Gulf Journal of Mathematics ISSN: 2309-4966
Vol 21, Issue 1 (2025) 104-118 https://doi.org/10.56947/gjom.v21i1.3302

# CHARACTERIZATION RESULTS AND RIESZ SPECTRAL THEORY FOR COMPACT LINEAR RELATIONS

ABDERRAHMANE ZOUAGUI[1] and ABDELLAH GHERBI[2*]

Abstract. This paper extends Riesz's theory of compact operators to the general setting of compact linear relations. Using compact selections and studying invariant subspaces, we develop new characterizations of compactness in the multivalued setting. In doing so, we lay the groundwork for a spectral theory of compact linear relations. Our investigation encompasses key structural properties, including the behavior of the range, Aronszajn–Smith and Riesz–Schauder type results, and spectral notions such as ascent, descent, isolated spectral points, the analytic core, and the quasinilpotent part. The results highlight both deep analogies and essential differences between single-valued compact operators and their multivalued counterparts.

*Keywords.* Compact linear relation, selection, invariant subspace, spectrum, analytic core, quasinilpotent part.
*2020 Mathematics Subject Classification.* 47A06, 47A10, 47B06.

## 1. Introduction

Since their introduction by F. Riesz in the early 20th century, compact operators have held a central place in functional analysis and operator theory. By mapping bounded subsets of normed spaces into relatively compact sets, they exhibit a remarkable algebraic structure: they form a closed two-sided ideal in the algebra of bounded operators, remain invariant under addition, and can be uniformly approximated by finite-rank operators. Their spectral theory is equally elegant: on an infinite-dimensional Banach space $X$, the spectrum of compact operator is at most countable with zero as the only possible accumulation point, and each nonzero spectral value appears as an eigenvalue of finite multiplicity, with a finite-dimensional spectral subspace. These properties underpin many applications in infinite-dimensional analysis.

Over the last few decades, operator theory has been extended to the broader framework of linear relations (or relations for short), i.e., multivalued linear operators. In this setting, notions such as compactness have been generalized, yet the classical spectral theory does not always carry over in a straightforward way.

*Date*: Received: Jul 22, 2025; Accepted: Aug 19, 2025.
* Corresponding author

Recently, in [1, Theorem 3.6], the authors established a spectral theorem for compact relations under certain restrictive conditions, but left open whether such properties persist in the absence of these constraints. Moreover, classical results such as the Aronszajn–Smith and Riesz–Schauder theorems require careful reformulation. As demonstrated in Section 4, the Riesz–Schauder theorem does not extend in full generality to compact linear relations. Furthermore, several spectral notions that play a central role in the single-valued theory—such as ascent, descent, isolated points of the spectrum, the analytic core, and the quasinilpotent part—have not, to the best of our knowledge, been extensively investigated in the framework of compact linear relations. Addressing this gap constitutes another central motivation for the present work.

This paper develops a systematic extension of Riesz's compact operator theory to the setting of compact linear relations. Our first main contribution is a set of fundamental characterizations of compact relations via compact selections (Theorem 3.10, Corollary 3.11), which provide a direct bridge between the multivalued theory and its single-valued counterpart. We then establish a new version of the Aronszajn–Smith theorem for compact linear relations (Theorem 3.15). By exploring the invariant subspace problem, we prove in Theorem 3.17 a new characterization of a compact linear relation $T$ via a specific induced operator. This result extends [1, Theorem 3.5], originally proved under the restrictive assumption that $T(0)$ is a $T$-invariant subspace, to the more general situation of any closed $T$-invariant subspace $M$.

We also investigate the distribution of the spectrum for compact relations. In particular, using the concept of compact selections, we show in Theorem 4.8 that the spectral description provided in [1, Theorem 3.6] remains valid without the $T(0)$-invariance assumption. This shows that the spectrum of a compact relation shares the same structure as in the single-valued case. In addition, we identify conditions under which the Riesz–Schauder theorem remains valid in the multivalued setting (Proposition 4.1).

Building on recent results in [5, 8, 10], we establish a concise description of the isolated spectrum of compact relations. As a concluding result, Proposition 4.16 offers a new spectral decomposition of the Banach space $X$ into a sum of the analytic core and quasinilpotent part, providing deeper insight into the fine spectral structure of compact relations.

While revealing clear parallels and important differences between single-valued and multivalued compactness, this paper extends classical results and develop some new spectral and structural features for compact linear relations.

## 2. Basic notions of linear relations

For the sake of a self-contained presentation, we recall in this section several fundamental notions of linear relations, drawing from [4]. Throughout this paper, all relations considered are assumed to be linear and defined on infinite-dimensional Banach spaces $X$ and $Y$. For brevity, we will omit the term "linear" when referring to such objects, and we will use the term relation to mean a linear relation unless stated otherwise. The collection of all such relations from $X$ to $Y$ will be denoted by $LR(X,Y)$.

**Definition 2.1.** [4] A relation $T \in LR(X,Y)$ is a linear set-valued mapping from $X$ into $2^Y$, assigning to each element $x \in D(T)$ a nonempty subset $T(x) \subset Y$. For $x \notin D(T)$, we set $T(x) = \emptyset$. Under this convention, we define the domain of $T$ as $D(T) := \{x \in X : T(x) \neq \emptyset\}$, while the range and kernel are given by $R(T) := \bigcup\limits_{x\in D(T)} T(x)$, $\ker T := \{x \in D(T) : 0 \in T(x)\}$, respectively. The multivalued part of $T$, is the set $T(0) := \{y \in X : (0,y) \in T\}$, it characterizes the ambiguity of $T$ as a multivalued operator.

Note that if $x \in D(T)$ and $y \in Y$, then $y \in T(x)$ if and only if $T(x) = y+T(0)$. Hence, $T$ is an operator if and only if $T(0) = \{0\}$. Every relation $T \in LR(X,Y)$ is uniquely determined by its graph $G(T) = \{(x,y) : \ x \in D(T),\ y \in T(x)\}$. The nullity, deficiency, and index of $T$ are $\alpha(T) = \dim\ker(T)$, $\beta(T) = \dim(X \setminus R(T))$ and $\kappa(T) = \alpha(T) - \beta(T)$, respectively. If both $\alpha(T)$ and $\beta(T)$ are infinite, then $\kappa(T)$ is undefined.

**Definition 2.2.** [4] A relation $T \in LR(X,Y)$ is called injective if $\ker(T) = \{0\}$, surjective if $R(T) = Y$, and invertible if it is both injective and surjective.

Independently of whether $T$ is invertible in this sense, one may always define the inverse relation $T^{-1} \in LR(Y,X)$ by reversing the graph:

$$G(T^{-1}) = \{(y,x) \in Y \times X : (x,y) \in G(T)\}.$$

This definition yields the identities $\ker(T^{-1}) = T(0)$ and $\ker(T) = T^{-1}(0)$, which reflect a duality between the injectivity of $T$ and the multivalued nature of its inverse. In particular, $T$ is injective if and only if $T^{-1}$ is single-valued operator.

**Definition 2.3.** [4] A relation $T \in LR(X,Y)$ is said to be closed if its graph $G(T)$ is closed subspace in $X \times Y$.

