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NEW VERSIONS OF SMALL, S-ESSENTIAL AND S-SMALL SOFT SUBMODULES

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Abstract. In this paper, we define a new version of small soft submodules by using the operation sums that were introduced by Türkmen and Pancar. Moreover, we introduce the concepts of S-small soft submodules and S-essential soft submodules. Moreover, we investigate their properties by focusing on mapping of each soft module.

Keywords. multiplicatively closed subsets, small soft modules, essential soft modules, S-essential soft submodules, S-small soft submodules
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1. Introduction

Soft set theory that deals with uncertainty in a parametric manner, is a generalization of fuzzy set theory, that was proposed by Molodtsov [4] in 1999. Later, new notions on soft set theory such as subset and super sets of soft sets, equality of two soft sets, a complement of a soft set, a null soft set, and an absolute soft set, with examples were defined in [3]. However, some operations in [3] are not generally true by counterexamples. Based on the analysis of several operations on soft sets, some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets were defined in [1]. The notion of modules is a generalization of the notion of vector spaces in which the field of scalars is replaced by a ring. Recent research in module theory appeared in [6] and [7]. In 2008, Sun et al. [8] first introduced the notion of soft modules and investigated their basic properties. Next, Türkmen and Pancar [9] defined the new sum operations in soft module theory. In 2022, Rajaee [5] generalized small and essential submodules of an R-module M to S-small and S-essential submodules of M, respectively, where $S \subseteq R$ is a multiplicatively closed subset.

Although Türkmen and Pancar introduced the concept of small soft submodules [9], we argue that their definition is not natural. In this paper, we define the notion of small soft modules by using the operation sum of soft modules [9] which

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was introduced by Türkmen and Pancar. Moreover, we define the concepts of S-essential soft submodules and S-small soft submodules. We study the properties of them. The methodology of this paper focuses on the parameter mappings of soft modules, providing a new way to explore their structure.

2. Preliminaries

Throughout this paper, we let R be an associative ring with identity and assume that all right R-modules are unitary right R-modules. Firstly, we recall some concepts of soft sets.

Definition 2.1 ([5]). A nonempty subset S of R is called a *multiplicatively closed subset* of R, if $0 \notin S, 1 \in S$ and $ss^{'} \in S$ for all $s, s^{'} \in S$.

Definition 2.2 ([4]). Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.3 ([1]). Let (F, A) and (G, B) be soft sets over a set M with $A \cap B \neq \emptyset$. The *restricted intersection* of (F, A) and (G, B) is denoted by $(F, A) \mathbin{\tilde{\cap}} (G, B)$ and is defined as $(F, A) \mathbin{\tilde{\cap}} (G, B) = (H, A \cap B)$ where $H(x) = F(x) \cap G(x)$ for all $x \in A \cap B$.

Now, we recall some concepts in soft module theory.

Definition 2.4 ([8]). A soft set (F, A) over a module M is said to be a *soft module* over M if $F(a) \leq M$ for all $a \in A$.

Definition 2.5 ([8]). Let (F, A) and (G, B) be soft modules over a module M. (G, B) is called a *soft submodule* of (F, A), denoted by $(G, B) \mathrel{\tilde{\leq}} (F, A)$ if $B \subseteq A$ and $G(b) \leq F(b)$ for all $b \in B$. If $(G, B) \mathrel{\tilde{\leq}} (F, A)$ and $(F, A) \mathrel{\tilde{\leq}} (G, B)$ two soft sets (F, A) and (G, B) over M are called *soft equal* and is written $(G, B) = (F, A)$.

Proposition 2.6. *Let (F, A) and (G, A) be soft modules over a module M. The $(F, A) \mathrel{\tilde{\leq}} (G, A)$ if and only if $F(a) \leq G(a)$ for each $a \in A$.*

Proof. It follows from Definition 2.5. □

Proposition 2.7. *Let (F, A) and (G, A) be soft modules over a module M. The $(F, A) = (G, A)$ if and only if $F = G$.*

Proof. It follows from Definition 2.5. □

Definition 2.8 ([9]). Let (F, A) be a soft module over a module M and let $\{(F_i, A_i)\}_{i\in I}$ be any collection of soft submodules $(F_i, A_i) \mathrel{\tilde{\leq}} (F, A)$, where I is a nonempty set. *The sum of soft submodules* (F_i, A_i) of (F, A) is defined as $\tilde{\sum}_{i\in I}(F_i, A_i) = (H, \bigcup_{i\in I} A_i)$ such that, for all $a \in \bigcup_{i\in I} A_i$, $H(a) = \sum_{i\in I(a)} F_i(a)$ where $I(a)$ is the set of element $i \in I$ such that $a \in A_i$.

3. Main results

Let M be a right R-module and A be a nonempty set. We defined $M : a \mapsto M$ for each $a \in A$, $0 : a \mapsto 0$ for each $a \in A$. For brevity, we will sometimes write M instead map M and write 0 instead map 0 it depended on context and situation.

Definition 3.1. Let (F_1, A_1) and (G, A_3) be soft modules over M such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. We called (F_1, A_1) is a *small soft submodule* of (G, A_3) and denoted by $(F_1, A_1) \tilde{\leq}_s (G, A_3)$ if for each soft submodule (F_2, A_2) of (G, A_3) if $(F_1, A_1) \tilde{+} (F_2, A_2) = (G|_{A_1 \cup A_2}, A_1 \cup A_2)$ then $(F_2, A_2) = (G|_{A_2}, A_2)$.

