

COEFFICIENT INEQUALITIES FOR A CLASS OF HARMONIC UNIVALENT FUNCTIONS

O.A. FADIPE-JOSEPH^{1*}, Y.O AFOLABI² AND B.O MOSES³

ABSTRACT. In this paper, a new class of harmonic univalent function using the modified Salagean differential operator is introduced. New results on the new class are obtained. Distortion bounds, Convolution and Convex combination in the new class are also established. The work extended some earlier results .

1. INTRODUCTION AND PRELIMINARIES

Let A_λ denote the class of functions of the form:

$$f(z) = z^{\lambda+1} + \sum_{k=2}^{\infty} a_k z^{\lambda+k}, \quad 0 \leq \lambda < 1 \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

If for convenience, we set $A_0 = A$, we see that A_0 denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U}$, normalized by $f(0) = 0$ and $f'(0) = 1$.

Moreover, we have

$$f(z) \in A \implies z^\lambda f(z) \in A_\lambda, \quad 0 \leq \lambda < 1.$$

Define the Salagean operator D^n as follows:

$$D^0 f(z) = f(z)$$

$$D' f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$$

Now expanding (1.1) by Binomial theorem, we define

$$f(z)^\beta = z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}, \quad \beta > 0, \quad 0 \leq \lambda < 1 \quad (1.2)$$

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* O.A. Fadipe-Joseph.

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and

$$D^n f(z)^\beta = z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \quad (1.3)$$

Remark 1.1. If $\lambda = 0$, then $D^n f(z)^\beta$ becomes the known Salagean differential operator.

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real hamonic in D . In any simply connected domain, we can write

$$f = h + \bar{g} \quad (1.4)$$

The form (1.4) is called the local decomposition of the harmonic function f , where h and g are analytic in D . We call h the analytic part and g the coanalytic part of f .

A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'| > |g'|$ in D .

Denote by S_U , the class of functions f of the form (1.4) that are harmonic univalent and sense-preserving in the unit disk U . The subclasses of harmonic univalent functions have been studied by some authors for different purposes and different properties (Jahangiri. [4], Al-shaqsi, et al. [5]).

In particular Ezhilarasi, et al. [2], introduced a class of harmonic functions that are harmonic univalent and sense preserving, the class defined was based on a generalized fractional differential operator. Pathak, et al. [6], studied a certain subclass of harmonic univalent functions associated with Dziok-Srivastava operator. Eljamal and Darus. [1], made use of the generalized derivative operator to introduce a new class of complex valued harmonic functions. Porwal et al. [9], studied a new class of harmonic univalent functions by using fractional calculus operator. The result obtained for the class reduced to the corresponding results for several known classes. In this work, a new class of harmonic univalent functions using the modified Salagean differential operator is introduced.

We may express the analytic functions h and g as:

$$\left. \begin{aligned} h(z)^\beta &= z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \\ g(z)^\beta &= \sum_{k=1}^{\infty} \beta b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}, \quad |b_1| < 1 \end{aligned} \right\} \quad (1.5)$$

So that,

$$f(z)^\beta = h(z)^\beta + \overline{g(z)^\beta} \quad (1.6)$$

We define the modified Salagean operator of f as:

$$D^n f(z)^\beta = D^n h(z)^\beta + (-1)^n \overline{D^n g(z)^\beta}, \quad (n \in \mathbb{N}_0) \quad (1.7)$$

where,

$$\left. \begin{aligned} D^n h(z)^\beta &= z^{\beta(\lambda+1)} + \sum_{K=2}^{\infty} \beta(\lambda+k)^n a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \\ D^n g(z)^\beta &= \sum_{K=1}^{\infty} \beta(\lambda+k)^n b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \end{aligned} \right\} \quad (1.8)$$

We let $\Phi_\lambda(n, \beta, \alpha)$ be the class of harmonic functions of the form (1.6) such that

$$\operatorname{Re} \left\{ \frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta} \right\} > \alpha, \quad \beta > 0, \quad 0 \leq \alpha < 1 \quad (n \in \mathbb{N}_0) \quad (1.9)$$

where $D^n f(z)^\beta$ is given by (1.7).

The Class $\Phi_\lambda(n, \beta, \alpha)$ includes a variety of well-known subclasses of S_U .

Remark 1.2. (1) If $n = 0$, $\lambda = 0$ and $\beta = 1$, then $\Phi_0(0, 1, \alpha) \equiv S^*(\alpha)$, which is the class of sense-preserving harmonic univalent functions f which are starlike of order α in U .