In this case, both $T(0)$ and $\ker T$ are closed. The collection of all closed relations from $X$ to $Y$ is denoted by $CR(X,Y)$. In the case $X = Y$, we adopt the notation $LR(X) = LR(X,X)$ and $CR(X) = CR(X,X)$.

The notions of adjoint, restriction, sum, and product are also well-defined for linear relations.

**Definition 2.4.** [4] Let $T \in LR(X)$.

The adjoint of $T$, denoted by $T'$, is the closed relation $T' \in CR(X')$ defined by $G(T') = \{(y',x') : \ y'(y) - x'(x) = 0 \text{ for all } (x,y) \in G(T)\}$, where $X'$ denotes the dual space of $X$.

If $M \subset D(T)$ is a linear subspace of $X$, then the restriction $T_{|M}$ is defined by $G(T_{|M}) = G(T) \cap (M \times X)$. When $D(T) \cap M \neq \emptyset$, we write $T_{|M\cap D(T)} = T_{|M}$.

Let $T_1, T_2 \in LR(X)$. The sum $T_1+T_2$ is the relation on $X$ whose graph is given by $G(T_1 + T_2) := \{(x,\, y + z) : (x,y) \in G(T_1),\ (x,z) \in G(T_2)\}$. For any scalar $\lambda \in \mathbb{C}$, the relation $\lambda - T$ is defined by $G(\lambda - T) := \{(x,\, \lambda x - y) : (x,y) \in G(T)\}$.

Moreover, if $\mathrm{Ran}(T_1) \cap D(T_2) \neq \emptyset$, the product $T_2T_1 \in LR(X)$ is defined by

$$G(T_2T_1) := \{(x,\, z) : \exists\, y \in X \text{ such that } (x,y) \in G(T_1),\ (y,z) \in G(T_2)\}.$$

Since the product is associative, the powers $T^n$ for $n \in \mathbb{Z}$ are defined recursively by $T^0 = I$, $T^{n+1} = TT^n$, where $I$ is the identity.

The sequences $(\ker T^n)_{n\in\mathbb{N}}$ and $(\mathrm{Ran}T^n)_{n\in\mathbb{N}}$ are respectively increasing and decreasing. If for some $m \in \mathbb{N}$, we have $\ker T^m = \ker T^{m+1}$, then $\ker T^m = \ker T^n$ for all $n \geq m$. Similarly, if $\mathrm{Ran}T^m = \mathrm{Ran}T^{m+1}$, then $\mathrm{Ran}T^m = \mathrm{Ran}T^n$ for all $n \geq m$. These observations give rise to the ascent and descent concepts.

**Definition 2.5.** [3] Let $T \in LR(X)$, The ascent and descent of $T$ are defined, respectively, by

$$asc(T) = \min\left\{k \in \mathbb{N} :\ \ker(T^k) = \ker(T^{k+1})\right\},$$
$$des(T) = \min\left\{k \in \mathbb{N} :\ \mathrm{Ran}(T^k) = \mathrm{Ran}(T^{k+1})\right\},$$

whenever these minima exist. If no such $k$ exists, the corresponding quantity is taken to be $\infty$.

We often work with single-valued operators associated with a given relation through a process of selection. This motivates the following definition.

**Definition 2.6.** [4] A selection of $T$ is a linear operator $A$ from $X$ to $Y$ such that $T = A + T - T$ and $D(T) = D(A)$. That is, $A$ is a single-valued representative of $T$ satisfying $G(T) = G(A)\dot{+}\{0\} \times T(0)$ where $\dot{+}$ denotes the componentwise sum.

**Proposition 2.7.** [3] *If $A_1$ and $A_2$ are selections of $T_1$ and $T_2$, respectively. Then, $A_1 + A_2$ is a selection of $T_1 + T_2$. In addition, if $X = Y$ and $RanT_1 \subset D(T_2)$, then $A_2A_1$ is a selection of $T_2T_1$. Furthermore, if $D(T) = X$, then, for all $n \in \mathbb{N}$, $A_1^n$ is a selection of $T_1^n$.*

To define a norm for $T \in LR(X,Y)$, let $Q_T$ be the natural quotient map from $Y$ to $Y/\overline{T(0)}$ . Obviously, $Q_TT$ is a single-valued operator, and we define $\|Tx\| = \|Q_TTx\|,\ x \in D(T)$ and $\|T\| = \|Q_TT\|$.

**Definition 2.8.** [4] A relation $T \in LR(X,Y)$ is continuous if $\|T\| < \infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, and bounded below if it is injective and open.

The class of all bounded closed relations is denoted by $BCR(X,Y)$ or $BCR(X)$ when $X = Y$.

**Definition 2.9.** [4] A relation $T \in LR(X,Y)$ is said to be semi Fredholm, if $\mathrm{Ran}T$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. If $\alpha(T)$ and $\beta(T)$ are both finite, then we say that $T$ is Fredholm relation.

We use the notation $\Phi(X,Y)$ when we deal with closed Fredholm relations.

**Definition 2.10.** [3] Given $T \in LR(X)$, a complex number $\lambda \in \mathbb{C}$ is called an eigenvalue of $T$ if there exists $u \in X$ such that $(u, \lambda u) \in G(T)$. The set of all such eigenvalues is denoted by $\sigma_p(T)$.

The following result holds:

**Lemma 2.11.** [13] *If $A$ is a selection of $T$, then $\sigma_p(T) = \sigma_p(A)$.*

If for some $\lambda \in \mathbb{C}$, the relation $R_T(\lambda) = (T - \lambda I)^{-1}$ is an everywhere defined operator, then $\lambda$ belongs to the resolvent set $\rho(T)$ of $T$. $R_T(\lambda)$ is called the resolvent of $T$ at $\lambda$. The spectrum $\sigma(T)$ of $T$ is the complement of $\rho(T)$ in $\mathbb{C}$.

**Definition 2.12.** [8] A point $\lambda \in \sigma(T)$ is said to be isolated if there exists $\gamma > 0$ such that the only spectral point of $T$ within the open disk $\{\mu \in \mathbb{C} : |\mu - \lambda| < \gamma\}$ is $\lambda$ itself. We denote the set of all isolated spectral points of $T$ by $\sigma_{\text{iso}}(T)$.

In the context of closed and everywhere defined relations, the isolated spectrum admit a more precise description. Specifically, it was established in [8, Theorem 3.1] that if $T \in BCR(X)$, then a point $\lambda \in \sigma(T)$ is isolated if and only if $K(\lambda - T)$ is closed and $X = H_0(\lambda - T) \oplus K(\lambda - T)$, where $K(\lambda - T)$ and $H_0(\lambda - T)$ are the analytic core and the quasinilpotent part of $\lambda - T$.