We use Proposition 2.7 to construct an equivalent definition of small soft submodule this theorem focus on map of soft module.

Theorem 3.2. *Let (F_1, A_1) and (G, A_3) be soft modules over M such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. Then $(F_1, A_1) \tilde{\leq}_s (G, A_3)$ if and only if for each soft submodule (F_2, A_2) of (G, A_3) if $\forall a \in A_1 \cup A_2$,*

$$G|_{A_1 \cup A_2}(a) = \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2 \\ F_2(a) & \text{if } a \in A_2 \setminus A_1 \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2, \end{cases}$$

then $\forall a \in A_2$, $G|_{A_2}(a) = F_2(a)$.

Proof. We assume that $(F_1, A_1) \tilde{\leq}_s (G, A_3)$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ and $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence $(F_1, A_1) \tilde{+} (F_2, A_2) = (G|_{A_1 \cup A_2}, A_1 \cup A_2)$. Since $(F_1, A_1) \tilde{\leq}_s (G, A_3)$, $(F_2, A_2) = (G|_{A_2}, A_2)$. Hence $\forall a \in A_2$, $G|_{A_2}(a) = F_2(a)$.

Conversely, we assume the hypothesis. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $(F_1, A_1) \tilde{+} (F_2, A_2) = (G|_{A_1 \cup A_2}, A_1 \cup A_2)$. Hence $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

By assumption, we have $\forall a \in A_2$, $G|_{A_2}(a) = F_2(a)$. Hence $(F_2, A_2) = (G|_{A_2}, A_2)$ as required. □

Example 3.3. Let (G, A_3) be a soft module over M. Then $(0, A_1) \tilde{\leq}_s (G, A_3)$.

Proof. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = \begin{cases} 0 & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ 0 + F_2(a) = F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence we have $\forall a \in A_2$, $G|_{A_2}(a) = F_2(a)$.

$\square$

Proposition 3.4. *Let M be an uniserial right R-module and $(F_1, A_1) \mathrel{\widetilde{\leq}} (G, A_3)$. If $F_1(a) \neq G(a)$ for each $a \in A_1$ then $(F_1, A_1) \mathrel{\widetilde{\leq}_s} (G, A_3)$.*

Proof. Let $(F_2, A_2) \mathrel{\widetilde{\leq}} (G, A_3)$ and there exists $a \in A_2$ such that $F_2(a) \neq G|_{A_2}(a)$. Hence $a \in A_1 \cap A_2$ or $a \in A_2 \setminus A_1$.

Case $a \in A_2 \setminus A_1$: We have $G|_{A_1 \cup A_2}(a) = G|_{A_2}(a) \neq F_2(a)$.

Case $a \in A_2 \cap A_1$: Since $F_1(a)$ and $F_2(a)$ are submodules of M and M is uniserial, $F_1(a) \leq F_2(a)$ or $F_2(a) \leq F_1(a)$.

If $F_2(a) \leq F_1(a)$, then $F_1(a) + F_2(a) = F_1(a) \neq (G(a) = G|_{A_1 \cup A_2}(a))$.

If $F_1(a) \leq F_2(a)$, then $F_1(a) + F_2(a) = F_2(a) \neq (G|_{A_2}(a) = G|_{A_1 \cup A_2}(a))$. $\square$

Proposition 3.5. *Let M be a right R-module and K be a submodule of M. Then K is a small submodule of M if and only if $(F_K, \{0\}) \mathrel{\widetilde{\leq}_s} (M, \{0\})$ where $F_K : 0 \mapsto K$.*

Proof. Assume that K is a small submodule of M. Let $(F_2, \{0\}) \mathrel{\widetilde{\leq}} (M, \{0\})$ such that $M = F_K(0) + F_2(0)$. Since $F_2(0) \leq M$ and $F_k(0) = K$, $F_2(0) = M$. To prove the converse, we assume that $(F_K, \{0\}) \mathrel{\widetilde{\leq}} (M, \{0\})$. Let $L \leq M$ such that $L + K = M$. So $F_L(0) + F_K(0) = M$ where $F_L : 0 \mapsto L$. It follows by assumption that $F_L(0) = M$. Hence $L = M$. $\square$

Example 3.6. Let $M = \mathbb{Z}_4$ by consider M as a right $\mathbb{Z}$-module and

$$H_1 : 0 \mapsto <\overline{2}>,$$
$$H_2 : 0 \mapsto <\overline{2}>, 1 \mapsto <\overline{2}>,$$
$$H_3 : 0 \mapsto \{\overline{0}\}, 1 \mapsto M,$$
$$H_4 : 0 \mapsto M, 1 \mapsto 0.$$

Then $(H_1, \{0\})$ and $(H_2, \{0, 1\})$ are soft small submodules of $(M, \{0, 1\})$ but $(H_3, \{0, 1\})$ is not a soft small submodule of $(M, \{0, 1\})$.

