L}} \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + d_{\mathfrak{R}} \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mu_2| \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
\mathring{\mho}_0 &= m_{\mathfrak{L}} \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + m_{\mathfrak{R}} \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + \varpi_1^* + \varpi_2^* \\
\mathring{\mho} &= max\{\mathring{\mho}_1, \mathring{\mho}_2\}
\end{aligned} \tag{3.15}$$

Theorem 3.2. *Suppose that $\mathbb{A}_1$ holds. Then the coupled system (1.1) has a unique solution on Ω_{b}, on the condition that*

$$\mathfrak{L}_1 + \mathfrak{R}_1 < 1 \tag{3.16}$$

where $\mathfrak{L}_1$ and $\mathfrak{R}_1$ are defined in (3.14).

Proof. We begin by proving that $\Lambda > 0$ is a real number such that

$$\Lambda > \frac{\mathfrak{L}_2 + \mathfrak{R}_2}{1 - \mathfrak{L}_1 - \mathfrak{R}_1}$$

First, we shall show that $\mathcal{T}\omega_\Lambda \subset \omega_\Lambda$, where $\mathcal{T}$ is defined in (3.11) as well as

$$\omega_\Lambda = \{(x,y) \in \mathcal{X} \times \mathcal{Y} : \|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \leq \Lambda\}$$

Let $\sup_{t\in\Omega_b}|\mathfrak{L}(t,0,0)|=\eta_1<\infty$ and $\sup_{t\in\Omega_b}|\mathfrak{R}(t,0,0)|=\eta_2<\infty$ and $\varpi_i^*=\sup_{t\in\Omega_b}|\varpi_i(t)|, i\in\{1,2\}$.

$$\begin{aligned}
|\mathcal{T}_1(x,y)(t)| \leq & \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}|\mathfrak{L}(r,x(r),y(r))|\,dr\right)ds \\
& + |\mu_1|\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}|x(s)|\right)ds + |\varpi_1(t)| \\
\leq & \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}\left(|\mathfrak{L}(r,x(r),y(r))-\mathfrak{L}(r,0,0)|+|\mathfrak{L}(r,0,0)|\right)dr\right)ds \\
& + |\mu_1|\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}|x(s)|\right)ds + |\varpi_1(t)| \\
\leq & \left(L_{\mathfrak{L}}\left(\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}\right)+\eta_1\right)\int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}dr\right)ds \\
& + |\mu_1|\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\|x\|\right)ds + \varpi_1^* \\
\leq & \Lambda\left[L_{\mathfrak{L}}\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)}+|\mu_1|\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]+\eta_1\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)}+\varpi_1^* \\
= & \Lambda\mathfrak{L}_1+\mathfrak{L}_2,
\end{aligned}$$

which implies that

$$\|\mathcal{T}_1(x,y)\|_{\mathcal{X}} \leq \Lambda\mathfrak{L}_1+\mathfrak{L}_2 \tag{3.17}$$

Similarly, we can find that

$$\begin{aligned}
\|\mathcal{T}_2(x,y)\|_{\mathcal{Y}} \leq & \Lambda\left[L_{\mathfrak{R}}\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mu_2|\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\right]+\eta_2\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)}+\varpi_2^* \\
= & \Lambda\mathfrak{R}_1+\mathfrak{R}_2.
\end{aligned} \tag{3.18}$$

Accordingly, from (3.17) and (3.18), we obtain

$$\begin{aligned}
\|\mathcal{T}(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \leq & \left(\mathfrak{L}_1+\mathfrak{R}_1\right)\Lambda+\mathfrak{L}_2+\mathfrak{R}_2 \\
\leq & \Lambda
\end{aligned}$$