According to [8], the analytic core $K(T)$ of $T \in BCR(X)$ is defined as the set of all $x \in X$ for which there exists $a > 0$ and a sequence $(u_n)_n \subset X$ satisfying $x = u_0$, $u_n \in Tu_{n+1}$ and $\text{dist}(u_n, T(0)) \leq a^n \text{dist}(x, T(0))$ for all $n \in \mathbb{N}$. While the quasinilpotent part of $T$, denoted by $H_0(T)$, is defined as the set of all $x \in X$ for which there exists a sequence $(u_n)_n \subset X$ such that $u_0 = x$, $u_{n+1} \in Tu_n$ for all $n \in \mathbb{N}$ and $\|u_n\|^{1/n} \to 0$.

**Proposition 2.13.** [10] *Let $T \in LR(X)$. The analytic core $K(T)$ and the quasinilpotent part $H_0(T)$ satisfy the following properties:*

(1) *$T(0) \subset K(T) \subseteq R^\infty(T)$, where $R^\infty(T) := \bigcup_{n \in \mathbb{N}} \text{Ran}(T^n)$.*
(2) *$T(K(T)) = K(T)$ and $T(H_0(T)) \subset H_0(T)$.*
(3) *$K(T)$ and $H_0(T)$ are generally not closed.*
(4) *If $F \subset X$ is a closed $T$-invariant subspace, then $F \subset K(T)$.*

## 3. Some characterizations of compact relations

This section investigates several characterizations of compact relations. We begin by establishing fundamental algebraic and topological properties. We then develop some characterization results in terms of compact selections and invariant subspaces.

### 3.1. **Definitions and immediately consequences.**

**Definition 3.1.** [4] A relation $T \in LR(X, Y)$ is said to be compact if $Q_T T$ is a compact operator from $X$ into $Y/\overline{T(0)}$.

It is often more convenient to consider precompact relations when $X$ and $Y$ are not complete.

**Definition 3.2.** [4] A relation $T \in LR(X, Y)$ is called precompact, if $Q_T T B_{D(T)}$ is totally bounded in $Y$, where $B_{D(T)}$ denotes the unit sphere in $D(T)$.

Equivalently, $T$ is precompact if and only if for any bounded sequence $(x_n)$ in $D(T) \subset X$, the sequence $(Q_T T x_n)$ has a Cauchy subsequence.

**Example 3.3.** Let $A$ and $B$ be two bounded operators from $X$ to $Y$, and consider the relation $T$ in $Y$ defined by $G(T) = \{(Ax, Bx) : x \in X\}$. It can be easily confirmed that $D(T) = \text{Ran}(A)$, $\text{Ran}(T) = \text{Ran}(B)$, and $T(0) = B(\ker(A))$.

If $B$ is compact, then $T$ is also compact. Indeed, let $(y_n)$ be a bounded sequence in $\mathrm{Ran}(A)$. Since $A$ is bounded, there exists a bounded sequence $(x_n)$ in $X$ such that $y_n = Ax_n$ for all $n \in \mathbb{N}$. Consequently, $Q_T T y_n = Q_T B x_n$ for all $n \in \mathbb{N}$. Since $B$ is compact, the operator $Q_T B$ is compact, as it is the product of a compact operator with a bounded one. So, the sequence $(Q_T T(y_n)) = (Q_T B x_n)$ has a convergent subsequence in $Y$, which implies that $T$ is compact.

**Example 3.4.** Continuous finite rank linear relations from $X$ to $Y$ are compact (See [4, Proposition V.1.3]. )

The following proposition shows that the class of compact relations in $BCR(X)$ behaves like an ideal.

**Proposition 3.5.** [1] *Let $T \in BCR(X)$ be compact. Then:*

(1) *$T$ is continuous. Moreover, if $D(T) = X$ then $T$ is bounded.*
(2) *Let $S, R \in BCR(X)$, if $R$ has a bounded selection then the relation $STR$ is compact.*
(3) *If $X$ is a Hilbert space, then the class of compact relations in $BCR(X)$ is a two-sided ideal.*
(4) *Scalar multiples of compact relations are compact.*
(5) *The sum of compact relations still compact.*

*Remark* 3.6. A compact relation $T$ can only be injective if $\dim D(T) < \infty$. Indeed, assume $T$ is injective. By Proposition 3.5, the identity $I_{D(T)} = T^{-1}T$ on $D(T)$ would then be compact, which is possible only when $D(T)$ is finite-dimensional.

**Proposition 3.7.** *Let $(T_n)$ be a sequence of compact relations converging to a continuous relation $T \in LR(X, Y)$ such that $\bigcup_{n=1}^{\infty} T_n(0) \subset \overline{T(0)}$. If $Y$ is complete, then $T$ is compact.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $X$. Since $T_1$ is compact, there exists a subsequence $(x_{1n})$ of $(x_n)$ for which $(Q_{T_1} T_1 x_{1n})$ converges. Applying the same argument to $T_2$, we obtain a further subsequence $(x_{2n}) \subset (x_{1n})$ such that $(Q_{T_2} T_2 x_{2n})$ converges. Continuing inductively, for each $k \in \mathbb{N}$ we construct a subsequence $(x_{kn})$ of $(x_{(k-1)n})$ for which $(Q_{T_k} T_k x_{kn})$ converges. Now define the diagonal sequence $(x_{nn})$. By construction, for each fixed $k$, the sequence $(x_{nn})$ coincides (up to a finite number of terms) with a subsequence of $(x_{kn})$, and hence $(Q_{T_k} T_k x_{nn})$ converges as $n \to \infty$. We claim that $(Q_T T x_{nn})$ also converges. To this end, it suffices to show that it is Cauchy in the quotient space $Y/\overline{T(0)}$, which is complete since $Y$ is complete. Let $\varepsilon > 0$ be fixed. Because $\|T_n - T\| \to 0$ and $\bigcup_{n=1}^{\infty} T_n(0) \subset \overline{T(0)}$, there exists $K \in \mathbb{N}$ such that $\|Q_T T - Q_{T_K} T_K\| < \frac{\varepsilon}{3M}$, where $M > 0$ is chosen so that $\|x_{nn}\| \leq M$ for all $n$. Moreover, for this $K$, there exists $N \in \mathbb{N}$ such that $\|Q_{T_K} T_K (x_{nn} - x_{mm})\| < \frac{\varepsilon}{3}$ whenever $m, n > N$. For $m, n > N$

we then have

$$\begin{aligned}\|Q_T T x_{nn} - Q_T T x_{mm}\| &\leq \|Q_T T x_{nn} - Q_{T_K} T_K x_{nn}\| + \|Q_{T_K} T_K (x_{nn} - x_{mm})\| \\ &\quad + \|Q_{T_K} T_K x_{mm} - Q_T T x_{mm}\| \\ &< \frac{\varepsilon}{3M} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} M = \varepsilon.\end{aligned}$$

Thus, $(Q_T T x_{nn})$ is a Cauchy sequence in $Y/\overline{T(0)}$, and hence convergent. We conclude that $Q_T T$ is compact, so $T$ is compact as claimed. □

In general, if $Y$ is not complete, the limit in $LR(X,Y)$ of compact relations does not need to be compact.