Proof. By Proposition 3.5, we have that $(H_1, \{0\})$ and $(H_2, \{0, 1\})$ are soft small submodules of $(M, \{0, 1\})$. Next, we will show that $(H_3, \{0, 1\})$ are not soft small submodule of $(M, \{0, 1\})$. We choose $(H_4, \{0, 1\})$. It is clear that $H_4 \neq M$. We observe that

$$M = \begin{cases} H(a) + H(a) = \mathbb{Z}_4 & \text{if } a = 0, \\ H(a) + H(a) = \mathbb{Z}_4 & \text{if } a = 1. \end{cases}$$

$\square$

Lemma 3.7. *Let M be a nonzero right R-module and $A \subseteq A_3$. If $G|_A \neq 0$, then $(G|_A, A) \mathrel{\widetilde{\not\leq}_s} (G, A_3)$.*

Proof. Assume that $G|_A \neq 0$. Hence there exists $a \in A$ such that $G|_A(a) \neq 0$. Let $F_2 : a \mapsto 0$. So $(F_2, \{a\}) \mathrel{\widetilde{\leq}} (G, A_3)$ and $(F_2, \{a\}) \neq (G, A_3)$. We observe that

$$G|_{A\cup\{a\}}(b) = \begin{cases} G(b) & \text{if } b \neq a, \\ G(b) + F_2(b) = G(b) + 0 = G(b) & \text{if } b = a. \end{cases}$$

We are done. □

Theorem 3.8. *Let M be a right R-module and $A \subseteq A_3$. Then $G|_A = 0$ if and only if $(G|_A, A) \tilde{\leq}_s (G, A_3)$.*

Proof. It is obvious since $(0, A) \tilde{\leq}_s (G, A_3)$ and by Lemma 3.7. □

Remark 3.9. For each $G \neq 0$, we obtain that $(G, A) \tilde{\not\leq}_s (G, A_3)$.

Remark 3.10. Let M be a right R-module and S be a multiplicatively closed subset of R. If (F, A) is a soft module over M and $s \in S$, then $(F, A)s = (Fs, A)$ where $Fs : a \mapsto F(a)s$.

Note when we deal with S-small soft submodules, we will consider R to be a commutative ring with identity.

Definition 3.11. Let M be a right R-module and S be a multiplicatively closed subset of a ring R. Let (F, A) be a soft module over M. We called (F_1, A_1) is an *S-small soft submodule* of (G, A_3) and denoted by $(F_1, A_1) \tilde{\leq}^s_s (G, A_3)$ if, for each $(F_2, A_2) \tilde{\leq} (G, A_3)$ and for each $s \in S$, $(G|_{A_1\cup A_2}, A_1 \cup A_2)s \tilde{\leq} (F_1, A_1) \tilde{+} (F_2, A_2)$ then there exists $t \in S$ such that$(G|_{A_2}, A_2)t \tilde{\leq} (F_2, A_2)$.

We use Proposition 2.6 to construct an equivalent definition of S-small soft submodule this theorem focus on map of soft module.

Theorem 3.12. *Let (F_1, A_1) and (G, A_3) be soft modules such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. Then $(F_1, A_1) \tilde{\leq}^s_s (G, A_3)$ if and only if for each soft submodule (F_2, A_2) of (G, A_3) for each $s \in S$ if $\forall a \in A_1 \cup A_2$,*

$$G|_{A_1\cup A_2}(a)s \leq \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2, \end{cases}$$

then $\exists t \in S \forall a \in A_2$, $G|_{A_2}(a)t \leq F_2(a)$.

Proof. We use similar technique of Theorem 3.2. □

Example 3.13. Let (G, A_3) be a soft module over M and S be a multiplicatively closed subset of R. Then $(0, A_1) \tilde{\leq}^s_s (G, A_3)$ where $A_1 \subseteq A_3$.

Proof. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ and $s \in S$ such that $\forall a \in A_1 \cup A_2$,

$$G|_{A_1\cup A_2}(a)s \leq \begin{cases} F_1(a) = 0 & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) = 0 + F_2(a) = F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence, we have $G|_{A_2}(a)s \leq F_2(a)$ for all $a \in A_2$. □

Example 3.14. Consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}_6$-module, $A_3 = \{\overline{0}, \overline{1}\}$ and $S = \{\overline{1}, \overline{2}, \overline{4}\}$. We have $(F_1, A_3) \overset{\sim s}{\leq_s} (M, A_3)$ and $(F_3, A_3) \overset{\sim s}{\nleq_s} (M, A_3)$ where $F_1 : 0 \mapsto <\overline{3}>, 1 \mapsto <\overline{3}>$ and $F_3 : 0 \mapsto <\overline{2}>, 1 \mapsto <\overline{2}>$.

Proof. First, we will show that $(F_1, A_3) \overset{\sim s}{\leq_s} (M, A_3)$. Let $(F_2, A_2) \tilde{\leq} (M, A_3)$ and $s \in S$ such that $\forall a \in A_1 \cup A_2$,

$$Ms \leq \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence in this case by condition of above hypothesis, it suffices to consider only case $A_2 = A_3$. Since $A_2 = A_3$, we reduce our hypothesis to $Ms \leq F_1(a) + F_2(a)$ for each $a \in A_3$. Since Ms can be M or $<\overline{2}>$, $F_i(j)$ can be M or $<\overline{2}>$ for $i, j \in \{1, 2\}$. Hence we can choose $t = \overline{2}$ so $Mt \leq F_2(a)$ for each $a \in A_2$. Second, we will show that $(F_3, A_3) \overset{\sim s}{\nleq_s} (M, A_3)$. We choose $(0, A_3)$ and $s = \overline{2}$ so for each $a \in A_3$, $Ms \leq F_3(a) + F_2(a)$. But for each $t \in S$, there exists $t \in A_2$ such that $Mt \nleq F_2(a)$. □