Hence, $\mathcal{T}\omega_\Lambda \subset \omega_\Lambda$ Now, for $(x_1, y_1), (x_2, y_2) \in \mathcal{X} \times \mathcal{X}$ and for each $t \in \Omega_{\mathrm{b}}$, we obtain

$$\begin{aligned}
&|\mathcal{T}_1(x_1,y_1)(t) - \mathcal{T}_1(x_2, y_2)(t)| \\
&\quad \leq \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2 - 1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1 - 1} |\mathfrak{L}(r, x_1(r), y_1(r)) - \mathfrak{L}(r, x_2(r), y_2(r))| \, dr \right) ds \\
&\qquad + |\mu_1| \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2 - 1} |x_1(s) - x_2(s)| \right) ds \\
&\quad \leq L_{\mathfrak{L}} \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2 - 1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1 - 1} (|x_1(r) - x_2(r)| + |y_1(r) - y_2(r)|) \, dr \right) ds \\
&\qquad + |\mu_1| \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2 - 1} |x_1(s) - x_2(s)| \right) ds \\
&\quad \leq \mathfrak{L}_1 (\|x_1 - x_2\| + \|y_1 - y_2\|)
\end{aligned}$$

and hence

$$\|\mathcal{T}_1(x_1, y_1) - \mathcal{T}_1(x_2, y_2)\|_{\mathcal{X}} \leq \mathfrak{L}_1 (\|x_1 - x_2\| + \|y_1 - y_2\|) \tag{3.19}$$

By analogoud reasoning, we obtain

$$\|\mathcal{T}_2(x_1, y_1) - \mathcal{T}_2(x_2, y_2)\|_{\mathcal{X}} \leq \mathfrak{R}_1 (\|x_1 - x_2\| + \|y_1 - y_2\|) \tag{3.20}$$

In view of (3.19) and (3.20), it follows that

$$\|\mathcal{T}(x_1, y_1) - \mathcal{T}(x_2, y_2)\|_{\mathcal{X} \times \mathcal{Y}} \leq (\mathfrak{L}_1 + \mathfrak{R}_1) (\|x_1 - x_2\| + \|y_1 - y_2\|)$$

Based on the above inequality, we conclude that $\mathcal{T}$ is a contraction, given condition (3.16). Hence, it follows using Banach fixed point theorem that there exists a unique fixed point for the operator $\mathcal{T}$, which corresponds to the unique solution of the coupled system (1.1) on Ω_{b}. Accordingly, the proof is complete. $\square$

We now state and prove an existence result for the coupled system (1.1), relying on the Leray-Schauder alternative

Theorem 3.3. *Suppose that $\mathbb{A}_2$ holds. If $\mathring{\mho} < 1$, then the coupled system (1.1) has at least one solution on Ω_{b}, where $\mathring{\mho}$ is given in (3.15).*

Proof. We begin by proving that the operator $\mathcal{T} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ is completely continuous. Due to the continuity of the functions $\mathfrak{L}$ and $\mathfrak{R}$, the operator $\mathcal{T}$ is continuous. Assume that $\Upsilon \subset X \times Y$ be bounded. Therefore, there exist positive constants ι_1 and ι_2 such that $|\mathfrak{L}(t, x(t), y(t))| \leq \iota_1$, $|\mathfrak{R}(t, x(t), y(t))| \leq \iota_2$, for all $(x, y) \in \Upsilon$.
For any $(x, y) \in \Upsilon$, we have

$$
\begin{aligned}
|\mathcal{T}_1(x,y)(t)| \leq & \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2-1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1-1} |\mathfrak{L}(r,x(r),y(r))| \, dr \right) ds \\
& + |\mu_1| \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t))^{\alpha_2-1} |x(s)| \right) ds + |\varpi_1(t)| \\
\leq & \iota_1 \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + |\mu_1| \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} |x(t)| + |\varpi_1(t)|
\end{aligned}
$$

which implies that

$$
\|\mathcal{T}_1(x,y)(t)\|_{\mathcal{X}} \leq \iota_1 \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + |\mu_1| \frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)} \|x(t)\| + \varpi_1^* \tag{3.21}
$$

Analogously, we obtain

$$
\|\mathcal{T}_2(x,y)(t)\|_{\mathcal{Y}} \leq \iota_2 \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)} + |\mu_2| \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)} \|y(t)\| + \varpi_2^* \tag{3.22}
$$