**Theorem 3.8.** *Let $X, Y$ be Banach spaces and $T \in LR(X,Y)$ be compact with closed range in $Y$, then $\dim Ran(T) < \infty$.*

*Proof.* Let $Y_0 = \mathrm{Ran}(T)$ and define $T_0 \in LR(X, Y_0)$ by $T_0 x = Tx$ for all $x \in X$. It is clear that $T_0$ is compact, and thus $T_0'$ is compact by [1, Proposition 3.2]. Since $\mathrm{Ran}(T_0) = Y_0$, and $Y_0$ is complete (being closed in the Banach space $Y$), it follows from [4, Proposition III.4.6(b)] that $T_0'$ is invertible. Therefore, by Remark 3.6, $\dim Y_0' < \infty$, and hence $\dim Y_0 < \infty$. □

*Remark* 3.9. When $\dim T(0) < \infty$, the relation $T$ has a closed range if and only if there exists a selection of $T$ whose range is closed.

3.2. **Characterization by means of compact selections.** We now emphasize the connection between compact operators and compact relations. Indeed, we investigate the compactness of linear relations through the concept of compact selection.

**Theorem 3.10.** *Let $T \in LR(X,Y)$. If $T(0)$ is topologically complemented in $Y$ with a topological complement $M$, then $T$ has a selection $A : X \to M$. Moreover $T$ is compact if and only if $A$ is a compact operator.*

*Proof.* If $T(0)$ is topologically complemented in $Y$, there exist a bounded projection $P$ on $Y$ and a subspace $M$ in $Y$ such that $T(0) = \ker P$ and $M = \mathrm{Ran}P$ is the topological complement of $T(0)$. Let $Q_T^M$ denote the restriction of $Q_T$ to $M$, clearly $Q_T^M : M \to Y/\overline{T(0)}$ is invertible with bounded inverse $\left(Q_T^M\right)^{-1}$. Let $A$ be a selection of $T$. Then, $Q_T T x = Q_T(Ax + T(0)) = Q_T A x = Q_T^M A x$ for all $x \in D(T)$ because $\mathrm{Ran}A \subset M$. Therefore,

$$Q_T T = Q_T^M A. \tag{3.1}$$

and

$$A = \left(Q_T^M\right)^{-1} Q_T T. \tag{3.2}$$

If $A$ is compact, then $T$ is compact by (3.1). Conversely, if $T$ is compact, then $A$ is also compact by (3.2), since the composition of a compact operator with a bounded operator remains compact. □

Theorem 3.10 reveals a number of significant observations. Firstly, as a consequence of statement (2) in Proposition 3.5, if $T$ is a compact relation with $T(0)$ topologically complemented, then the selection $PT$ is compact. Here, $P$

is the bounded projection whose kernel satisfies $\ker(P) = T(0)$. Note that $P$ is guaranteed to exist because $T(0)$ is topologically complemented.

Secondly, since every finite-dimensional subspace and every closed subspace of finite codimension in a normed space is topologically complemented, we obtain the following corollary.

**Corollary 3.11.** *Let $T \in LR(X, Y)$, if $T(0)$ is either finite-dimensional or a closed subspace of finite codimension in $Y$, then $T$ is compact if and only if it has a compact selection.*

Thirdly, in the context of Hilbert space, every closed subspace is topologically complemented by its orthogonal complement. More generally, a Banach space $X$ has the property that all its closed subspaces are complemented if and only if $X$ is isomorphic to a Hilbert space (see Paragraph II.4.3 in [4]). So, for every closed relation $T$ that acts between such spaces, $T(0)$ is topologically complemented. This observation gives rise to the following result.

**Corollary 3.12.** *Assume that $Y$ is a Hilbert space and let $T \in CR(X, Y)$, then $T$ is compact if and only if it has a compact selection. In particular, if $P$ is a continuous projection with domain Ran$(T)$ and kernel $T(0)$, then $T$ is compact if and only if the selection $PT$ is compact.*

The following theorem provides a similar result to the previous theorem when $X$ and $Y$ are arbitrary normed spaces.

**Theorem 3.13.** *Let $T \in LR(X, Y)$ with $T(0)$ compact. If $A$ is a selection of $T$, then $T$ is compact if and only if $A$ is compact.*

The proof of this theorem relies on the following lemma.

**Lemma 3.14.** [3] *Let $F \subset X$ be a compact linear subspace. Let $(x_n)$ in $X$ be a sequence such that $(Q_F x_n)$ is a convergent sequence, then $(x_n)$ has a convergent subsequence.*

*Proof of Theorem 3.13.* Let $A$ be a selection of $T$. If $A$ is compact, then $T$ is compact. Indeed, since $T(x) = A(x) + T(0)$ for all $x \in X$, we have

$$Q_T T(B_X) = Q_T\big(A(B_X) + T(0)\big) = Q_T A(B_X).$$

Since $A$ is compact, $A(B_X)$ is relatively compact, and thus so is $Q_T T(B_X)$, implying that $T$ is compact. Conversely, suppose that $T \in LR(X, Y)$ is compact. For any bounded sequence $(x_n)$ in $X$, there exists a subsequence $(x_{\varphi(n)})$ for which the sequence $(Q_T T x_{\varphi(n)})$, defined by $Q_T T x_{\varphi(n)} = Q_T(Ax_{\varphi(n)} + T(0)) = Q_T(Ax_{\varphi(n)})$ for all $n, \varphi(n) \in \mathbb{N}$, converges. Since $T(0)$ is compact, it follows from Lemma 3.14 that $(Ax_{\varphi(n)})$ has a convergent subsequence. Hence, $A$ is compact.

□

3.3. **Characterization by means of invariant subspaces.** The classic Aronszajn and Smith theorem states that every compact operator has a nontrivial closed invariant subspace. In this paragraph, our goal is to present a weaker version of this theorem for compact relations.

A subspace $M \subset X$ is invariant under a relation $T \in LR(X)$, or $T$-invariant, if $T(M) \subset M$. Similarly, $M$ is called weakly invariant under $T$, or $T$-weakly invariant, if $T(M) \cap M \neq \emptyset$. Clearly, every $T-$invariant subspace is $T-$weakly invariant. However the converse is not true. Since $T(0)$ is a linear subspace of $X$, it is trivially $T-$weakly invariant subspace. So, a $T-$weakly invariant subspace is said to be nontrivial if it is not reduced to $\{0\}$ and is not equal to $T(0)$. When $T(0)$ is invariant under $T$, we say that the relation $T$ satisfies the stabilization criterion.

**Theorem 3.15.** *Let $T \in LR(X)$ be compact. If $T(0)$ is topologically complemented in $X$, then $T$ has a nontrivial weakly invariant subspace.*

*Proof.* Let $A = PT$ be a selection of $T$, where $P$ is a bounded projection such that $\ker(P) = T(0)$. By Theorem 3.10, since $T$ is compact, the selection $A$ is compact as well. Thus, by Aronszajn's theorem, $A$ admits a nontrivial closed invariant subspace $M \subset X$, that is, $A(M) \subset M$. Since $A(x) \in T(x)$ for all $x \in X$, it follows that $A(M) \subset T(M)$, and thus $M$ is a $T$-weakly invariant subspace. Moreover, $M$ is nontrivial because $A(M) \subset \mathrm{Ran}(A) \subset \mathrm{Ran}(P)$, and by construction $\mathrm{Ran}(P)$ is a topological complement of $T(0)$. Therefore, $M$ is not contained in $T(0)$. $\square$

To further explore the compactness of a relation $T$, an alternative approach based on induced relations is adopted, under the assumption that $T$ admits a closed invariant subspace.