Proposition 3.15. *Let S be a multiplicatively closed subset of a ring R and K be a submodule of M. Then K is an S-small submodule of M if and only if $(F_K, \{0\}) \overset{\sim s}{\leq_s} (M, \{0\})$ where $F_K : 0 \mapsto K$.*

Proof. Let K be a S-small submodule of M. Let $(F_2, \{0\}) \tilde{\leq} (M, \{0\})$ such that $Ms \leq F_K(0) + F_2(0)$. Since $F_2(0) \leq M$ and $F_2(0) = K$, there exists $t \in S$ such that $F_2(0) = M$. Conversely, we assume $(F_K, \{0\}) \overset{\sim s}{\leq_s} (M, \{0\})$. Let $L \leq M$ and $s \in S$ such that $Ms \leq L + K$ so $Ms \leq F_L(0) + F_K(0)$ where $F_L : 0 \mapsto L$. It follows by assumption that there exists $t \in S$ such that $Mt \leq F_L(0)$. Hence $Mt \leq L$.

□

Proposition 3.16. *Let M be a right R-module, S be a multiplicatively closed subset of R and $S \subseteq U(R)$ where $U(R)$ is the set of all unit elements in R. Then $(F_1, A_1) \tilde{\leq}_s (G, A_3)$ if and only if $(F_1, A_1) \overset{\sim s}{\leq_s} (G, A_3)$.*

Proof. We will show that $(F_1, A_3) \overset{\sim s}{\leq_s} (G, A_3)$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ and $s \in S$ such that $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a)s \leq \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2. \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Since $S \subseteq U(R)$, $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = G|_{A_1 \cup A_2}(a)s \leq \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

It follows from assumption that $\exists t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a)t = F_2(a)$. Hence there exists $t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a)t \leq F_2(a)$. Conversely, we will show that $(F_1, A_3) \mathrel{\tilde{\leq}_s} (G, A_3)$. Let $(F_2, A_2) \mathrel{\tilde{\leq}} (G, A_3)$ and $s \in S$ such that $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a) = \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

Hence $\forall a \in A_1 \cup A_2$,

$$G|_{A_1 \cup A_2}(a)(1) \leq \begin{cases} F_1(a) & \text{if } a \in A_1 \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A_1, \\ F_1(a) + F_2(a) & \text{if } a \in A_1 \cap A_2. \end{cases}$$

It follows from assumption that there exists $t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a)t \leq F_2(a)$. Since $S \subseteq U(R)$, $\forall a \in A_2$, $G|_{A_2}(a) = G|_{A_2}(a)t \leq F_2(a)$. Thus there exists $t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a) = F_2(a)$.

□

Theorem 3.17. *Let M be a right R-module, S be a multiplicatively closed subset of R and (G, A_3) be a soft module over M. Then the following statements are equivalent.*

(i) $(G, A_3) \mathrel{\tilde{\leq}_s^s} (G, A_3)$.
(ii) $\forall (\emptyset \neq A \subseteq A_3), (G|_A, A) \mathrel{\tilde{\leq}_s^s} (G, A_3)$.
(iii) $\forall ((F, A) \mathrel{\tilde{\leq}} (G, A_3)), (F, A) \mathrel{\tilde{\leq}_s^s} (G, A_3)$.
(iv) $\forall ((0, A) \neq (F, A) \mathrel{\tilde{\leq}} (G, A_3)), (F, A) \mathrel{\tilde{\leq}_s^s} (G, A_3)$.
(v) $\forall ((F, A) \mathrel{\tilde{\leq}} (G, A_3)) \exists t \in S, (G|_A, A)t \mathrel{\tilde{\leq}} (F, A)$.
(vi) $\forall (\emptyset \neq A \subseteq A_3), \bigcap_{a \in A} Ann_R(G|_A(a)) \cap S \neq 0$.
(vii) $\bigcap_{a \in A} Ann_R(G(a)) \cap S \neq 0$.

Proof. (i) → (ii): Assume (i). Let $(G, A_3) \mathrel{\tilde{\leq}_s^s} (G, A_3)$ and $\emptyset \neq A \subseteq A_3$. Let $(F_2, A_2) \mathrel{\tilde{\leq}} (G, A_3)$ and $s \in S$ such that $\forall a \in A \cup A_2$,

$$G|_{A \cup A_2}(a)s \leq \begin{cases} G(a) & \text{if } a \in A \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A, \\ G(a) + F_2(a) = G(a) & \text{if } a \in A \cap A_2 = A_2. \end{cases}$$

By assumption, we have $\forall a \in A_3$,

$$G|_{A_3 \cup A_2}(a)s \leq \begin{cases} G(a) & \text{if } a \in A_3 \setminus A_2, \\ G(a) + F_2(a) = G(a) & \text{if } a \in A_3 \cap A_2. \end{cases}$$

Hence there exists $t \in S$ such that $\forall a \in A_2$, $(G|_{A_2}(a)t \leq (F_2, A_2)$.