Using (3.21) and (3.22), therefore $\mathcal{T}$ is uniformly bounded.
We now establish that the operator $\mathcal{T}$ is equicontinuous. Let $t_1; t_2 \in \Omega_{\mathrm{b}}$ with $t_1 < t_2$. Then we get

$$
\begin{aligned}
& |\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)| \\
& \quad \leq | \int_{t_1}^{t_2} \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t_2))^{\alpha_2-1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1-1} \mathfrak{L}(r,x(r),y(r)) \, dr \right) ds. \\
& \quad + \int_b^{t_1} \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t_2))^{\alpha_2-1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1-1} \mathfrak{L}(r,x(r),y(r)) \, dr \right) ds. \\
& \quad - \int_b^{t_1} \left(\frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t_1))^{\alpha_2-1} \int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)} (\phi_r(s))^{\alpha_1-1} \mathfrak{L}(r,x(r),y(r)) \, dr \right) ds|. \\
& \quad + |\mu_1| \int_b^{t_1} \frac{\psi'(s)}{\Gamma(\alpha_2)} \left((\psi_s(t_2))^{\alpha_2-1} - (\psi_s(t_1))^{\alpha_2-1} \right) |x(s)| \, ds + |\varpi_1(t_2) - \varpi_1(t_1)| \\
& \quad + |\mu_1| \int_{t_1}^{t_2} \frac{\psi'(s)}{\Gamma(\alpha_2)} (\psi_s(t_2))^{\alpha_2-1} |x(s)| \, ds \\
& \leq \frac{\iota_1 (\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)} \left[2 (\psi_{t_1}(t_2))^{\alpha_2} + (\psi_b(t_2))^{\alpha_2} - (\psi_b(t_1))^{\alpha_2} \right]. \\
& \quad + \frac{|\mu_1| \, \|x\|}{\Gamma(\alpha_2+1)} \left[(\psi_b(t_2))^{\alpha_2} - (\psi_b(t_1))^{\alpha_2} + 2 (\psi_{t_1}(t_2))^{\alpha_2} \right] + |\varpi_1(t_2) - \varpi_1(t_1)|
\end{aligned}
$$

If $t_1 \longrightarrow t_2$, then $\|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)\| \longrightarrow 0$ independently of the values of $(x,y) \in \Upsilon$. Moreover, we obtain

$$|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)| \leq \frac{\iota_2\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\left[2\left(\tilde{\psi}_{t_1}(t_2)\right)^{\alpha_4}+\left(\tilde{\psi}_b(t_2)\right)^{\alpha_4}-\left(\tilde{\psi}_b(t_1)\right)^{\alpha_4}\right].$$
$$+\frac{|\mu_2|\,\|y\|}{\Gamma(\alpha_4+1)}\left[\left(\tilde{\psi}_b(t_2)\right)^{\alpha_4}-\left(\tilde{\psi}_b(t_1)\right)^{\alpha_4}+2\left(\tilde{\psi}_{t_1}(t_2)\right)^{\alpha_4}\right]+|\varpi_2(t_2)-\varpi_2(t_1)|$$

If $t_1 \longrightarrow t_2$, then $\|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)\| \longrightarrow 0$ independently of the values of $(x,y) \in \Upsilon$. Therefore, $\mathcal{T}$ is equicontinuous. Hence, by Arzelà-Ascoli theorem, we conclude that the operator $\mathcal{T}$ is completely continuous.
To conclude, we establish that the set

$$\Theta = \{(x,y) \in \mathcal{X}\times\mathcal{Y} : (x,y) = \varrho\mathcal{T}(x,y), 0<\varrho<1\}$$

is bounded. Let $(x,y) \in \Theta$. Then $(x,y) = \varrho\mathcal{T}(x,y)$. For any $t \in \Omega_{\mathrm{b}}$, we get