(2) If $n = 1$, $\lambda = 0$ and $\beta = 1$, then $\Phi_0(1, 1, \alpha) \equiv K(\alpha)$, which is the class of sense-preserving harmonic univalent functions f which are convex of order α in U .

We also let the subclass $\overline{\Phi_\lambda(n, \beta, \alpha)}$ consists of harmonic functions $f_n(z)^\beta = h(z)^\beta + \overline{g_n(z)^\beta}$ in $\Phi_\lambda(n, \beta, \alpha)$ so that h and g are of the form:

$$\left. \begin{aligned} h(z)^\beta &= z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} \\ g_n(z)^\beta &= (-1)^n \sum_{k=1}^{\infty} \beta |b_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)}, \quad |b_1| < 1 \end{aligned} \right\} \quad (1.10)$$

2. MAIN RESULTS

We now consider some theorems on the class $\Phi_\lambda(n, \beta, \alpha)$ and its subclass.

Theorem 2.1. Let $f(z)^\beta = h(z)^\beta + \overline{g(z)^\beta}$, where $h(z)^\beta, g(z)^\beta$ are given by (1.5).

If

$\sum_{k=1}^{\infty} [(\lambda + k - \alpha)|a_k| + (\lambda + k + \alpha)|b_k|] \beta (\lambda + k)^n \leq (1 + \beta)(1 - \alpha)$
where $a_1 = 1$, $n \in \mathbb{N}_0$, $\beta \geq 1$ and $0 \leq \alpha < 1$, then $f(z)^\beta$ is univalent, sense - preserving in U and $f(z)^\beta \in \Phi_\lambda(n, \beta, \alpha)$.

Proof. Suppose $z_1^\beta \neq z_2^\beta$, then

$$\begin{aligned} \left| \frac{f(z_1)^\beta - f(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta} \right| &= \left| \frac{h(z_1)^\beta + \overline{g(z_1)^\beta} - (h(z_2)^\beta + \overline{g(z_2)^\beta})}{h(z_1)^\beta - h(z_2)^\beta} \right| \\ &= \left| \frac{h(z_1)^\beta - h(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta} + \frac{\overline{g(z_1)^\beta} - \overline{g(z_2)^\beta}}{h(z_1)^\beta - h(z_2)^\beta} \right| \\ &= \left| 1 + \frac{\overline{g(z_1)^\beta} - \overline{g(z_2)^\beta}}{h(z_1)^\beta - h(z_2)^\beta} \right| \\ &\geq 1 - \left| \frac{\overline{g(z_1)^\beta} - \overline{g(z_2)^\beta}}{h(z_1)^\beta - h(z_2)^\beta} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} [(\beta - 1)(\lambda + 1) + (\lambda + k)] b_k}{1 - \sum_{k=2}^{\infty} [(\beta - 1)(\lambda + 1) + (\lambda + k)] a_k} \\ &\geq 1 - \frac{\sum_{k=1}^{\infty} \beta (\lambda + k)^n \left(\frac{\lambda + k + \alpha}{1 - \alpha} \right) |b_k|}{1 + \beta \left[1 - (\lambda + 1)^n \left(\frac{\lambda + 1 - \alpha}{1 - \alpha} \right) \right] - \sum_{k=2}^{\infty} \beta (\lambda + k)^n \left(\frac{\lambda + k - \alpha}{1 - \alpha} \right) |a_k|} \geq 0 \end{aligned}$$

$$\implies \left| \frac{f(z_1)^\beta - f(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta} \right| > 0$$

Hence, $f(z_1)^\beta \neq f(z_2)^\beta$, which proves the univalence.

It is sufficient to show that $|h'(z)^\beta| > |g'(z)^\beta|$ for $f(z)^\beta$ to be sense-preserving. Now,

$$\begin{aligned} |h'(z)^\beta| &= \left| \left(z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right)' \right| \\ &> \beta \left((\lambda+1) - \sum_{k=2}^{\infty} \frac{(\lambda+k-\alpha)}{(1-\alpha)} \beta(\lambda+k)^n |a_k| \right) \\ &\geq \beta \left(\beta \left[\left(\frac{\lambda+1-\alpha}{1-\alpha} \right) (\lambda+1)^n - 1 \right] + \lambda + \sum_{k=1}^{\infty} \frac{(\lambda+k+\alpha)}{(1-\alpha)} \beta(\lambda+k)^n |b_k| \right) \\ &\geq \sum_{k=1}^{\infty} \beta [(\beta-1)(\lambda+1) + (\lambda+k)] |b_k| |z|^{[(\beta-1)(\lambda+1)+(\lambda+k)]-1} \geq |g'(z)^\beta| \\ &\implies |h'(z)^\beta| > |g'(z)^\beta| \end{aligned}$$

