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STUDY OF A LANE-EMDEN TYPE EQUATION UNDER NAVIER BOUNDARY CONDITIONS AND TOPOLOGICAL INSIGHT USING PERSISTENT HOMOLOGY

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Abstract. In this paper, we investigate a Lane-Emden type equation under Navier boundary conditions. Under a given set of assumptions, we prove the existence and uniqueness of a weak solution to the problem using variational methods. Subsequently, we use persistent homology to analyze the topological features of the solution and to track the behavior of its critical points, capturing their birth, death, and structural persistence across multiple scales.

Keywords. Lane-Emden nonlinearity, Fourth-order elliptic, Navier boundary value problem, Weak solution, Variational method, Numerical simulation, Homology, Persistence homology, Vietoris-Rips complex, Critical points
2020 Mathematics Subject Classification. Primary 46L55; Secondary 44B20

1. Introduction

This work addresses the given fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u(\mathbf{X}) - a(\mathbf{X})\Delta u(\mathbf{X}) = \mu f(\mathbf{X})u(\mathbf{X}) + \lambda g(\mathbf{X})|u(\mathbf{X})|^{p-1}u(\mathbf{X}) + h(\mathbf{X}), & \text{in } \Omega \\ \Delta u(\mathbf{X}) = u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N\,(N \geq 1)$ denotes a bounded domain with a smooth boundary $\partial\Omega$, and Δ^2 represents the biharmonic operator. The functions a, f and g are usually serve as weights, and $|u(\mathbf{X})|^{p-1}u(\mathbf{X})$ represents the so-called Lane–Emden nonlinearity. Here, $\lambda < 0$, $\mu > 0$, and $1 < p < 2$.

In this work, the exact assumptions on a, f, g, and h are as follows:

(a) $a(\mathbf{X}) > 0$ for all $\mathbf{X}$ in Ω.
(f) $f \in L^\infty(\Omega)$.
(g) $g \in L^\infty(\Omega)$.
(h) $h \in L^\infty(\Omega)$.

The given functions f, g, h are assumed to be nonnegative.

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In recent decades, the existence and multiplicity of solutions for the following fourth-order boundary value problems:

$$\begin{cases} \alpha\Delta^2 u(\mathbf{X}) + \beta\Delta u(\mathbf{X}) = \lambda f(\mathbf{X}, u) & \text{in } \Omega, \\ u(\mathbf{X}) = \Delta u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

where f satisfies certain conditions and $c < \mu_1$, or sometimes $c > \mu_1$, with μ_1 being the first eigenvalue of $-\Delta u = \lambda u$ under Dirichlet boundary conditions, have been extensively discussed by many authors. Equation (1.2) arises in the study of several problems stemming from various areas, such as microelectromechanical systems (MEMS), surface diffusion on solids, and flow in Hele-Shaw cells (see [8]) and references therein. Furthermore, such equations arise in modeling the static deformation of beams and the dynamics of rigid bodies. Let us recall some forms of equation (1.2) that have been studied in recent years.
In [15], Pistoia and Micheletti determined a geometrical structure for the following equation

$$\Delta^2 u(\mathbf{X}) + c\Delta u(\mathbf{X}) = bg(\mathbf{X}, u),$$

Analogously to the linking theorem, we proceed under the assumptions stated below:

(1) $2G(\mathbf{X}, s) \leq s^2$,
(2) $\limsup_{s\to-\infty} G(\mathbf{X}, s)/s^2 \leq 0$,
(3) $\liminf_{s\to 0} G(\mathbf{X}, s)/s^2 = l(\mathbf{X})$.

where $G(\mathbf{X}, u) = \int_0^u g(\mathbf{X}, s)ds$.
Moreover, they established multiplicity results concerning the existence of solutions.

In [3], using arguments involving continua of fixed points of suitable nonlinear compact operators and the Lyapunov–Schmidt method, A. Correa proved the existence and multiplicity of solutions for the following non-homogeneous resonant elliptic problems:

$$\begin{cases} \alpha\Delta^2 u(\mathbf{X}) + \beta\Delta u(\mathbf{X}) - \mu_1 u(\mathbf{X}) + g(\mathbf{X}, u) = f(\mathbf{X}), & \text{in } \Omega, \\ \Delta u(\mathbf{X}) = u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In [19], Tang et al. used the Ekeland variational principle along with the mountain pass theorem to investigate a fourth-order Navier boundary value problem with an indefinite nonlinearity:

$$\begin{cases} \Delta^2 u(\mathbf{X}) + c\Delta u(\mathbf{X}) = a(\mathbf{X})|u(\mathbf{X})|^{s-2}u(\mathbf{X}) + f(\mathbf{X}, u) & \text{in } \Omega, \\ u(\mathbf{X}) = \Delta u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

and under certain conditions on the weight function a and the nonlinearity f, proved the existence and multiplicity of solutions.