$$x(t) = \varrho\mathcal{T}_1(x,y)(t), y(t) = \varrho\mathcal{T}_1(x,y)(t)$$

It follows that

$$\begin{aligned}
|x(t)| &= |\varrho\mathcal{T}_1(x,y)(t)| \leq |\mathcal{T}_1(x,y)(t)| \\
&\leq \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}(c_{\mathfrak{L}}|x_1(r)|+d_{\mathfrak{L}}|x_2(r)|+m_{\mathfrak{L}})\,dr\right)ds \\
&+|\mu_1|\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}|x(s)|\right)ds+|\varpi_1(t)| \\
&\leq (c_{\mathfrak{L}}|x_1|+d_{\mathfrak{L}}|x_2|+m_{\mathfrak{L}})\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)}+|\mu_1|\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}|x(t)|+|\varpi_1(t)|
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &= |\varrho\mathcal{T}_2(x,y)(t)| \leq |\mathcal{T}_2(x,y)(t)| \\
&\leq (c_{\mathfrak{R}}|x_1|+d_{\mathfrak{R}}|x_2|+m_{\mathfrak{R}})\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mu_2|\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}|y(t)|+|\varpi_2(t)|
\end{aligned}$$

Hence, we get

$$\|x\|_{\mathcal{X}} \leq (c_{\mathfrak{L}}\|x\|_{\mathcal{X}}+d_{\mathfrak{L}}\|y\|_{\mathcal{Y}}+m_{\mathfrak{L}})\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)}+|\mu_1|\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\|x\|_{\mathcal{X}}+\varpi_1^*$$

and

$$\|y\|_{\mathcal{Y}} \leq (c_{\mathfrak{R}}\|x\|_{\mathcal{X}}+d_{\mathfrak{R}}\|y\|_{\mathcal{Y}}+m_{\mathfrak{R}})\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mu_2|\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\|y\|_{\mathcal{Y}}+\varpi_2^*$$

which imply that

$$\begin{aligned}
\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}} &\leq \mathring{\mho}_0+\mathring{\mho}_1\|x\|_{\mathcal{X}}+\mathring{\mho}_2\|y\|_{\mathcal{Y}} \\
&\leq \mathring{\mho}_0+\mathring{\mho}\|x+y\|_{\mathcal{X}\times\mathcal{Y}}
\end{aligned}$$

where $\mathring{\mho}_0, \mathring{\mho}_1, \mathring{\mho}_2$ and $\mathring{\mho}$ are given in (3.15). Consequently, we get

$$\|x+y\|_{\mathcal{X}\times\mathcal{Y}} \leq \frac{\mathring{\mho}_0}{1-\mathring{\mho}}$$

thus proving that the set Θ is bounded. Thus, by Lemma (2.4), the operator $\mathcal{T}$ has at least one fixed point. concequently, the coupled system (1.1) has at least one solution on Ω_{b}. Thus, the proof is complete. □

4. Stability results for problem (1.1)

In this section, we investigate the Hyers-Ulam stability of the proposed system. To this end, we begin by considering the following inequality:

$$\begin{cases} {}^{RL}\mathcal{D}_{a^+}^{\alpha_1;\phi}\left({}^{C}\mathcal{D}_{a^+}^{\alpha_2;\psi}-\mu_1\right)x(t)-\mathfrak{L}(t,x(t),y(t)) \leq \xi_1, & t\in\Omega_{\mathrm{b}}, \\ {}^{C}\mathcal{D}_{a^+}^{\alpha_3;\tilde{\phi}}\left({}^{RL}\mathcal{D}_{a^+}^{\alpha_4;\tilde{\psi}}-\mu_2\right)y(t)-\mathfrak{R}(t,x(t),y(t)) \leq \xi_2, & t\in\Omega_{\mathrm{b}}, \end{cases} \tag{4.1}$$

where ξ_1, ξ_2 are given two positive real numbers.