By (1.9),

$$Re \left\{ \frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta} \right\} = Re \left\{ \frac{D^{n+1}h(z)^\beta + (-1)^{n+1} \overline{D^{n+1}g(z)^\beta}}{D^n h(z)^\beta + (-1)^n \overline{D^n g(z)^\beta}} \right\} > \alpha$$

Using the fact that $Re\{w\} > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that

$$\begin{aligned} &\left| 1 - \alpha + \frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta} \right| - \left| 1 + \alpha - \frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta} \right| \\ &= \left| \frac{D^{n+1}f(z)^\beta + (1-\alpha)D^n f(z)^\beta}{D^n f(z)^\beta} \right| - \left| \frac{-D^{n+1}f(z)^\beta + (1+\alpha)D^n f(z)^\beta}{D^n f(z)^\beta} \right| \geq 0 \\ &\implies |D^{n+1}f(z)^\beta + (1-\alpha)D^n f(z)^\beta| - |D^{n+1}f(z)^\beta - (1+\alpha)D^n f(z)^\beta| \geq 0 \end{aligned}$$

Now,

$$\begin{aligned} &|D^{n+1}f(z)^\beta + (1-\alpha)D^n f(z)^\beta| - |D^{n+1}f(z)^\beta - (1+\alpha)D^n f(z)^\beta| \\ &= \left| D^{n+1}h(z)^\beta + (-1)^{n+1} \overline{D^{n+1}g(z)^\beta} + (1-\alpha) \left[D^n h(z)^\beta + (-1)^n \overline{D^n g(z)^\beta} \right] \right| \\ &\quad - \left| D^{n+1}h(z)^\beta + (-1)^{n+1} \overline{D^{n+1}g(z)^\beta} - (1+\alpha) \left[D^n h(z)^\beta + (-1)^n \overline{D^n g(z)^\beta} \right] \right| \\ &= \left| z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^{n+1} a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^{n+1} \sum_{k=1}^{\infty} \overline{\beta(\lambda+k)^{n+1} b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right. \\ &\quad \left. + (1-\alpha) \left[z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \overline{\beta(\lambda+k)^n b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right] \right| \\ &\quad - \left| z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^{n+1} a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^{n+1} \sum_{k=1}^{\infty} \overline{\beta(\lambda+k)^{n+1} b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right. \\ &\quad \left. - (1+\alpha) \left[z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \overline{\beta(\lambda+k)^n b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right] \right| \end{aligned}$$

$$\begin{aligned}
& -(1+\alpha) \left[z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \overline{\beta(\lambda+k)^n b_k z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right] \\
& = \left| (2-\lambda) z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k+1-\alpha) a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right. \\
& \quad \left. - (-1)^n \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k-1+\alpha) \overline{b_k} \overline{z}^{(\beta-1)(\lambda+1)+(\lambda+k)} \right| \\
& = \left| (-\lambda) z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k-1-\alpha) a_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right. \\
& \quad \left. - (-1)^n \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k+1+\alpha) \overline{b_k} \overline{z}^{(\beta-1)(\lambda+1)+(\lambda+k)} \right| \\
& \geq (2-\alpha) |z|^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k+1-\alpha) |a_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& \quad - \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k-1+\alpha) |b_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& \quad - \alpha |z|^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k-1-\alpha) |a_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& \quad - \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k+1+\alpha) |b_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& = 2(1-\alpha) |z|^{\beta(\lambda+1)} - 2 \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k-\alpha) |a_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& \quad - 2 \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k+\alpha) |b_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
& = 2(1-\alpha) - 2 \sum_{k=2}^{\infty} \beta(\lambda+k)^n (\lambda+k-\alpha) |a_k| - 2 \sum_{k=1}^{\infty} \beta(\lambda+k)^n (\lambda+k+\alpha) |b_k| \\
& = 2(1-\alpha) \left[1 - \sum_{k=2}^{\infty} \beta(\lambda+k)^n \left(\frac{\lambda+k-\alpha}{1-\alpha} \right) |a_k| - \sum_{k=1}^{\infty} \beta(\lambda+k)^n \left(\frac{\lambda+k+\alpha}{1-\alpha} \right) |b_k| \right]
\end{aligned}$$