In [20], the equation $\Delta^2 u + c\Delta u = \lambda u + f(u)$ was studied under Navier boundary conditions, where f satisfies a subcritical growth condition, i.e.,

$$|f(s)| \leq d_1|s| + d_2|s|^{p-1}$$

for some $p \in [2, 2^*)$ and constants $d_1, d_2 > 0$, by applying topological degree theory.

In [4], Silva et al. studied the following class of fourth-order superlinear elliptic problems:

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = f(\mathbf{X}, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 4$ is a smooth bounded domain, $\alpha > 0$, $\beta \in (-\infty, \alpha\lambda_1)$ under a suitable assumptions, the authors proved the existence of solutions and then demonstrated that multiple solutions exist.

In a recent article, S. Saiedinezhad [17] considered the following problem:

$$\begin{cases} \Delta^2 u(\mathbf{X}) - a(\mathbf{X})\Delta u(\mathbf{X}) = b(\mathbf{X})|u(\mathbf{X})|^{p-2}u(\mathbf{X}), & \mathbf{X} \in \Omega, \\ \Delta u(\mathbf{X}) = u(\mathbf{X}) = 0, & \mathbf{X} \in \partial\Omega. \end{cases} \tag{1.6}$$

where Ω is a bounded subset of $\mathbb{R}^N$, with $N > 4$ and $2 < p < 2^* = \frac{2N}{N-2}$. By using the Mountain Pass Theorem under the Cerami condition, the author established results on the existence and multiplicity of solutions under specific assumptions on a and b.

Looking at (1.1), it is referred to as the Lane–Emden equation because it is a natural extension of the given well-known equation:

$$\begin{cases} -\Delta u(\mathbf{X}) = |u(\mathbf{X})|^p, & \text{in } \Omega \\ u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

for u to be strictly positive on the ball of radius R in $\mathbb{R}^3$, arising in astronomy, which is known in physics as the Lane–Emden equation of index p.
In addition, in 1983, Brézis and Nirenberg considered the following linear perturbation of the Lane–Emden equation [1]:

$$\begin{cases} -\Delta u(\mathbf{X}) = \lambda u(\mathbf{X}) + |u(\mathbf{X})|^{p-1}u(\mathbf{X}), & \text{in } \Omega \\ u(\mathbf{X}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

where Ω is a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

According to V. D. *Rădulescu* [16], the Lane–Emden equation naturally describes many physical phenomena. For example, this equation also arises in the study of cellular automata and interacting particle systems with self-organized criticality, as well as in the modeling of fluid flow over an impermeable plate. These types of problems have been extensively studied in the case of the Laplacian or the fourth-order operator. For more details, see [16, 17, 19, 20] and the references therein.

The second part requires a refresher on persistent homology. We begin with a brief overview of the main concepts and objectives. Persistent homology (PH) [13] is a central tool in topological data analysis (TDA) that captures the evolution of topological features such as connected components, loops, and voids-across multiple scales. By assigning birth and death times to these features through a filtration process, PH provides a compact, noise-tolerant summary of the data's shape. This makes it particularly valuable for studying complex or high-dimensional structures that are difficult to interpret using traditional numerical methods.

In the context of nonlinear partial differential equations (PDEs), solutions often exhibit intricate spatial patterns and critical behaviors, especially under nontrivial boundary conditions. While classical numerical methods can approximate these solutions, they typically offer limited insight into the deeper topological structures embedded in the solution field. Here we propose a complementary approach that combines numerical simulation with persistent homology to investigate the global structure of approximate solutions to a nonlinear equation.

This topological perspective provides a robust, scale-aware analysis that distinguishes meaningful geometric structures from numerical artifacts. Using persistence diagrams and barcode representations, we study how the critical points evolve with spatial and parametric variation. In doing so, we reveal structural patterns that enrich the classical interpretation and demonstrate the potential of persistent homology as a powerful tool for the qualitative analysis of nonlinear PDEs.