Definition 4.1. Problem (1.1) is Hyers-Ulam stable if there exist $\pi_i > 0; i = 1; 2;$ such that for a given ξ_1, ξ_2 and for each solution $(x^*; y^*) \in \mathcal{X}\times\mathcal{Y}$ of inequality (4.1), there exists a solution $(x; y) \in \mathcal{X}\times\mathcal{Y}$ of problem (1.1) with

$$|(x,y)-(x^*,y^*)| \leq \pi_1\xi_1+\pi_2\xi_2, t\in\Omega_{\mathrm{b}}, \tag{4.2}$$

Remark 4.2. $(x^*; y^*)$ is a solution of inequality (4.1) if there exist functions $\Psi \in C(\Omega_{\mathrm{b}},\mathbb{R}), i = 1, 2,$ which depend upon $x^*; y^*$, respectively, such that

$$i)|\Psi_1(x,y)| \leq \xi_1, \quad ii)|\Psi_2(x,y)| \leq \xi_2, t\in\Omega_{\mathrm{b}}, \tag{4.3}$$

$$\begin{cases} {}^{RL}\mathcal{D}_{a^+}^{\alpha_1;\phi}\left({}^{C}\mathcal{D}_{a^+}^{\alpha_2;\psi}-\mu_1\right)x(t)=\mathfrak{L}(t,x(t),y(t))+\Psi_1(x,y)(t), & t\in\Omega_{\mathrm{b}}, \\ {}^{C}\mathcal{D}_{a^+}^{\alpha_3;\tilde{\phi}}\left({}^{RL}\mathcal{D}_{a^+}^{\alpha_4;\tilde{\psi}}-\mu_2\right)y(t)=\mathfrak{R}(t,x(t),y(t))+\Psi_2(x,y)(t), & t\in\Omega_{\mathrm{b}} \end{cases} \tag{4.4}$$

Theorem 4.3. *Suppose that $\mathbb{A}_1, \mathbb{A}_2$ and (3.16) hold true. The coupled system (1.1) is H-U stable.*

Proof. Let $(x^*; y^*)$, be any solution of inequality (4.1), Thus,

$$x^*(t) = \mathcal{T}_1(x^*,y^*)(t)+\int_b^t\left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s\frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}\Psi_1(x,y)(r)dr\right)ds$$

and

$$y^*(t) = \mathcal{T}_2(x^*,y^*)(t)+\int_b^t\left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)}\left(\tilde{\psi}_s(t)\right)^{\alpha_4-1}\int_b^s\frac{\tilde{\phi}'(r)}{\Gamma(\alpha_3)}\left(\tilde{\phi}_r(s)\right)^{\alpha_3-1}\Psi_2(x,y)(r)dr\right)ds$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined by (3.12) and (3.13). It follows that

$$\begin{aligned}|\mathcal{T}_1(x^*,y^*)(t)-x^*(t)| \leq& \int_b^t \left(\frac{\psi'(s)}{\Gamma(\alpha_2)}(\psi_s(t))^{\alpha_2-1}\int_b^s \frac{\phi'(r)}{\Gamma(\alpha_1)}(\phi_r(s))^{\alpha_1-1}|\Psi_1(x,y)(r)|\,dr\right)ds\\ \leq& \xi_1\frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)}\end{aligned}$$

and

$$\begin{aligned}|\mathcal{T}_2(x^*,y^*)(t)-y^*(t)| \leq& \int_b^t \left(\frac{\tilde{\psi}'(s)}{\Gamma(\alpha_4)}\left(\tilde{\psi}_s(t)\right)^{\alpha_4-1}\int_b^s \frac{\tilde{\phi}'(r)}{\Gamma(\alpha_3)}\left(\tilde{\phi}_r(s)\right)^{\alpha_3-1}|\Psi_2(x,y)(r)|\,dr\right)ds\\ \leq& \xi_2\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\end{aligned}$$