Let $M$ be a closed $T$-invariant subspace of $X$, i.e., $T(M) \subset M$. Denoting by $X/M$ the quotient space, we can associate with $M$ the relation

$$[T]_M : X/M \to X/M \quad \text{such that} \quad [T]_M([x]_M) = [y]_M,$$

where $[x]_M = x + M$ and $y \in Tx$.

**Lemma 3.16.** *Let $M$ be a closed $T$-invariant subspace of a Banach space $X$, and let $T \in BCR(X)$ be a compact relation. Then $Q_M T : X \to X/M$ is a compact operator.*

*Proof.* Let $(x_n)$ be a bounded sequence of $X$. Since $T$ is compact, there exists a subsequence $(x_{n_k})$ such that $(Q_T T x_{n_k})$ converges.That is, there exists $y \in X$ such that $\|Q_T T x_{n_k} - Q_T(y)\| \to 0$. Since $M$ is closed $T-$invariant subspace, we obtain $T(0) \subset M$, and $Q_M = Q_{M/T(0)}^{X/T(0)} Q_T$ (see [4, Lemma IV.5.2]). Thus, $(Q_M T x_{n_k})$ converges to $Q_M(y)$ due to the continuity of $Q_{M/T(0)}^{X/T(0)}$. Consequently, $Q_M T$ is a compact operator. $\square$

**Theorem 3.17.** *Let $M$ be a closed $T$-invariant subspace of a Banach space $X$, and let $T \in BCR(X)$. Then $T$ is compact if and only if $[T]_M$ is compact.*

*Proof.* Assume that $T$ is compact and let $([x_n])$ be a bounded sequence in $X/M$. Then, there exists $\alpha > 0$ such that $\|[x_n]\| \leq \alpha$ for all $n$. Thus, for each $n$, there exists $u_n \in M$ such that $\|x_n - u_n\| \leq \|[x_n]\| + \frac{1}{2^n} \leq \alpha + \frac{1}{2^n}$. Hence, the sequence $(x_n - u_n)$ is bounded in $X$. Since $T$ is compact, there exists a subsequence $(x_{n_k} - u_{n_k})$ such that $Q_M T(x_{n_k} - u_{n_k}) = Q_M T x_{n_k}$ converges. As $Q_M T(x_{n_k}) = [T]_M([x_{n_k}]_M)$, there is a subsequence $([x_{n_k}])$ such that $[T]_M([x_{n_k}]_M)$

converges. Therefore $[T]_M$ is compact. Conversely, suppose $[T]_M$ is compact and let $(x_n)$ be a bounded sequence in $X$. Then $([x_n]_M)$ is a bounded sequence in $X/M$ since $0 \in M$ and $\|[x_n]\| \leq \|x_n - 0\| = \|x_n\|$. Since $[T]_M$ is compact, there exists a subsequence $([x_{n_k}])$ such that $[T]_M([x_{n_k}])$ converges. For each $n_k$, $[T]_M([x_{n_k}]) = Q_M T(x_{n_k}) = Q^{X/\overline{T(0)}}_{M/\overline{T(0)}} Q_T T(x_{n_k})$, thus there is a subsequence $(x_{n_k})$ such that $Q_T T(x_{n_k})$ converges. Therefore $T$ is compact. □

## 4. Spectral theory of compact linear relations

4.1. **Riesz–Schauder Theorem.** The Riesz–Schauder spectral theorem for compact operators stands as one of the central achievements in the functional analysis of Banach spaces. This naturally raises the question of whether a similar result holds for relations. Recall that the Riesz-Schauder theorem asserts that if $T$ is compact operator, then for all $\lambda \neq 0$, $\lambda - T$ is Fredholm. In [1], the authors suggest that this result may extend to closed compact relations. However, this is not the case. To illustrate this, let $X$ be an infinite-dimensional Banach space and consider the relation $T$ defined by $G(T) = X \times X$. Since $D(T) \times X = G(T) + (\{0\} \times X)$, $\{0\} \times T(0) = G(T) \cap (\{0\} \times X)$ and $\ker(T) \times \{0\} = G(T) \cap (X \times \{0\})$, we obtain the following: on the one hand, $D(T) = \ker(T) = T(0) = X$, , implying that $T$ is bounded, compact and closed relation. In the other hand, since $T = I + T$, it follows that $I + T \notin \Phi(X)$ because $\alpha(I + T) = \infty$.

The following proposition provides a necessary and sufficient condition for which the Riesz-Schauder theorem remains valid for compact relations.

**Proposition 4.1.** *Let $T \in LR(X)$ be compact, and let $\lambda \neq 0$. Then,*

$$\lambda - T \in \Phi(X) \text{ if and only if } \dim T(0) < +\infty.$$

*Proof.* Assume that $\lambda - T \in \Phi(X)$, then $Q_T(\lambda - T) \in \Phi(X, X/\overline{T(0)})$. Hence, the operator $Q_T = \frac{1}{\lambda}(Q_T T + Q_T(\lambda - T)$ is also Fredholm. Thus, $\ker(Q_T) = \overline{T(0)}$ is finite-dimensional. Conversely, suppose that $\dim T(0) < \infty$ and let $A$ be a selection of $T$. It is easy to show that $\lambda - A$ is a selection of $\lambda - T$. Since $T$ is compact, it follows from Corollary 3.11 that $A$ is compact operator, hence for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\mathrm{Ran}(\lambda - A)$ is closed. Therefore, $\mathrm{Ran}(\lambda - T) = \mathrm{Ran}(\lambda - A) + T(0)$ is also closed since $\dim T(0) < \infty$. Now, let's check that both of $\alpha(\lambda - T)$ and $\beta(\lambda - T)$ are finite. We have in one hand,

$$\ker(\lambda - T) \subset \ker[Q_{\lambda-T}(\lambda - T)] = \ker(Q_{\lambda-T}) \cap R(\lambda - T) = T(0) \cap R(\lambda - T) \subset T(0).$$

Hence, $\alpha(\lambda - T)$ is finite. In the other hand, Since $\mathrm{Ran}(\lambda - T)$ is closed it follows from (i) of [7, Lemma 2.1.3], $\mathrm{Ran}(Q_T(\lambda - T))$ is closed. Hence, $\beta(Q_T(\lambda - T)) < \infty$. By the statement (ii) of the same lemma, we get $\beta(\lambda - T) = \beta(Q_T(\lambda - T)) < \infty$. Consequently, $\lambda - T \in \Phi(X)$. □

**Corollary 4.2.** *If $T$ is finite rank relation, then for all $\lambda \neq 0$, $\lambda - T \in \Phi(X)$.*

*Remark* 4.3. In the previous example, we observe that $T(0) = X$ is infinite-dimensional. This reveals a distinctive feature of compact relations: unlike compact operators, they may map bounded subsets into infinite-dimensional ranges due to $T(0)$. This represents a fundamental structural distinction from the single-valued case.