Our main purpose in the present work has two objectives. The first is to study a general class of Lane–Emden equations involving the eigenvalue problem for the operator $\Delta^2 - a(x)\Delta$. Motivated by the proof techniques used in the aforementioned works, we present similar results regarding the existence of weak solutions to problem (1.1), but through a different approach. The second objective is to analyze approximate solutions by applying persistent homology to investigate the topological features of the solution and to track the behavior of its critical points, capturing their birth, death, and structural persistence across multiple scales.

This paper is organized as follows. In Section 2, we present the variational framework for the given problem (1.1). Section 3 is devoted to the proof of our main Theorem 2.1. Then, in Section 4, we approximate the solution and analyze its behavior using persistence barcodes and diagrams, offering new insights into the geometry and complexity of the solution.

2. Variational framework and main results

This section presents the basic mathematical material required to prove the theorems of existence and uniqueness for the Lane–Emden Type Equation under Navier Boundary given in (1.1).

Throughout this article, Ω denotes an open, smooth, and bounded subset of $\mathbb{R}^N$. We denote the norm of u in $L^p(\Omega)$ by $\|u\|_p = (\int_\Omega |u(\mathbf{X})|^p \, d\mathbf{X})^{\frac{1}{p}}$, and we use $\|.\|$ to refer to the relevant norm in context of this work. The symbol $\rightharpoonup$ denotes weak convergence, while $\to$ denotes strong convergence.

We consider the following Sobolev space:

$$\mathcal{X} = H^2(\Omega) \cap H_0^1(\Omega)$$

where H^1, H_0^1, H^2 are the Sobolev space defined by:

$$\begin{aligned}
&H^1(\Omega) := \{u \in L^2(\Omega) : u_{\mathbf{X}_i} \in L^2(\Omega), 1 \le i \le N\},\\
&H_0^1(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\},\\
&H^2(\Omega) := \{u \in L^2(\Omega) : u_{\mathbf{X}_i}, u_{\mathbf{X}_i\mathbf{X}_j} \in L^2(\Omega),\ 1 \le i, j \le N\}.
\end{aligned}$$

The space $\mathcal{X}$ is endowed with the standard inner product:

$$\langle u, v\rangle = \int_\Omega (\triangle u \triangle v + a(\mathbf{X})\nabla u \nabla v) d\mathbf{X},$$

and equipped with the induced norm:

$$\|u\| = \Big(\int_\Omega \big(|\Delta u|^2 + a(\mathbf{X})|\nabla u|^2\big)\, d\mathbf{X}\Big)^{1/2}, \quad u \in \mathcal{X}.$$

In addition $\mathcal{X}$ is a separable and reflexive real Banach space.

By the Sobolev embedding theorem, there exists a positive constant C such that

$$\|u\|_{L^\theta} \leq C\|u\|$$

where $\|u\|_{L^\theta}$ denotes the norm in L^θ for $1 \leq \theta \leq \frac{2N}{N-4}$.
More precisely, according to [19], we have the following inequality:

$$\|u\|^2_{L^2} \leq \frac{1}{\mu_1}\|u\|^2$$

where μ_1 denotes the first eigenvalue of the following eigenvalue problem:

$$\begin{cases} \Delta^2 u - a(\mathbf{X})\Delta u = \mu f(\mathbf{X})u, & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Among the main results of this paper, we highlight the following theorems.

Theorem 2.1. *Assume that* $(a), (f), (g)$ *and* (h) *hold. Then, for each* $\lambda < 0$ *and* $\mu_1 \geq \mu\|f\|_\infty$, *problem* (1.1) *admits a unique nontrivial weak solution.*

To prove the existence of solutions to problem (1.1), we recall the following result from monotone operator theory (see [21]).

Theorem 2.2. *Let* $(\mathcal{X}, \|.\|)$ *be a real reflexive Banach space and let* $I : \mathcal{X} \to \mathbb{R}$ *be a Gâteaux differentiable functional with the following properties:*

(1) I' *is strictly monotone,*
(2) I *is weakly coercive.*

Then

a- *The operator equation* $I'(u) = 0$ *for* $u \in \mathcal{X}$ *has a unique solution* $u \in \mathcal{X}$, *i.e., there exists a unique* $u \in \mathcal{X}$ *such that* $\langle I'(u), v\rangle = 0$ *for all* $v \in \mathcal{X}$*;*
b- $I(u) = \inf_{v\in\mathcal{X}} I(v)$.

3. Proof of existence and uniqueness

To establish the existence of a unique solution for problem (1.1), we recall in this section some definitions and results that will be used later.