The last expression is nonnegative by the hypothesis of the theorem. Thus the prove. $\square$

Corollary 2.2. *Let $f = h + \overline{g}$. If*

*$\sum_{k=1}^{\infty} [(\lambda+k-\alpha)|a_k| + (\lambda+k+\alpha)|b_k|](\lambda+k)^n \leq 2(1-\alpha)$
where $a_1 = 1$, $n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then $f(z) \in \Phi_{\lambda}(n, 1, \alpha)$.*

Proof. Put $\beta = 1$ in Theorem 2.1. $\square$

Corollary 2.3. Let $f(z)^\beta = h(z)^\beta + \overline{g(z)^\beta}$, where $h(z)^\beta, g(z)^\beta$ are given by (2.2). If $\sum_{k=1}^{\infty} [(k - \alpha)|a_k| + (k + \alpha)|b_k|] \beta k^n \leq (1 + \beta)(1 - \alpha)$ where $a_1 = 1, n \in \mathbb{N}_0, \beta \geq 1$ and $0 \leq \alpha < 1$, then $f(z)^\beta \in \Phi_0(n, \beta, \alpha)$.

Proof. Put $\lambda = 0$ in Theorem 2.1. □

Remark 2.4. This is the result obtained by Al-Shaqsi, et al. [5].

Corollary 2.5. Let $f = h + \bar{g}$. If $\sum_{k=1}^{\infty} [(k - \alpha)|a_k| + (k + \alpha)|b_k|] k^n \leq 2(1 - \alpha)$ where $a_1 = 1, n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then $f(z) \in \Phi_0(n, 1, \alpha)$.

Proof. Put $\beta = 1, \lambda = 0$ in Theorem 2.1. □

Remark 2.6. This is the result obtained by Jahangiri [4].

Corollary 2.7. Let $f = h + \bar{g}$. If $\sum_{k=1}^{\infty} [(\lambda + k - \alpha)|a_k| + (\lambda + k + \alpha)|b_k|] (\lambda + k)^n \leq 2(1 - \alpha)$ where $a_1 = 1, n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then $f(z)$ is univalent in U .

Proof. Put $\beta = 1$ in Theorem 2.1. □

Corollary 2.8. Let $f = h + \bar{g}$. If $\sum_{k=1}^{\infty} [(\lambda + k - \alpha)|a_k| + (\lambda + k + \alpha)|b_k|] (\lambda + k)^n \leq 2(1 - \alpha)$ where $a_1 = 1, n \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then $f(z)$ is sense-preserving in U .

Proof. Put $\beta = 1$ in Theorem 2.1. □

In the following theorem, it is also shown that the inequality condition in Theorem 3.1 is also necessary for functions in the subclass $\overline{\Phi_\lambda(n, \beta, \alpha)}$.

Theorem 2.9. Let $f_n(z)^\beta = h(z)^\beta + \overline{g_n(z)^\beta}$, where $h(z)^\beta, g_n(z)^\beta$ are given by (1.10). Then $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$ if and only if

$$\sum_{k=1}^{\infty} [(\lambda + k - \alpha)|a_k| + (\lambda + k + \alpha)|b_k|] \beta (\lambda + k)^n \leq (1 + \beta)(1 - \alpha) \quad (2.1)$$

where $a_1 = 1, n \in \mathbb{N}_0, \beta \geq 1$ and $0 \leq \alpha < 1$.

Proof. Since $\overline{\Phi_\lambda(n, \beta, \alpha)} \subset \Phi_\lambda(n, \beta, \alpha)$, we only need to prove the "only if" part of the theorem. To this end, for $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$.

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{D^{n+1} f_n(z)^\beta}{D^n f_n(z)^\beta} \right\} > \alpha \\ &= \operatorname{Re} \left\{ \frac{D^{n+1} f_n(z)^\beta}{D^n f_n(z)^\beta} - \alpha \right\} \geq 0 \end{aligned}$$

Here,

$$\begin{aligned} & D^n f_n(z)^\beta = D^n h(z)^\beta + (-1)^n \overline{D^n g_n(z)^\beta} \\ &= z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta(\lambda+k)^n |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^{2n} \sum_{k=1}^{\infty} \beta(\lambda+k)^n |b_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \end{aligned}$$

For the "only if" part, we show that $f_n(z)^\beta \notin \overline{\Phi_\lambda(n, \beta, \alpha)}$ if the condition (2.1) does not hold.