Since

$$\begin{aligned}|x(t)-x^*(t)| =&|x(t)-\mathcal{T}_1(x^*,y^*)(t)+\mathcal{T}_1(x^*,y^*)(t)-x^*(t)|\\ \leq&|x(t)-\mathcal{T}_1(x^*,y^*)(t)|+|\mathcal{T}_1(x^*,y^*)(t)-x^*(t)|\\ \leq&\left[L_{\mathfrak{L}}\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\frac{(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_1+1)}+|\mu_1|\frac{(\psi_b(\ell_0))^{\alpha_2}}{\Gamma(\alpha_2+1)}\right](|x(t)-x^*(t)|+|y(t)-y_*(t)|)\\ &+\xi_1\frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)}\\ \leq&\mathfrak{L}_1(\|x(t)-x^*(t)\|+\|y(t)-y^*(t)\|)+\xi_1\frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)}\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}|y(t)-y^*(t)| =&|y(t)-\mathcal{T}_2(x^*,y^*)(t)+\mathcal{T}_2(x^*,y^*)(t)-y^*(t)|\\ \leq&|y(t)-\mathcal{T}_2(x^*(t),y^*)(t)|+|\mathcal{T}_2(x^*(t),y^*)(t)-y^*(t)|\\ \leq&\left[L_{\mathfrak{R}}\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\frac{\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_3+1)}+|\mu_2|\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}}{\Gamma(\alpha_4+1)}\right](|x(t)-x^*(t)|+|y(t)-y_*(t)|)\\ &+\xi_2\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\\ \leq&\mathfrak{R}_1(\|x(t)-x^*(t)\|+\|y(t)-y^*(t)\|)+\xi_2\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\end{aligned}\tag{4.6}$$

it can be deduced from (4.5), (4.6) that

$$\begin{aligned}\|(x,y)-(x^*,y^*)\| \leq& \xi_1\frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)}+\xi_2\frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\\ &+\mathfrak{L}_1(\|x(t)-x^*(t)\|+\|y(t)-y^*(t)\|)+\mathfrak{R}_1(\|x(t)-x^*(t)\|+\|y(t)-y^*(t)\|)\end{aligned}$$

from which we obtain

$$\begin{aligned}\|(x,y)-(x^*,y^*)\| \leq & \xi_1 \frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)} + \xi_2 \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)} \\ = & \pi_1\xi_1 + \pi_2\xi_2\end{aligned}$$

where

$$\begin{aligned}\pi_1 = & \frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\Gamma(\alpha_2+1)\Gamma(\alpha_1+1)} \\ \pi_2 = & \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\Gamma(\alpha_4+1)\Gamma(\alpha_3+1)}\end{aligned}$$

Therefore, coupled system (1.1) has a H-U stable solution via Definition 4.1. □

5. Examples

Finally, we give an example to illustrate our results.

Example 5.1. We consider the nonlinear fractional differential equation

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{0.31;\phi}\left({}^{C}\mathcal{D}_{0^+}^{0.42;\psi} - \frac{1}{10}\right)x(t) = \frac{1}{4}\arctan(\frac{1}{4}x(t)) + \frac{t^3}{16+t^2}\sin(y(t)) + \frac{t^2}{t^3+9}, 0<t<1, \\ {}^{C}\mathcal{D}_{0^+}^{0.53;\tilde{\phi}}\left({}^{RL}\mathcal{D}_{0^+}^{0.64;\tilde{\psi}} - \frac{2}{10}\right)y(t) = \frac{1}{8}\left(\cos(x(t)) + y(t)\right) + \frac{2\cos(t)}{3t^2+16}, 0<t<1, \\ x_{\phi}^{[0]}(0) = 0, {}^{C}\mathcal{D}_{0^+}^{0.42;\phi}x(0) = 1, x_{\psi}^{[0]}(0) = 0, \\ y_{\tilde{\phi}}^{[0]}(0) = 0, {}^{RL}\mathcal{D}_{0^+}^{0.64;\tilde{\phi}}y(0) = 1, y_{\tilde{\psi}}^{[0]}(0) = 0, \end{cases} \tag{5.1}$$

we have $\alpha_1 = 0.31, \alpha_2 = 0.42, \alpha_3 = 0.53, \alpha_4 = 0.64$, $\psi = \frac{t}{2}+2, \tilde{\psi} = \frac{t}{3}+3, \phi = \frac{t}{4}+4, \tilde{\phi} = \frac{t}{5}+5$, $\mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{2}$ and

$$\begin{aligned}\mathfrak{L}(t,x(t),y(t)) &= \frac{1}{4}\arctan\left(\frac{1}{4}x(t)\right) + \frac{t^3}{16+t^2}\sin(y(t)) + \frac{t^2}{t^3+9}, \\ \mathfrak{R}(t,x(t),y(t)) &= \frac{1}{8}\left(\cos\left(x(t)\right) + y(t)\right) + \frac{2\cos(t)}{3t^2+16}.\end{aligned}$$