A function $u \in \mathcal{X}$ is called a weak solution of problem (1.1) if it satisfies

$$\int_\Omega (\triangle u \triangle v + a(\mathbf{X})\nabla u \nabla v) d\mathbf{X} = \mu \int_\Omega f(\mathbf{X})uv\, d\mathbf{X} + \lambda \int_\Omega g(\mathbf{X})|u|^p uv\, d\mathbf{X} + \int_\Omega h(\mathbf{X})v\, d\mathbf{X}, \text{ for all } v \in X$$

To establish the existence of solutions to problem (1.1), we employ a variational method.
The energy functional associated with problem (1.1) is defined by

$$I(u) = \frac{1}{2}\int_\Omega (|\triangle u|^2 + a(\mathbf{X})|\nabla u|^2)d\mathbf{X} - \frac{\mu}{2}\int_\Omega g(\mathbf{X})|u|^2 d\mathbf{X} - \frac{\lambda}{p+1}\int_\Omega g(\mathbf{X})|u|^{p+1}d\mathbf{X} - \int_\Omega h(\mathbf{X})u\, d\mathbf{X},\ u \in X \tag{3.1}$$

It is well known that the functional I is well-defined and belongs to the class $C^1(\Omega, \mathbb{R})$. Moreover, for all $u,\ v \in X$, we have

$$(I'(u), v) = \int_\Omega (\triangle u \triangle v + a(\mathbf{X})\nabla u \nabla v)d\mathbf{X} - \mu\int_\Omega f(\mathbf{X})uv\, d\mathbf{X} - \lambda\int_\Omega g(\mathbf{X})|u|^p uv\, d\mathbf{X} - \int_\Omega h(\mathbf{X})v\, d\mathbf{X}. \tag{3.2}$$

It is clear that any nonzero critical point of I corresponds to a nontrivial weak solution of problem (1.1).

Proof of Theorem 2.1. In the remainder of this section, we prove **Theorem** (2.1). Throughout, we assume that conditions $(a), (f), (g)$ and (h) hold.
Recall the following well-known inequalities, stated in the lemma below. (see, for example, [5]).

Lemma 3.1. *For any vectors $A, B \in \mathbb{R}^n$, the following inequalities hold:*

$$C_p\langle |A|^{p-2}A - |B|^{p-2}B, A - B\rangle \geq \begin{cases} |A-B|^p & \text{if } p \geq 2 \\ \frac{|A-B|^2}{(|A|+|B|)^{2-p}} & \text{if } 1 \leq p \leq 2, \end{cases} \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ stands for the usual dot product in $\mathbb{R}^n$, and C_p denotes a constant depending on p.

In order to prove Theorem(2.1), we first state and prove the following two lemmas.

Lemma 3.2. *$I' : \mathcal{X} \to \mathcal{X}^*$ is strictly monotone.*

Proof. For any $u_1, u_2 \in \mathcal{X}$, from (3.2) and the assumption that $\lambda < 0$, $\mu_1\, \mu\|f\|_\infty$ and $1 < p < 2$, we have

$$\begin{aligned} \langle I'(u_1) - I'(u_2), u_1 - u_2\rangle &= \int_\Omega (\Delta u_1 - \Delta u_2)\Delta(u_1 - u_2)\, d\mathbf{X} + \int_\Omega a(\mathbf{X})(\nabla u_1 - \nabla u_2)\nabla(u_1 - u_2)\, d\mathbf{X} \\ &- \mu\int_\Omega f(\mathbf{X})|u_1 - u_2|^2\, d\mathbf{X} - \lambda\int_\Omega g(\mathbf{X})(|u_1|^p u_1 - |u_2|^p u_2)(u_1 - u_2)\, d\mathbf{X} \\ &\geq (\mu_1 - \mu\|f\|_\infty)\|u_1 - u_2\|^2. \end{aligned}$$

Then $\langle I'(u_1) - I'(u_2), u_1 - u_2\rangle > 0$ for all $u_1, u_2 \in \mathcal{X}$ with $u_1 \neq u_2$. Thus, I' is strictly monotone. This completes the proof of lemma (3.2). ☐

Lemma 3.3. *$I : \mathcal{X} \to \mathbb{R}$ is weakly coercive.*

Proof.