Now,

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{D^{n+1} f_n(z)^\beta - \alpha D^n f_n(z)^\beta}{D^n f_n(z)^\beta} \right\} \\ &= \operatorname{Re} \left\{ \frac{(1-\alpha)z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta(\lambda+k)^n(\lambda+k-\alpha)|a_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta(\lambda+k)^n|a_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right. \\ & \quad \left. - \frac{(-1)^{2n} \sum_{k=1}^{\infty} \beta(\lambda+k)^n(\lambda+k+\alpha)|b_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{+(-1)^{2n} \sum_{k=1}^{\infty} \beta(\lambda+k)^n|b_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right\} \\ &= \operatorname{Re} \left\{ \frac{(1-\alpha) - \sum_{k=2}^{\infty} \beta(\lambda+k)^n(\lambda+k-\alpha)|a_k| \frac{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{z^{\beta(\lambda+1)}}}{1 - \sum_{k=2}^{\infty} \beta(\lambda+k)^n|a_k| \frac{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{z^{\beta(\lambda+1)}}} \right. \\ & \quad \left. - \frac{(-1)^{2n} \sum_{k=1}^{\infty} \beta(\lambda+k)^n(\lambda+k+\alpha)|b_k| \frac{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{z^{\beta(\lambda+1)}}}{+(-1)^{2n} \sum_{k=1}^{\infty} \beta(\lambda+k)^n|b_k| \frac{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}{z^{\beta(\lambda+1)}}} \right\} \geq 0 \end{aligned}$$

The above required condition must hold for all values of z in U . If we choose the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum_{k=2}^{\infty} \beta(\lambda+k)^n(\lambda+k-\alpha)|a_k|r^{k-1} - \sum_{k=1}^{\infty} \beta(\lambda+k)^n(\lambda+k+\alpha)|b_k|r^{k-1}}{1 - \sum_{k=2}^{\infty} \beta(\lambda+k)^n|a_k|r^{k-1} + \sum_{k=1}^{\infty} \beta(\lambda+k)^n|b_k|r^{k-1}} \geq 0$$

If the condition (2.1) does not hold, then the numerator in the last inequality is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in the last inequality is negative. This contradicts the required condition for

$f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$ and so the proof is complete. $\square$

3. DISTORTION BOUNDS

Theorem 3.1. Let $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$. Then for $|z| = r < 1$, we have

$$\begin{aligned} & f_n(z)^\beta \leq (1 + \beta b_1) r^{\beta(\lambda+1)} \\ & + \frac{(1-\alpha)\beta}{(\lambda+2-\alpha)\beta(\lambda+2)^n} \left(1 + \beta - \left[\frac{\lambda+1-\alpha}{1-\alpha} + \frac{\lambda+1+\alpha}{1-\alpha} |b_1| \right] \beta(\lambda+1)^n \right) r^{(\beta-1)(\lambda+1)+(\lambda+2)} \end{aligned}$$

and

$$\begin{aligned} & f_n(z)^\beta \geq (1 - \beta b_1) r^{\beta(\lambda+1)} \\ & - \frac{(1-\alpha)\beta}{(\lambda+2-\alpha)\beta(\lambda+2)^n} \left(1 + \beta - \left[\frac{\lambda+1-\alpha}{1-\alpha} + \frac{\lambda+1+\alpha}{1-\alpha} |b_1| \right] \beta(\lambda+1)^n \right) r^{(\beta-1)(\lambda+1)+(\lambda+2)} \end{aligned}$$

Proof. We only prove the right-hand inequality, the prove of the left-hand inequality is similar and will be omitted.

Let $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$, on taking the absolute value, we have

$$|f_n(z)^\beta| = \left| z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta|a_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta|b_k|z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right|$$