Obviously, on can find that

$$\begin{aligned}|\mathfrak{L}(t,x_1,y_1) - \mathfrak{L}(t,x_2,y_2)| &\leq \frac{1}{16}\left(|x_1-x_2| + |y_1-y_2|\right) \\ |\mathfrak{R}(t,x_1,y_1) - \mathfrak{R}(t,x_2,y_2)| &\leq \frac{1}{8}\left(|x_1-x_2| + |y_1-y_2|\right),\end{aligned}$$

from which, we get $L_{\mathfrak{L}} = \frac{1}{16}$ and $L_{\mathfrak{R}} = \frac{1}{8}$

According the given data, the condition (3.16) reduces to

$$\begin{aligned}\mathfrak{L}_1 + \mathfrak{R}_1 &= 0.45990072062 + 0.3084997071 \\ &= 0.8014609817 < 1.\end{aligned}$$

Hence, all the assumptions of Theorem 3.2 hold. Thus, it follows that the coupled system (5.1) has a unique solution on [0; 1]. Hence condition (4.2) and (3.16) becomes

$$\pi_1 = \frac{(\psi_b(\ell_0))^{\alpha_2} (\phi_b(\ell_0))^{\alpha_1}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\,\Gamma(\alpha_2+1)\,\Gamma(\alpha_1+1)} = 3.08435823972$$
$$\pi_2 = \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4} \left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\,\Gamma(\alpha_4+1)\,\Gamma(\alpha_3+1)} = 1.33215341878.$$

It follows from Theorem 4.3 that the coupled system (5.1) is H-U stable.

Example 5.2. We consider the nonlinear fractional differential equation

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{1.22;\phi}\left({}^{C}\mathcal{D}_{0^+}^{1.44;\psi} - \frac{1}{10}\right)x(t) = \frac{1}{4+t^2}\frac{|x(t)|}{1+|x(t)|} + \frac{t}{4}\arctan(y(t)) + \frac{1}{\sqrt{400+t}}, 0<t<1,\\ {}^{C}\mathcal{D}_{0^+}^{1.66;\tilde{\phi}}\left({}^{RL}\mathcal{D}_{0^+}^{1.88;\tilde{\psi}} - \frac{2}{10}\right)y(t) = \frac{\exp^{-2t}}{25+t}\left(\sin(x(t)) + |y(t)| + \frac{1}{3}\right), 0<t<1,\\ x_{\phi}^{[0]}(0)=0, x_{\phi}^{[1]}(0)=b_1, {}^{C}\mathcal{D}_{0^+}^{1.44;\phi}x(0)=1, {}^{C}\mathcal{D}_{0^+}^{2.44;\phi}x(0)=\ell_1, x_{\psi}^{[0]}(0)=0, x_{\psi}^{[1]}(0)=b_1^*,\\ y_{\tilde{\phi}}^{[0]}(0)=0, y_{\tilde{\phi}}^{[1]}(0)=\tilde{b}_1, {}^{RL}\mathcal{D}_{0^+}^{1.88;\tilde{\phi}}y(0)=1, {}^{RL}\mathcal{D}_{0^+}^{2.88;\tilde{\phi}}y(0)=\tilde{\ell}_1, y_{\tilde{\psi}}^{[0]}(0)=0, y_{\tilde{\psi}}^{[1]}(0)=\tilde{b}_1^*, \end{cases} \tag{5.2}$$

we have $\alpha_1 = 1.22, \alpha_2 = 1.44, \alpha_3 = 1.66, \alpha_4 = 1.88$, $\psi = \frac{1}{5}(t+1)^2, \tilde{\psi} = \frac{1}{5}(t+2)^2, \phi = \frac{1}{5}(t+3)^2, \tilde{\phi} = \frac{1}{5}(t+4)^2$, $\mu_1 = \frac{3}{4}, \mu_2 = \frac{2}{3}$ and