(ii) $\to$ (iii): Assume (ii). Suppose that $(F, A) \tilde{\leq} (G, A_3)$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$, $s \in S$ such that $\forall a \in A \cup A_2$,

$$G|_{A \cup A_2}(a)s \leq \begin{cases} F(a) & \text{if } a \in A \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A, \\ F(a) + F_2(a) & \text{if } a \in A \cap A_2. \end{cases}$$

Since $(G|_A, A) \tilde{\leq}_s^s (G, A_3)$ and $\forall a \in A \cup A_2$,

$$G|_{A \cup A_2}(a)s \leq \begin{cases} G(a) & \text{if } a \in A \setminus A_2, \\ F_2(a) & \text{if } a \in A_2 \setminus A, \\ G(a) + F_2(a) = G|_A(a) & \text{if } a \in A \cap A_2, \end{cases}$$

there exists $t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a)t \leq F_2(a)$.

(iii) $\to$ (iv): It is obvious.

(iv) $\to$ (v): Assume (iv). Let $(F_2, A_2) \tilde{\leq} (G, A_3)$. If $G = 0$ we are done. We consider $G \neq 0$. By assumption, we have $(G, A_3) \tilde{\leq}_s^s (G, A_3)$. Since

$$G(a)s \leq \begin{cases} G(a) & \text{if } a \in A_3 \setminus A_2, \\ G(a) + F_2(a) = G(a) & \text{if } a \in A_3 \cap A_2, \end{cases}$$

there exists $t \in S$ such that $\forall a \in A_2$, $G|_{A_2}(a)t \leq F_2(a)$. Hence $(G|_{A_2}, A)t \tilde{\leq} (F, A_2)$.

(v) $\to$ (vi): Assume (v). Let $\emptyset \neq A \subseteq A_3$. By (vi) there exists $t \in S$ such that $G|_A(a)t = 0$ for each $a \in A$. We are done.

(vi) $\to$ (vii): It is clear by choose $A = A_3$.

(vii) $\to$ (i): Assume (vi) It follows by assumption that there exists $t \in S$ such that $G(a)t = 0$ for each $a \in A_3$ so we have $G|_A(a)t \leq L$ for each $a \in A$ for each $L \leq G(a)$ where $A \subseteq A_3$. $\square$

Example 3.18. Let $M = \mathbb{Z}_6$ as a right $\mathbb{Z}$-module and $S = \{1, 6, 9, 12, 15, 18, ...\}$ be a multiplicatively closed subset of $\mathbb{Z}$. Let $G : 0 \mapsto < \overline{3} >$, $1 \mapsto < \overline{3} >$. Hence $(G, \{0, 1\}) \tilde{\leq}_s^s (G, \{0, 1\})$.

Proof. Since $\bigcap_{a \in A} Ann_R(G(a)) \cap S \neq 0$ where $A = \{0, 1\}$, by Theorem 3.17, $(G, \{0, 1\}) \tilde{\leq}_s^s (G, \{0, 1\})$. $\square$

Remark 3.19. Example 3.18 show that (G, A_3) can be S-small soft submodule of (G, A_3) this generalization of small soft submodule can do but in small soft module (G, A_3) can not be small soft submodule of (G, A_3) when $G \neq 0$.

Now, we recall the definition of soft essential submodules in [2].

Definition 3.20. [2] Let M be a right R-module, and (F_1, A_1) and (G, A_3) be soft modules over M such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. We called (F_1, A_1) is a *soft essential submodule* of (G, A_3) and denoted by $(F_1, A_1) \tilde{\leq}_e (G, A_3)$ if for each soft submodule (F_2, A_2) of (G, A_3) such that $A_2 \cap A_1 \neq \emptyset$ if $(F_1, A_1) \tilde{\cap} (F_2, A_2) = (0|_{A_1 \cap A_2}, A_1 \cap A_2)$ then $(F_2, A_2) = (0|_{A_2}, A_2)$.

Theorem 3.21. *Let M be a right R-module, and (F_1, A_1) and (G, A_3) be soft modules over M such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. Then $(F_1, A_1) \tilde{\leq}_e (G, A_3)$ if and only if for each soft submodule (F_2, A_2) of (G, A_3) such that $A_2 \cap A_1 \neq \emptyset$ if $\forall a \in A_1 \cap A_2$, $F_1(a) \cap F_2(a) = 0$ then $\forall a \in A_2$, $F_2(a) = 0$.*

Proof. We use similar technique of Theorem 3.2. □

Example 3.22. Let M be a right R-module and (G, A_3) be a soft module over M. Then $(G, A_3) \tilde{\leq}_e (G, A_3)$.

Proof. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A_2 \cap A_3 \neq \emptyset$ and $\forall a \in A_3 \cap A_2 = A_2$, $F_2(a) = G(a) \cap F_2(a) = 0$. Hence we have $F_2(a) = 0$ for all $a \in A_2$.

□

Proposition 3.23. *Let M be a nonzero uniserial right R-module. Assume that $(F, A) \tilde{\leq} (G, A)$ are soft modules over M. If $\forall a \in A$, $G(a) \neq 0$ and for each $a \in A$, $F_1 : a \mapsto 0 \neq L_a \leq G(a)$ then $(F_1, A) \tilde{\leq}_e (G, A)$.*

Proof. Let $(F_2, A_2) \tilde{\leq} (G, A)$ such that $A \cap A_2 \neq \emptyset$. We assume that $\forall a \in A \cap A_2 = A_2$, $F_1(a) \cap F_2(a) = 0$. Since $F_1(a) = L_a$ and $0 \neq L_a \leq G(a) \leq M$, $F_2(a) \leq L_a$. Hence $F_2(a) = 0$ for each $a \in A_2$.