$$\begin{aligned}
&= \left| z^{\beta(\lambda+1)} + (-1)^n \beta |b_1| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+1)}} - \sum_{k=2}^{\infty} \beta |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right. \\
&\quad \left. + (-1)^n \sum_{k=2}^{\infty} \beta |b_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right| \\
&\leq |z|^{\beta(\lambda+1)} + \beta |b_1| |z|^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta |a_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} + \sum_{k=2}^{\infty} \beta |b_k| |z|^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
&= (1 + \beta |b_1|) r^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta (|a_k| + |b_k|) r^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
&\leq (1 + \beta |b_1|) r^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} \beta (|a_k| + |b_k|) r^{(\beta-1)(\lambda+1)+(\lambda+2)} \\
&\leq (1 + \beta |b_1|) r^{\beta(\lambda+1)} + \\
&\quad \frac{(1-\alpha)\beta}{(\lambda+2-\alpha)\beta(\lambda+2)^n} \sum_{k=2}^{\infty} \left(\frac{(\lambda+2-\alpha)\beta(\lambda+2)^n}{1-\alpha} |a_k| + \frac{(\lambda+2+\alpha)\beta(\lambda+2)^n}{1-\alpha} |b_k| \right) r^{(\beta-1)(\lambda+1)+(\lambda+2)} \\
&\leq (1 + \beta |b_1|) r^{\beta(\lambda+1)} + \\
&\quad \frac{(1-\alpha)\beta}{(\lambda+2-\alpha)\beta(\lambda+2)^n} \sum_{k=2}^{\infty} \left(\frac{(\lambda+k-\alpha)\beta(\lambda+k)^n}{1-\alpha} |a_k| + \frac{(\lambda+k+\alpha)\beta(\lambda+k)^n}{1-\alpha} |b_k| \right) r^{(\beta-1)(\lambda+1)+(\lambda+2)} \\
&\leq (1 + \beta |b_1|) r^{\beta(\lambda+1)} + \\
&\quad \frac{(1-\alpha)\beta}{(\lambda+2-\alpha)\beta(\lambda+2)^n} \left(1 + \beta - \left[\frac{(\lambda+1-\alpha)}{1-\alpha} + \frac{(\lambda+1+\alpha)}{1-\alpha} |b_1| \right] \beta (\lambda+1)^n \right) r^{(\beta-1)(\lambda+1)+(\lambda+2)}
\end{aligned}$$

□

Theorem 3.2. Let $f_n(z)^\beta = h(z)^\beta + \overline{g_n(z)^\beta}$, where $h(z)^\beta, g(z)^\beta$ are given by (1.10). Then $f_n(z)^\beta \in \Phi_\lambda(n, \beta, \alpha)$ if and only if

$$f_n(z)^\beta = \sum_{k=1}^{\infty} (X_k h_k(z)^\beta + Y_k g_{nk}(z)^\beta),$$

where

$$\begin{aligned}
h_1(z)^\beta &= z^{\beta(\lambda+1)} \\
h_k(z)^\beta &= z^{\beta(\lambda+1)} - \frac{(1-\alpha)}{(\lambda+k-\alpha)(\lambda+k)^n} z^{(\beta-1)(\lambda+1)+(\lambda+k)} \quad k = 2, 3, \dots \\
g_{nk}(z)^\beta &= z^{\beta(\lambda+1)} + (-1)^n \frac{(1-\alpha)}{(\lambda+k+\alpha)(\lambda+k)^n} \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \quad k = 1, 2, \dots
\end{aligned}$$

and

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1 \quad X_k \geq 0, Y_k \geq 0$$

Proof. For functions $f_n(z)^\beta = h(z)^\beta + \overline{g_n(z)^\beta}$, we have