$$\mathfrak{L}(t,x(t),y(t)) = \frac{1}{4+t^2}\frac{|x(t)|}{1+|x(t)|} + \frac{t}{4}\arctan(y(t)) + \frac{1}{\sqrt{400+t}},$$
$$\mathfrak{R}(t,x(t),y(t)) = \frac{\exp^{-2t}}{25+t}\left(\sin(x(t)) + |y(t)| + \frac{1}{3}\right).$$

After the calculation, we have

$$|\mathfrak{L}(t,x(t),y(t))| \le \frac{1}{4}|x| + \frac{1}{4}|y| + \frac{1}{20}$$
$$|\mathfrak{R}(t,x(t),y(t))| \le \frac{1}{25}|x| + \frac{1}{25}|y| + \frac{1}{75},$$

where $c_{\mathfrak{L}} = \frac{1}{4}, d_{\mathfrak{L}} = \frac{1}{4}, m_{\mathfrak{L}} = \frac{1}{20}, c_{\mathfrak{R}} = \frac{1}{25}, d_{\mathfrak{R}} = \frac{1}{25}, m_{\mathfrak{R}} = \frac{1}{75}$. Using (3.15), we find that

$$\mathring{\mho} = max\{\mathring{\mho}_1, \mathring{\mho}_2\} = max\{0.448368117, 0.5378874924\} = 0.5378874924 < 1.$$

Therefore, the assumptions of Theorem (3.3) are satisfied. Hence, the coupled system (5.2) has at least one solution on $[0;1]$.

Therefore, we have

$$|\mathfrak{L}(t,x_1,y_1) - \mathfrak{L}(t,x_2,y_2)| \le \frac{1}{4}\left(|x_1-x_2| + |y_1-y_2|\right)$$
$$|\mathfrak{R}(t,x_1,y_1) - \mathfrak{R}(t,x_2,y_2)| \le \frac{1}{25}\left(|x_1-x_2| + |y_1-y_2|\right),$$

from which, we get $L_{\mathfrak{L}} = \frac{1}{4}$ and $L_{\mathfrak{R}} = \frac{1}{25}$

From the given data, the condition (3.16) becomes

$$\begin{aligned} \mathfrak{L}_1 + \mathfrak{R}_1 &= 0.40888022176 + 0.41077964615 \\ &= 0.81965986791 < 1. \end{aligned}$$

Hence, all the assumptions of Theorem 3.2 are met. Thus it follows that the coupled system (5.2) has a unique solution on $[0;1]$. Hence condition (4.2) and (3.16) reduces to

$$\pi_1 = \frac{(\psi_b(\ell_0))^{\alpha_2}(\phi_b(\ell_0))^{\alpha_1}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\,\Gamma(\alpha_2+1)\,\Gamma(\alpha_1+1)} = 2.81929140823$$
$$\pi_2 = \frac{\left(\tilde{\psi}_b(\ell_0)\right)^{\alpha_4}\left(\tilde{\phi}_b(\ell_0)\right)^{\alpha_3}}{(1-\mathfrak{L}_1-\mathfrak{R}_1)\,\Gamma(\alpha_4+1)\,\Gamma(\alpha_3+1)} = 5.47408593156.$$

It follows from Theorem 4.3 that the coupled system (5.2) is H-U stable.

6. Conclusion

In this study, we have investigated a class of coupled nonlinear Langevin equations involving mixed generalized fractional derivatives, specifically governed by the ψ-Caputo and ϕ-Riemann–Liouville operators under mixed boundary conditions. By employing fixed point techniques, we established sufficient conditions for the existence and uniqueness of solutions. Additionally, we investigated the Hyers-Ulam stability of the considered system, thereby ensuring the robustness of the obtained solutions. Two examples are provided to verify the validity of the reported theoretical results. Hence, the results obtained in the current research extend several results reported in the previous works [25, 6] and include them as particular cases under certain conditions.

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