□

Proposition 3.24. *Let M be a right R-module and K be a submodule of M. Then K is an essential submodule of M if and only if $(F_K, \{0\}) \tilde{\leq}_e (M, \{0\})$ where $F_K : 0 \mapsto K$.*

Proof. Let K be an essential submodule of M. Let $(F_2, \{0\}) \tilde{\leq} (M, \{0\})$ such that $0 = F_K(0) \cap F_2(0)$. Since $F_2(0) \leq M$ and $F_k(0) = K$, $F_2(0) = K$. Conversely, let $(F_K, \{0\}) \tilde{\leq} (M, \{0\})$. Let $L \leq M$ such that $L \cap K = 0$ so $F_L(0) \cap F_K(0) = 0$ where $F_L : 0 \mapsto L$. It follows by assumption that $F_L(0) = 0$. Hence $L = 0$. □

Example 3.25. Let $M = \mathbb{Z}_{12}$ be a right $\mathbb{Z}$-module. Then $(F_1, \{0, 1\}) \tilde{\leq}_e (M, \{0, 1\})$ and $(F_3, \{0\}) \tilde{\nleq}_e (M, \{0, 1\})$ where $F_1 : 0 \mapsto < \overline{6} >$, $1 \mapsto < \overline{6} >$, $F_3 : 0 \mapsto < \overline{2} >$.

Proof. We will show that $(F_1, \{0, 1\}) \tilde{\leq}_e (M, \{0, 1\})$. Let $(F_2, A_2) \tilde{\leq} (M, \{0, 1\})$ such that $A_2 \cap \{0, 1\} \neq \emptyset$. We assume that $\forall a \in \{0, 1\} \cap A_2$, $F_1(a) \cap F_2(a) = 0$. Since $< \overline{6} >$ is an essential submodule of $\mathbb{Z}_{12}$, $F_2(a) = 0$ for each $a \in A_2$. Next, we will show that $(F_3, \{0\}) \tilde{\nleq}_e (M, \{0, 1\})$. We can choose $(F_2, \{0\}) \tilde{\leq} (M, \{0, 1\})$ where $F_2 : 0 \mapsto < \overline{3} >$. Hence $\forall a \in \{0\}$, $F_2(a) \cap F_3(a) = 0$ but $F_2(a) \neq 0$ for each $a \in \{0\}$. □

Theorem 3.26. *Let M be a right R-module and $A \subseteq A_3$. Then $G = 0$ if and only if $(0, A) \tilde{\leq}_e (G, A_3)$.*

Proof. $(\rightarrow)$ It is obvious.
$(\leftarrow)$ Let $(0, A) \tilde{\leq}_e (G, A_3)$. Since $(G, A_3) \tilde{\leq} (G, A_3)$ and for each $a \in A \cap A_3 = A$, $0 \cap G(a) = 0$, $G(a) = 0$ for each $a \in A_3$. □

Remark 3.27. For each $G \neq 0$ we obtain $(0, A_3) \tilde{\nleq}_e (G, A_3)$.

Definition 3.28. Let M be a right R-module and S be a multiplicatively closed subset of R. Let (F, A) be a soft module over M. We call that (F_1, A_1) is an S-*essential soft submodule* of (G, A_3) and denoted by $(F_1, A_1) \tilde{\leq}_e^s (G, A_3)$ if for each $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A_1 \cap A_2 \neq \emptyset$ for each $s \in S$ if $(F_1, A_1) \tilde{\cap} (F_2, A_2) = (0|_{A_1 \cap A_2}, A_1 \cap A_2)$ then there exists $t \in S$ such that $(F_2, A_2)t = (0|_{A_2}, A_2)$.

We use Proposition 2.7 to construct an equivalent definition of small soft submodules. The next theorem focus on map of soft modules.

Theorem 3.29. *Let (F_1, A_1) and (G, A_3) be soft modules over M such that $(F_1, A_1) \tilde{\leq} (G, A_3)$. Then $(F_1, A_1) \tilde{\leq}_e^s (G, A_3)$ if and only if for each soft submodule (F_2, A_2) of (G, A_3) such that $A_1 \cap A_2 \neq \emptyset$ if $\forall a \in A_1 \cap A_2$, $F_1(a) \cap F_2(a) = 0$ then $\exists t \in S \forall a \in A_2$, $F_2(a)t = 0$.*

Proof. We use similar technique of Theorem 3.2.

□

Proposition 3.30. *Let M be a right R-module, and (F_1, A_1) and (G, A_3) be soft modules over M. If $(F_1, A_1) \tilde{\leq}_e (G, A_3)$ then $(F_1, A_1) \tilde{\leq}_e^s (G, A_3)$.*

Proof. Let $(F_1, A_1) \tilde{\leq}_e (G, A_3)$ and $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A_1 \cap A_3 \neq \emptyset$. We assume that $\forall a \in A_1 \cap A_2$, $F_1(a) \cap F_2(a) = 0$. It follows by assumption that $\forall a \in A_2$, $F_2(a)(1) = 0$.

□

Example 3.31. Let M be a right R-module, and (F_1, A_1) and (G, A_3) be soft modules over M. Then $(G, A_3) \tilde{\leq}_e^s (G, A_3)$.