$$\begin{aligned}
f_n(z)^\beta &= \sum_{k=1}^{\infty} (X_k h_k(z)^\beta + Y_k g_{nk}(z)^\beta) \\
&= X_1 h_1(z)^\beta + Y_1 g_{n1}(z)^\beta + \sum_{k=2}^{\infty} (X_k h_k(z)^\beta + Y_k g_{nk}(z)^\beta) \\
&= X_1 z^{\beta(\lambda+1)} + Y_1 \left(z^{\beta(\lambda+1)} + (-1)^n \frac{(1-\alpha)}{(\lambda+1+\alpha)(\lambda+1)^n} \overline{z^{\beta(\lambda+1)}} \right) \\
&\quad + \sum_{k=2}^{\infty} \left(X_k \left[z^{\beta(\lambda+1)} - \frac{(1-\alpha)}{(\lambda+k-\alpha)(\lambda+k)^n} z^{(\beta-1)(\lambda+1)+(\lambda+k)} \right] \right. \\
&\quad \left. + Y_k \left[z^{\beta(\lambda+1)} + (-1)^n \frac{(1-\alpha)}{(\lambda+k+\alpha)(\lambda+k)^n} \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right] \right) \\
&= X_1 z^{\beta(\lambda+1)} + Y_1 z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} X_k z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} Y_k z^{\beta(\lambda+1)} \\
&\quad - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(\lambda+k-\alpha)(\lambda+k)^n} X_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \frac{(1-\alpha)}{(\lambda+1+\alpha)(\lambda+1)^n} Y_1 \overline{z^{\beta(\lambda+1)}} \\
&\quad + (-1)^n \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(\lambda+k+\alpha)(\lambda+k)^n} Y_k \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \\
&= \sum_{k=1}^{\infty} (X_k + Y_k) z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{(\lambda+k-\alpha)(\lambda+k)^n} X_k z^{(\beta-1)(\lambda+1)+(\lambda+k)} \\
&\quad + (-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)}{(\lambda+k+\alpha)(\lambda+k)^n} Y_k \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(\lambda+k-\alpha)}{1-\alpha} \beta(\lambda+k)^n |a_k| + \sum_{k=1}^{\infty} \frac{(\lambda+k+\alpha)}{1-\alpha} \beta(\lambda+k)^n |b_k| \\
&= \sum_{k=2}^{\infty} \frac{(\lambda+k-\alpha)}{1-\alpha} \beta(\lambda+k)^n \left| \frac{(1-\alpha)X_k}{(\lambda+k-\alpha)\beta(\lambda+k)^n} \right| \\
&\quad + \sum_{k=1}^{\infty} \frac{(\lambda+k+\alpha)}{1-\alpha} \beta(\lambda+k)^n \left| \frac{(1-\alpha)Y_k}{(\lambda+k+\alpha)\beta(\lambda+k)^n} \right| \\
&= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1
\end{aligned}$$

Conversely, suppose that $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$.

By setting

$$X_k = \frac{(\lambda+k-\alpha)}{1-\alpha} \beta(\lambda+k)^n |a_k|, \quad 0 \leq X_k \leq 1 \quad k = 2, 3, \dots$$

$$Y_k = \frac{(\lambda + k + \alpha)}{1 - \alpha} \beta(\lambda + k)^n |b_k|, \quad 0 \leq Y_k \leq 1 \quad k = 1, 2, \dots$$

and

$$X_1 = 1 - \sum_{k=1}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$$

therefore $f_n(z)^\beta$ can be written as

$$\begin{aligned} f_n(z)^\beta &= z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \\ &= z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta \frac{(1-\alpha)X_k}{(\lambda+k-\alpha)\beta(\lambda+k)^n} z^{(\beta-1)(\lambda+1)+(\lambda+k)} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \beta \frac{(1-\alpha)Y_k}{(\lambda+k+\alpha)\beta(\lambda+k)^n} \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \\ &= z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} (h_k(z)^\beta - z^{\beta(\lambda+1)}) X_k + \sum_{k=1}^{\infty} (g_{nk}(z)^\beta - z^{\beta(\lambda+1)}) Y_k \\ &= \sum_{k=2}^{\infty} h_k(z)^\beta X_k + \sum_{k=1}^{\infty} g_{nk}(z)^\beta Y_k + z^{\beta(\lambda+1)} \left(1 - \sum_{k=1}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \right) \\ &= X_1 z^{\beta(\lambda+1)} + \sum_{k=2}^{\infty} h_k(z)^\beta X_k + \sum_{k=1}^{\infty} g_{nk}(z)^\beta Y_k \\ &\implies f_n(z)^\beta = \sum_{k=1}^{\infty} (X_k h_k(z)^\beta + Y_k g_{nk}(z)^\beta) \end{aligned}$$

as required □

4. CONVOLUTION AND CONVEX COMBINATION

For harmonic functions

$$f_n(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}$$

and

$$F_n(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |A_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |B_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}$$

The convolution or hadamard product of $f_n(z)^\beta$ and $F_n(z)^\beta$ is given by

$$f_n(z)^\beta * F_n(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_k| |A_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| |B_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \quad (4.1)$$

Theorem 4.1. *Let $0 \leq \gamma \leq \alpha < 1$, $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$ and $F_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \gamma)}$. Then $f_n(z)^\beta * F_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)} \subset \overline{\Phi_\lambda(n, \beta, \gamma)}$*

Proof. Let the functions

$$f_n(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}$$

be in the class $\overline{\Phi_\lambda(n, \beta, \alpha)}$ and

$$F_n(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |A_k| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |B_k| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}}$$

be in the class $\overline{\Phi_\lambda(n, \beta, \gamma)}$. Then the convolution of the two functions is given by (4.1).

We wish to show that the coefficients of $f_n(z)^\beta * F_n(z)^\beta$ satisfy the required condition given in the Theorem 2.1.