4.2. **Ascent and descent.** Throughout this paragraph, we are concerned with the study ascent and the descent of $\lambda - T$ when $T \in LR(X)$ is compact and $\lambda \neq 0$. The following proposition will be essential to prove Theorem 4.6 below.

**Proposition 4.4.** *Let $T \in LR(X)$ be compact. There exists no infinite chain of closed subspaces $M_n$ of $X$ such that*

$$M_n \subsetneq M_{n+1} \text{ and } (\lambda - T)(M_{n+1}) \subset M_n. \tag{4.1}$$

The proof of this result is based on the following lemma.

**Lemma 4.5.** *Let $M$ and $N$ be subspaces of a Banach space $X$, with $M$ closed in $X$ and $M \subsetneq N$. Then, for every $\epsilon > 0$, there exists a vector $y \in N$ with $\|y\| = 1$ such that* $\operatorname{dist}(y, M) > 1 - \epsilon$.

Indeed, let $y_0 \in N \setminus M$. Since $M$ is closed and $y_0 \notin M$, there exists $x_0 \in M$ such that $\alpha = \|y_0 - x_0\| > \frac{\delta}{1-\epsilon}$, where $\delta = \operatorname{dist}(y_0, M)$. Define $y = \frac{1}{\alpha}(y_0 - x_0)$. Then $y \in N$, $\|y\| = 1$, and $\operatorname{dist}(y, M) > 1 - \epsilon$.

*Proof of Proposition 4.4.* We prove the result for $n \geq 0$; the case $n < 0$ can be treated analogously.

Assume, by contradiction, that there exists an infinite chain of closed subspaces $(M_n)_{n \geq 0}$ of $X$ satisfying (4.1). By Lemma 4.5, for each $n \geq 0$ and for any $\epsilon > 0$, there exists $x_{n+1} \in M_{n+1}$ with $\|x_{n+1}\| = 1$ and $\operatorname{dist}(x_{n+1}, M_n) > 1 - \epsilon$. Since $(\lambda - T)(M_{n+1}) \subset M_n \subset M_{n+1}$, it follows that $T(M_{n+1}) \subset M_{n+1}$, i.e., each $M_n$ is invariant under $T$. Now, fix integers $0 < k < l$, and let $x_k \in M_k$, $x_l \in M_l$ be such that $\|x_k\| = \|x_l\| = 1$, and $\operatorname{dist}(x_l, M_{l-1}) > 1 - \epsilon$. We have, $(\lambda - T)x_l \subset M_{l-1}$, and $Tx_k \subset M_k \subset M_{l-1}$. Thus, $(\lambda - T)x_l \cup Tx_k \subset M_{l-1}$, and so $\operatorname{dist}(x_l, (\lambda - T)x_l \cup Tx_k) \geq \operatorname{dist}(x_l, M_{l-1}) > 1 - \epsilon$. Therefore, for all $y_k \in Tx_k$ and $y_l \in Tx_l$, we have $\|y_k - y_l\| \geq \|x_l - y_l\| - \|x_l - y_k\| > 1 - \epsilon$. This implies that $T(\overline{B_X})$ contains an infinite sequence of points with mutual distances greater or equals to $1 - \epsilon$, contradicting the compactness of $T$. Hence, no such infinite chain $(M_n)$ can exist. □

**Theorem 4.6.** *Let $T \in BCR(X)$ be compact. Then $asc(\lambda - T) < \infty$. If, in addition, $T(0)$ is relatively compact, then $des(\lambda - T) < \infty$. Furthermore, if $\rho(T) \neq \emptyset$, then $asc(\lambda - T) = des(\lambda - T) = q < \infty$.*

*Proof.* Since $T \in BCR(X)$ is compact, we obtain, due to the Newton's formula and Proposition 3.5, for every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$,

$$(\lambda - T)^n = \lambda^n - T_n, \text{ where } T_n \text{ is compact relation on } X. \tag{4.2}$$

For the clarity of the proof, let's split it into three parts.

*First part.* We prove that $M_n = \ker(\lambda - T)^n$ is closed subspace of $X$ for all $n \in \mathbb{N}$. In view of (4.2), it suffices, without loss of generality, to establish that $M_1$ is closed. Let $x \in \overline{\ker(\lambda - T)}$. Then there exists $x_m \in \ker(\lambda - T)$ such that $x_m$ converges to $x$. Since $x_m \in \ker(\lambda - T)$, we have $(x_m, 0) \in G(\lambda - T)$, and thus $(x_m, \lambda x_m) \in G(T)$. Therefore, $(x_m, \lambda x_m)$ converges to $(x, \lambda x)$. Since $T$ is closed, it follows that $(x, \lambda x) \in G(T)$. Hence $(x, 0) \in G(\lambda - T)$ and $M_1$ is a closed subspace of $X$. Consequently, $M_n$ is a chain of closed subspaces of $X$ that satisfy the conditions (4.1). If this chain is not stationary then it would be contradict the proposition 4.4. Hence there exists a smallest integer $p \in \mathbb{N} \cup \{0\}$ for which $M_{p+1} = M_p$. Therefore, $asc(\lambda - T)$ is finite.

*Second part.* Firstly, we show that $\mathrm{Ran}((\lambda - T)^n)$, $n \in \mathbb{N}$ is closed. Since $T \in BCR(X)$ is compact and $T(0)$ is relatively compact, we obtain immediately $\dim T(0) = \dim \ker(T^{-1}) < \infty$, so, by [11, Lemma 5.4 (i)] we have for every $n \in \mathbb{N}$, $\dim \ker((T^{-1})^n) = \dim T^n(0) < \infty$. Now, applying the identity (4.2) we conclude, via Proposition 4.1, that $\mathrm{Ran}((\lambda - T)^n)$ is closed for all $n \in \mathbb{N}$. To prove that $des(\lambda - T)$ is finite, we put for all $n \geq 0$, $M_{-n} = \mathrm{Ran}((\lambda - T)^n)$ and the claim follows in a similar fashion as the first part.

*Third part.* If $\rho(T) \neq \emptyset$, then by combining [12, Lemma 6.1], [2, Lemma 2.4], and statement (i) of [11, Theorem 5.7], we obtain $asc(\lambda - T) \leq des(\lambda - T)$. Since $D(T) = X$, we may further apply statement (ii) of [11, Theorem 5.7] to conclude the desired equality. □

4.3. **Spectrum of compact relation.** In this paragraph, we present some results concerning the structure and distribution of the spectrum of compact relations in Banach spaces.

**Proposition 4.7.** *Let $T$ be a compact relation on $X$. Then:*

(1) *For any nonzero eigenvalue $\lambda$ of $T$, the corresponding spectral subspace $N_\lambda = \{x \in X : (x, \lambda x) \in G(T)\}$ is finite-dimensional.*
(2) *For any $k > 0$, the set of eigenvalues of $T$ with modulus greater than $k$, is finite.*

*Proof.* (1) Suppose that $\dim N_\lambda = \infty$ and let $x_1 \in N_\lambda$ with $\|x_1\| = 1$. By Lemma 4.5, we can inductively construct an infinite sequence $(x_n) \subset N_\lambda$ such that $\|x_n\| = 1$ for all $n$ and the distance from $x_n$ to the linear span of $\{x_j : 1 \leq j \leq n-1\}$ is greater than $1 - \epsilon$. It follows that the sequence $(\lambda x_n)$ has no convergent subsequence, since $\|\lambda x_n - \lambda x_m\| > (1-\epsilon)|\lambda|, \quad m \neq n$. This cannot occur if $T$ is compact.