Proof. It follow by Proposition 3.30 and remark 3.22. □

Example 3.32. Let $M = \mathbb{Z}_6$ as a right $\mathbb{Z}$-module and $S = \{1, 3^1, 3^2, 3^3, 3^4, 3^5, ...\}$ be a multiplicatively closed subset of $\mathbb{Z}$. Let $F_1 : 0 \mapsto < \overline{3} >$. Hence, $(F_1, \{0\}) \tilde{\leq}_e^s (M, \{0, 1\})$. Let $F_3 : 0 \mapsto < \overline{2} >$. Hence, $(F_3, \{0\}) \tilde{\nleq}_e^s (M, \{0, 1\})$.

Proof. We will show that $F_1 : 0 \mapsto < \overline{3} >$. Let $(F_2, A_2) \tilde{\leq} (M, \{0, 1\})$ such that $\{0\} \cap A_2 \neq \emptyset$. We assume that $\forall a \in \{0\} \cap A_2$, $F_1(a) \cap F_2(a) = 0$. Hence, $F_2(0) = 0$ or $F_2(0) = < \overline{2} >$. So we can choose $t = 3$ such that $F_2(0)t = 0$.
We will show that $F_3 : 0 \mapsto < \overline{2} > \tilde{\nleq}_e^s (M, \{0, 1\})$. We choose $F_1 : 0 \mapsto < \overline{3} >$. Thus $(F_2, \{0\}) \tilde{\leq} (F_1, \{0\}) = (0, \{0\})$ but $\forall t \in S \forall a \in \{0\}$, $F_2(a)t = < \overline{3} > \neq 0$. □

Remark 3.33. Example 3.32 show the converse of Proposition 3.30 is not necessarily true in general. Let $M = \mathbb{Z}_6$ as a right $\mathbb{Z}$-module, $S = \{1, 3^1, 3^2, 3^3, 3^4, 3^5, ...\}$ be a multiplicatively closed subset of $\mathbb{Z}$. Let $F_1 : 0 \mapsto <\overline{3}>$, $F_2 : 0 \mapsto <\overline{2}>$. Hence, $(F_1, \{0\}) \tilde{\nleq}_e (M, \{0, 1\})$ because $(F_1, \{0\}) \cap (F_2, \{0\}) \neq (0, \{0\})$ but $(F_2, \{0\}) = (0, \{0\})$.

Proposition 3.34. *Let M be a right R-module, S be a multiplicatively closed subset of a ring R and K be a submodule of M. Then K is an S-essential submodule of M if and only if $(F_K, \{0\}) \tilde{\leq}_e^s (M, \{0\})$ where $F_K : 0 \mapsto K$.*

Proof. Let K be an essential submodule of M and $(F_2, \{0\}) \tilde{\leq} (M, \{0\})$. We assume that $F_K(0) \cap F_2(0) = 0$. Since $F_2(0) \leq M$ and $F_2(0) = K$, there exists $t = 1 \in S$ such that $F_2(0)t = 0$. Conversely, let $(F_K, \{0\}) \tilde{\leq}_e^s (M, \{0\})$. Let $L \leq M$ such that $L \cap K = 0$. Then $F_L(0) \cap F_K(0) = 0$ where $F_L : 0 \mapsto L$. It follows by assumption that there exists $t \in S$ such that $F_2(0)t = 0$. □

Proposition 3.35. *Let M be a right R-module and S be a multiplicatively closed subset of R. If $S \subseteq U(R)$ where $U(R)$ is the set of all units in R, then $(F_1, A_1) \tilde{\leq}_e^s (G, A_3)$ if and only if $(F_1, A_1) \tilde{\leq}_e (G, A_3)$.*

Proof. We assume that $(F_1, A_1) \tilde{\leq}_e^s (G, A_3)$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A \cap A_2 \neq \emptyset$. We assume $\forall a \in A \cap A_2$, $F_1(a) \cap F_2(a) = 0$. It follows by assumption that $\exists t \in S \subseteq U(R)$, $\forall a \in A_2$, $F_2(a)t = 0$. Hence, $\forall a \in A_2$, $F_2(a) = 0$. Conversely, It follows from Proposition 3.30. □

Theorem 3.36. *Let M be a right R-module, S be a multiplicatively closed subset of R and (G, A_3) be a soft module over M. Then the following statements are equivalent.*

(i) $\forall a \in A_3$, $(0|_{\{a\}}, \{a\}) \tilde{\leq}_e^s (G, A_3)$.
(ii) $\forall(\emptyset \neq A \subseteq A_3), (0|_A, A) \tilde{\leq}_e^s (G, A_3)$.
(iii) $\forall((F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_e^s (G, A_3)$.
(iv) $\forall((G|_A, A) \neq (F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_e^s (G, A_3)$.
(v) $\forall((F, A) \tilde{\leq} (G, A_3)) \exists t \in S, (F, A)t = (0, A)$.
(vii) $\bigcap_{a \in A} Ann_R(G(a)) \cap S \neq 0$.
(viii) $(0, A_3) \tilde{\leq}_e^s (G, A_3)$.

Proof. (i) $\to$ (ii): Assume (i). Let $\emptyset \neq A \subseteq A_3$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A \cap A_2 \neq \emptyset$. We assume that $\forall a \in A \cap A_2$, $0 \cap F_2(a) = 0$. We can choose $b \in A \cap A_2$. Since $(0|_{\{b\}}, \{b\}) \tilde{\leq}_e^s (G, A_3)$ and $\forall a \in \{b\} \cap A_2$, $0 \cap F_2(a) = 0$, $\exists t \in S \forall a \in A_2$, $F_2(a)t = 0$.