For $F_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$.

Now, in view of the Theorem 2.1, for the convolution function $f_n(z)^\beta * F_n(z)^\beta$ we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{(\lambda+k-\gamma)}{1-\gamma} \beta(\lambda+k)^n |a_k| |A_k| + \sum_{k=1}^{\infty} \frac{(\lambda+k+\gamma)}{1-\gamma} \beta(\lambda+k)^n |b_k| |B_k| \\ & \leq \sum_{k=2}^{\infty} \frac{(\lambda+k-\gamma)}{1-\gamma} \beta(\lambda+k)^n |a_k| + \sum_{k=1}^{\infty} \frac{(\lambda+k+\gamma)}{1-\gamma} \beta(\lambda+k)^n |b_k| \\ & \leq \sum_{k=2}^{\infty} \frac{(\lambda+k-\alpha)}{1-\alpha} \beta(\lambda+k)^n |a_k| + \sum_{k=1}^{\infty} \frac{(\lambda+k+\alpha)}{1-\alpha} \beta(\lambda+k)^n |b_k| \leq 1 \end{aligned}$$

Since $0 \leq \gamma \leq \alpha < 1$ and $f_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)}$.

Therefore $f_n(z)^\beta * F_n(z)^\beta \in \overline{\Phi_\lambda(n, \beta, \alpha)} \subset \overline{\Phi_\lambda(n, \beta, \gamma)}$ □

We now investigate the convex combination of members of $\overline{\Phi_\lambda(n, \beta, \alpha)}$.

Let the functions $f_{ni}(z)^\beta$ be defined, for $i = 1, 2, \dots$, by

$$f_{ni}(z)^\beta = z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_{ki}| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_{ki}| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \quad (4.2)$$

Theorem 4.2. *Let the functions $f_{ni}(z)^\beta$ defined by (4.2) be in the class $\overline{\Phi_\lambda(n, \beta, \alpha)}$ for every $i = 1, 2, \dots$. Then the function $t_i(z)^\beta$ define by*

$$t_i(z)^\beta = \sum_{i=1}^m c_i f_{ni}(z)^\beta \quad (0 \leq c_i \leq 1)$$

is also in the class $\overline{\Phi_\lambda(n, \beta, \alpha)}$ where $\sum_{i=1}^m c_i = 1$

Proof. According to the definition of $t_i(z)^\beta$, we can write

$$\begin{aligned} t_i(z)^\beta &= \sum_{i=1}^m c_i f_{ni}(z)^\beta \\ &= \sum_{i=1}^m c_i \left[z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta |a_{ki}| z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta |b_{ki}| \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \right] \\ &= z^{\beta(\lambda+1)} - \sum_{k=2}^{\infty} \beta \left(\sum_{i=1}^m c_i |a_{ki}| \right) z^{(\beta-1)(\lambda+1)+(\lambda+k)} + (-1)^n \sum_{k=1}^{\infty} \beta \left(\sum_{i=1}^m c_i |b_{ki}| \right) \overline{z^{(\beta-1)(\lambda+1)+(\lambda+k)}} \end{aligned}$$

Since $f_{ni}(z)^\beta$ for every $(i = 1, 2, \dots, m)$ are in $\overline{\Phi_\lambda(n, \beta, \alpha)}$, then by (3.1), we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \left\{ \left[(\lambda + k - \alpha) \left(\sum_{i=1}^m c_i |a_{ki}| \right) + (\lambda + k + \alpha) \left(\sum_{i=1}^m c_i |b_{ki}| \right) \right] \beta (\lambda + k)^n \right\} \\ &= \sum_{i=1}^m c_i \left(\sum_{k=1}^{\infty} [(\lambda + k - \alpha) |a_{ki}| + (\lambda + k + \alpha) |b_{ki}|] \right) \\ &\leq \sum_{i=1}^m c_i (1 + \beta)(1 - \alpha) \\ &\leq (1 + \beta)(1 - \alpha) \end{aligned}$$

□

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¹ DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, ILORIN, NIGERIA.

E-mail address: famelov@unilorin.edu.ng

² DEPARTMENT OF MATHEMATICS, SOKOTO STATE UNIVERSITY, SOKOTO, NIGERIA.

E-mail address: afolaba4success@gmail.com

³ DEPARTMENT OF INDUSTRIAL MATHEMATICS, LANDMARK UNIVERSITY, OMU ARAN, NIGERIA.

E-mail address: moses.bolade@lmu.edu.ng