(2) Suppose there is an infinite sequence $(\lambda_n)$ of distinct eigenvalues of $T$ with $|\lambda_n| > k > 0$. Define $M_n = N_{\lambda_1} + N_{\lambda_2} + \cdots + N_{\lambda_n}$. Then by the first statement, each subspace $M_n$ is finite-dimensional. Also, for $n \geq 2$, $M_{n-1}$ is a proper subspace of $M_n$. To see this, suppose that $x_j \in N_{\lambda_j}$ and $\|x_j\| = 1$ $(j = 1, 2, \cdots, n)$. Suppose also that $x_1, \cdots, x_{n-1}$ are linearly independent. If $x_n = \sum_{j=1}^{n-1} \alpha_j x_j$ then $\sum_{j=1}^{n-1} \alpha_j (\lambda_n - \lambda_j) x_j \in (\lambda_n - T) x_n$, thus, $\sum_{j=1}^{n-1} \alpha_j (\lambda_n - \lambda_j) Q_T x_j = Q_T (\lambda_n - T) x_n = [0]$, since $\lambda_n - \lambda_j \neq 0$ for $j \neq n$ we have $\alpha_j = 0$ $(j = 1, 2, \cdots, n)$ and $x_n = 0$, a contradiction. By Lemma 4.5 we can choose $x_n$ in $M_n$ with $\|x_n\| = 1$ and $\mathrm{dist}(x_n, M_{n-1}) > 1 - \epsilon$ $(n \geq 2)$. Now, the sequence $(\lambda_n x_n)_{n \geq 2}$ cannot have a convergent subsequence. Indeed, for $n > m$, we have

$$\|\lambda_n x_n - \lambda_m x_m\| = |\lambda_n| \|x_n - \lambda_m \lambda_n^{-1} x_m\| \geq (1-\epsilon)|\lambda_n| > (1-\epsilon)k.$$

This contradicts the compactness of $T$, and the proof is complete. □

**Theorem 4.8.** *Let $T \in BCR(X)$ be compact. Then every nonzero point in $\sigma(T)$ is an eigenvalue of $T$, that is, $\sigma(T) = \sigma_p(T) \cup \{0\}$.*

*Proof.* We begin by noting that $0 \in \sigma(T)$. Indeed, if this were not the case, then $T$ would be invertible with bounded inverse. By Remark 3.6, this would imply that the identity operator $I = T^{-1}T$ is compact on $X$, which is impossible since $X$ is infinite-dimensional. Now, since $T \in BCR(X)$ is compact, it follows from [4, Proposition II.4.7] and Proposition 3.5 that the operator $A = PT$, where $P$ is a linear projection with domain $\mathrm{Ran}(T)$ and kernel $T(0)$, is a compact selection of $T$. Therefore, by the spectral theorem for compact operators together with Lemma 2.11, we obtain $\sigma(T) \supset \sigma(A) \supset \sigma(A) \setminus \{0\} = \sigma_p(A) = \sigma_p(T)$. □

**Corollary 4.9.** *If $X$ is a non separable Hilbert space and $T \in BCR(X)$ is compact, then $0 \in \sigma_p(T)$.*

*Proof.* The claim follows directly from Lemma 2.11 and Corollary 3.11 since the range of any compact operator is separable. □

4.4. **Analytic core and quasinilpotent part.** We now extend to compact relations a result initially established by Gong and Wang in [5] for compact operators. Our analysis begins with the following theorem.

**Theorem 4.10.** *Let $T \in BCR(X)$. If $\lambda_0 \in \sigma_{\mathrm{iso}}(T)$, then for every $\lambda \neq \lambda_0$ we have $\{0\} \subset H_0(\lambda_0 - T) \subseteq K(\lambda - T)$.*

*Proof.* Since $\lambda_0 \in \sigma_{\mathrm{iso}}(T)$, [8, Theorem 3.1 and Remark 3.1] ensure that

$K(T - \lambda_0 I)$ is closed and $X = K(\lambda_0 - T) \oplus H_0(\lambda_0 - T) = \ker(P_{\lambda_0}) \oplus \mathrm{Ran}(P_{\lambda_0})$.

Thus, for any $\lambda \neq \lambda_0$, one has $\lambda \in \rho(T_{\mathrm{Ran}(P_{\lambda_0})})$, which by [6, Theorem 49.1] implies that $(\lambda - T)_{\mathrm{Ran}(P_{\lambda_0})}$ is invertible. Hence $\mathrm{Ran}(P_{\lambda_0})$ is invariant under $\lambda - T$. Since $\mathrm{Ran}(P_{\lambda_0})$ is also closed, [10, Lemma 2.8] yields

$$\{0\} \subset H_0(\lambda_0 - T) = \mathrm{Ran}(P_{\lambda_0}) \subset K(\lambda - T).$$

□

*Remark* 4.11. Let $T \in BCR(X)$. If $\lambda \in \sigma_{\mathrm{iso}}(T) \setminus \{0\}$, then $H_0(\lambda - T)$ is invariant under $T$, i.e., $T[H_0(\lambda - T)] = H_0(\lambda - T)$.

**Proposition 4.12.** *Let $T \in BCR(X)$. If $\lambda_0 \in \sigma(T)$ satisfies $K(\lambda_0 - T) = \{0\}$, then $\lambda_0$ is the only possible isolated point in $\sigma(T)$.*

*Proof.* Suppose that $\lambda_1$ is an other isolated point such that$\lambda_1 \neq \lambda_0$. Then, by Theorem 4.10, $\{0\} \subset H_0(\lambda_1 - T) \subset K(\lambda_0 - T)$ which contradicts the hypothesis. □

This result reveals a simple condition that must be satisfied for $\sigma(T)$ to have more than one isolated point.

**Corollary 4.13.** *Let $T \in BCR(X)$. If $\sigma(T)$ has more than one isolated point, then for all $\lambda \in \mathbb{C}$, $\{0\} \subset K(\lambda - T)$.*

**Theorem 4.14.** *If $T \in BCR(X)$ is a compact, then the following are equivalent:*

(1) $0 \in \sigma_{\text{iso}}(T)$;
(2) $K(T)$ *is closed;*
(3) $\dim K(T) < \infty$;
(4) $K(T')$ *is closed;*
(5) $\dim K(T') < \infty$.

*Proof.* (1) $\implies$ (2) follows immediately from [8, Theorem 3.1]. (2) $\Rightarrow$ (3) Since $K(T)$ is closed, the restriction $T_{|K(T)}$ is compact. By [10, Proposition 2.7 (2)], this implies that $T_{|K(T)}$ is surjective. Therefore, Theorem 3.8 ensures that $K(T)$ is finite-dimensional.