(ii) $\to$ (iii): Assume (ii). Let $(F, A) \tilde{\leq} (G, A_3)$. Let $(F_2, A_2) \tilde{\leq} (G, A_3)$ such that $A \cap A_2 \neq \emptyset$. We assume that $\forall a \in A \cap A_2$, $F(a) \cap F_2(a) = 0$. Since $\emptyset \neq A \cap A_2 \subseteq A_3 \cap A_2$, $(0, A_3) \tilde{\leq}_e^s (G, A_3)$ and $\forall a \in A_3 \cap A_2$, $0 \cap F_2(a) = 0$, $\exists t \in S \forall a \in A_2$, $F_2(a)t = 0$.

(iii) $\to$ (iv): It is obvious.

(iv) $\to$ (v): Assume (iii). Let $(F, A) \tilde{\leq} (G, A_3)$. Since $\emptyset \neq A \cap A_3$, $(0, A_3) \tilde{\leq}_e^s (G, A_3)$ and $\forall a \in A_3 \cap A$, $0 \cap F_2(a) = 0$, $\exists t \in S \forall a \in A$, $F_2(a)t = 0$. Hence $(F, A)t = (0, A)$.

(v) $\to$ (vi): Assume (iv). We choose $(F_2, A_2) = (G, A_3)$. It follows by assumption that $\exists t \in S \forall a \in A_3$, $G(a)t = 0$. We are done.

(vi) $\to$ (i): Assume (vi). It follows by assumption that there exists $t \in S$ such that $G(a)t = 0$ for each $a \in A_3$ so we have $Lt \leq G|_A(a)t = 0$ for each $a \in A$ for each $L \leq G(a)$ where $A \subseteq A_3$.

(viii) $\to$ (ii): Assume (viii). Let $\emptyset \neq A \subseteq A_3$. Let (F_2, A_2) such that $A \cap A_2 \neq$ and $\forall a \in A \cap A_2$, $0|_A(a) \cap F_2(a) = 0$. Since $\forall a \in A_3 \cap A_2 \supseteq A \cap A_2$, $0(a) \cap F_2(a) = 0$ and $(0, A_3) \tilde{\leq}_e^s (G, A_3)$, $\exists t \in S$, $F_2(a)t = 0$.

(ii) $\to$ (viii): We choose $A = A_3$. $\square$

Example 3.37. Let $M = \mathbb{Z}_6$ as a right $\mathbb{Z}$-module, $S = \{1, 6, 9, 12, 15, 18, ...\}$ be a multiplicatively closed subset of $\mathbb{Z}$. Let $G : 0 \mapsto < \overline{3} >$, $1 \mapsto < \overline{3} >$. Hence, $(0, \{0, 1\}) \tilde{\leq}_e^s (G, \{0, 1\})$.

Proof. Since $\bigcap_{a \in A} Ann_R(G(a)) \cap S \neq 0$ where $A = \{0, 1\}$, by Theorem 3.36, $(0, \{0, 1\}) \tilde{\leq}_e^s (G, \{0, 1\})$. $\square$

Remark 3.38. Example 3.37 show that $(0, A_3)$ can be S-essential soft submodule of (G, A_3) this generalization of essential soft submodule can do but in essential soft module $(0, A_3)$ can not be essential soft submodule of (G, A_3) when $G \neq 0$.

Theorem 3.39. *Let R be a commutative ring with identity, M be a right R-module, S be a multiplicatively closed subset of R and (G, A_3) be a soft module over M. Then the following statements are equivalent.*

(i) $(G, A_3) \tilde{\leq}_s^s (G, A_3)$.

(ii) $\forall(\emptyset \neq A \subseteq A_3), (G|_A, A) \tilde{\leq}_s^s (G, A_3)$.

(iii) $\forall((F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_s^s (G, A_3)$.

(iv) $\forall((0, A) \neq (F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_s^s (G, A_3)$.

(v) $\forall((F, A) \tilde{\leq} (G, A_3)) \exists t \in S, (G|_A, A)t \tilde{\leq} (F, A)$.

(vi) $\forall(\emptyset \neq A \subseteq A_3), \bigcap_{a \in A} Ann_R(G|_A(a)) \cap S \neq 0$.

(vii) $\bigcap_{a \in A} Ann_R(G(a)) \cap S \neq 0$.

(viii) $\forall a \in A_3$, $(0|_{\{a\}}, \{a\}) \tilde{\leq}_e^s (G, A_3)$.

(ix) $\forall(\emptyset \neq A \subseteq A_3), (0|_A, A) \tilde{\leq}_e^s (G, A_3)$.

(x) $\forall((F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_e^s (G, A_3)$.

(xi) $\forall((G|_A, A) \neq (F, A) \tilde{\leq} (G, A_3)), (F, A) \tilde{\leq}_e^s (G, A_3)$.

(xii) $\forall((F,A) \mathrel{\tilde{\leq}_s} (G,A_3))\exists t \in S, (F,A)t = (0,A)$.

(xiii) $(0,A_3) \mathrel{\tilde{\leq}_e} (G,A_3)$.

Proof. It follows by Theorem 3.36 and 3.17. □

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