(3) $\implies$ (1) Since $T_{|K(T)}$ is surjective and $K(T)$ is finite-dimensional, $T_{|K(T)}$ is invertible. So there exists $\delta > 0$ such that $(\lambda - T)_{|K(T)}$ is invertible for $|\lambda| < \delta$. On the other hand, for $\lambda \neq 0$, $\ker(\lambda - T) \subset K(T)$, so we can assert

$$\ker(\lambda - T) = \ker((\lambda - T)_{K(T)}) = \{0\} \text{ if } 0 < |\lambda| < \delta.$$

Thus, by the Fredholm Alternative we have $\text{Ran}(\lambda - T) = X$ if $0 < |\lambda| < \delta$. Consequently, $\lambda \in \rho(T)$ if $0 < |\lambda| < \delta$. Hence $0 \in \sigma_{\text{iso}}(T)$.

The equivalence with (4) and (5) follows in the same way, using [1, Proposition 3.2] and [4, Proposition VI.1.11]. □

**Corollary 4.15.** *Let $T \in BCR(X)$ be compact. Then, $\sigma(T) = \{0\}$ if and only if $K(T) = \{0\}$.*

*Proof.* If $\sigma(T) = \{0\}$, then by [9, Remark 1.1], it follows that $H_0(T) = X$. Moreover, by [8, Theorem 3.1], we have $H_0(T) \cap K(T) = \{0\}$, which implies that $K(T) = \{0\}$. Conversely, if $K(T) = \{0\}$, then by Proposition 4.12, the only possible isolated point in $\sigma(T)$ is 0. Since $T$ is compact, it follows that $\sigma(T) = \{0\}$.

□

If $T$ is a compact relation with closed range, then $0 \in \sigma_{\text{iso}}(T)$, since $\text{Ran}(T)$ is finite-dimensional by Theorem 3.8. However, the converse does not hold in general. There exist compact relations $T$ such that $\sigma(T) = \{0\}$, while the range of $T$ is not closed. For example, consider the compact operator $T$ on $\ell^1$ defined by $T(x_1, x_2, x_3, \dots) = \left(0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right)$, for all $(x_1, x_2, x_3, \dots) \in \ell^1$. This operator is injective and quasinilpotent, and thus its range is not closed. Nevertheless, 0 is the only spectral value of $T$, and it is isolated, since $K(T) = \{0\}$.

**Proposition 4.16.** *Let $T \in BCR(X)$ be compact. If there exists $\lambda_0 \neq 0$ such that $H_0(\lambda_0 - T) + H_0(T) = X$, then $0 \in \sigma_{iso}(T)$.*

*Proof.* By [8, Remark 3.1 and Theorem 4.1] together with Proposition 4.7, for any compact relation $T \in BCR(X)$ and $\lambda_0 \neq 0$, there exists an integer $\nu(T) \geq 1$ such that $K(\lambda_0 - T) = \text{ran}((\lambda_0 - T)^{\nu(T)})$ and $H_0(\lambda_0 - T) = \ker((\lambda_0 - T)^{\nu(T)})$, where $H_0(\lambda_0 - T)$ is a finite-dimensional, $T$-invariant subspace of $X$. Hence, there exists $\delta > 0$ such that $(\lambda - T)H_0(\lambda_0 - T) = H_0(\lambda_0 - T)$ for all $|\lambda - \lambda_0| < \delta$. Since for all $\lambda \neq 0$, we have $H_0(T) \subset \operatorname{ran}(\lambda - T)$, it follows that

$$X = H_0(\lambda_0 - T) + H_0(T) \subset \operatorname{ran}(\lambda - T) \quad \text{for } 0 < |\lambda - \lambda_0| < \delta.$$

Applying the Fredholm Alternative, we conclude that $\ker(\lambda - T) = \{0\}$, and hence $\lambda \in \rho(T)$ for all $\lambda$ with $0 < |\lambda - \lambda_0| < \delta$. Therefore, $\lambda_0$ is an isolated point of the spectrum $\sigma(T)$.

□

**Acknowledgement.** This work was supported by the Algerian Research Project: PRFU, no. C00L03EP310220230001.

## References

1. Acharya, K. R. & McBride, M. (2023). Compact linear relations and their spectral properties. Electronic Journal of Mathematical Analysis and Applications, 11(1), 116-126. https://doi.org/10.21608/ejmaa.2023.284576
2. Alvarez, T. (2017). Left right Browder linear relations and Riesz perturbations. Acta Mathematica Scientia, 37(5), 1-16. https://doi.org/10.1016/S0252-9602(17)30083-8
3. Ammar, A & Jeribi, A. (2021). Spectral theory of multivalued linear operators. Apple Academic Press, New York. https://doi.org/10.1201/9781003131120
4. Cross, R.W. (1998). Multivalued Linear Operators, Marcel Dekker, New York.
5. Gong, W. & Wang, L. (2002). Mbekhta's subspaces and a spectral theory of compact operators. Proceedings of the American Mathematical Society, 131(2), 587–592. https://doi.org/10.1090/S0002-9939-02-06639-X
6. Heuser, H. (1981). Functional analysis. First edition. Wiley, New York.
7. Labuschagne, L. E. (1992). Certain norm related quantities of unbounded linear operators. Mathematische Nachrichten, 157, 137-162. https://doi.org/10.1002/mana.19921570111
8. Lajnaf, M. & Mnif, M. (2020). Isolated spectral points of linear relation. Monatshefte für Mathematik., 191, 595-614. https://doi.org/10.1007/s00605-019-01366-7
9. Mbekhta, M. (1990). Sur la theorie spectrale locale et limite des nilpotents. Proceedings of the American Mathematical Society, 110, 621-631. https://doi.org/10.2307/2047902
10. Mnif, M. & Ouled-Hamed, A. (2018). Analytic core and quasi-nilpotent part of linear relations in Banach spaces. Filomat, 32(7), 2499-2515. https://doi.org/10.2298/FIL1807499M
11. Sandovici, A. & De Snoo, H. & Winkler, H. (2007). Ascent, descent,nullity, defect, and related notions for linear relations in linear space. Linear Algebra and its Applications, 423, 456-497. https://doi.org/10.1016/j.laa.2007.01.024
12. Sandovici, A. & De Snoo, H.S.V. (2009). An index formula for the product of linear relations. Linear Algebra and its Applications, 431, 2160-2171. https://doi.org/10.1016/j.laa.2009.07.011
13. Wilcox, D. (2002). Multivalued Semi Fredholm operators in normed linear spaces. Ph.D. Thesis, Univ. Cape Town.

[1] Laboratory of Fundamental and Applicable Mathematics of Oran, University of Oran 1, Ahmed Benbella, Algeria.

*Email address*: zouagui.abderrahmane@edu.univ-oran1.dz

[2] Oran's Higher School of Electrical and Energetic Engineering- Algeria. Laboratory of Fundamental and Applicable Mathematics of Oran,, University of Oran 1, Ahmed Benbella, Algeria.

*Email address*: gherbi_abdellah@esgee-oran.dz