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega (|\triangle u|^2 + a(\mathbf{X})|\nabla u|^2)d\mathbf{X} - \frac{\mu}{2}\int_\Omega g(\mathbf{X})|u|^2 d\mathbf{X} - \frac{\lambda}{p+1}\int_\Omega g(\mathbf{X})|u|^{p+1}d\mathbf{X} \\ &\geq \frac{\|u\|^2}{2} - \frac{\mu\|f\|_\infty}{2\mu_1}\|u\|^2 - \frac{\lambda\|g\|_\infty}{p+1}\|u\|^{p+1} \\ &\geq (1 - \frac{\mu\|f\|_\infty}{\mu_1})\frac{\|u\|^2}{2} \end{aligned}$$

Hence, I is weakly coercive, which completes the proof of the lemma (3.3). □

In summary, the application of Lemmas (3.2) and (3.3) in Theorem(2.2) leads to the existence and uniqueness of a weak solution, as stated in Theorem(2.1). Since the lane emden quation arises in the study of several problems specially the static deformation of beams or the motion of rigid bodies. we approximate the solution under assumptions and we studied the topological insight using the TDA to describe the behavior of the numerical solution. Here in , we propose the persistent homology as a tool to analyze the numerical solution.
Since the lane emden quation arises in the study of several problems specially in the 2 dimensions, we approximate the solution under assumptions by using python tools and we studied the topological insight using the TDA to describe the behavior of the numerical solution. Here in , we propose the persistent homology as a tool to analyze the numerical solution in the next section.

4. Numerical Solution and Topological Analysis

In this section, we present a numerical solution to the nonlinear biharmonic equation introduced earlier, followed by a topological analysis of its critical point structure using tools from persistent homology. Our goal is to provide a deeper qualitative insight into the complexity of the obtained solutions.

4.1. **Preliminaries on Persistent Homology and Persistence Modules.**

In this section, we introduce the key concepts required for computing persistent homology, including simplicial complexes, filtrations, homology groups, and the notion of persistence modules. For further details, we refer the reader to [7, 9].

4.1.1. *Simplicial Complexes.* A **simplicial complex** is a combinatorial structure that models a topological space using simple building blocks called simplices (e.g., points, edges, triangles, tetrahedra). Formally, it is a collection of finite sets closed under the operation of taking subsets.

This discrete structure allows us to compute topological invariants algorithmically.

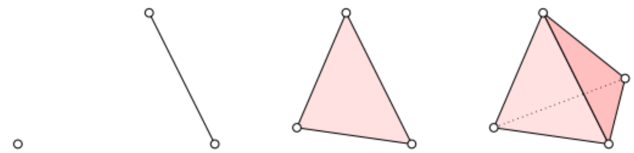

FIGURE 1. Geometric progression of simplexes in increasing dimensions: a point (0-simplex), a line segment (1-simplex), a triangle (2-simplex), and a tetrahedron (3-simplex)

4.1.2. *Filtrations.* A **filtration** is a sequence of nested simplicial complexes that reflects the growth of a space over a parameter (typically a scale or time parameter):

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n. \tag{4.1}$$

Such a sequence models how new simplices are added, allowing us to track the appearance and disappearance of topological features.

4.1.3. *Simplicial Homology and Topological Features.* Homology is an algebraic tool that associates a sequence of abelian groups $H_k(K)$ to a simplicial complex K, where each group encodes k-dimensional holes:

H_0: connected components,
H_1: 1-dimensional cycles (loops),
H_2: voids or cavities,
H_k: higher-dimensional analogues of holes.

The computation of homology across a filtration gives rise to persistent homology.

4.1.4. *Persistence Modules.* Let $\mathbb{F}$ be a field (typically $\mathbb{F} = \mathbb{Z}_2$), and let $\{K_i\}_{i\in\mathbb{Z}}$ be a filtration of simplicial complexes. Applying homology functor $H_k(-;\mathbb{F})$ to this filtration yields a sequence of vector spaces and linear maps:

$$V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} \dots \xrightarrow{\phi_{n-1}} V_n, \tag{4.2}$$

where $V_i = H_k(K_i;\mathbb{F})$ and ϕ_i is induced by the inclusion $K_i \hookrightarrow K_{i+1}$.

Definition 4.1. A persistence module (in degree k) is a sequence of vector spaces $\{V_i\}_{i\in\mathbb{Z}}$ indexed by integers, along with linear maps $\phi_i : V_i \to V_{i+1}$, such that the composition $\phi_j \circ \cdots \circ \phi_i$ respects the inclusion of the filtration.

The persistence module encodes how k-dimensional topological features (e.g., connected components, loops) appear and persist across scales. This information is summarized by a **persistence diagram** or **barcode**, which records the birth and death of each feature.

4.1.5. *Vietoris-Rips Complex.* Given a finite metric space (X, d), the Vietoris-Rips complex is a commonly used method to build a simplicial complex from a point cloud.

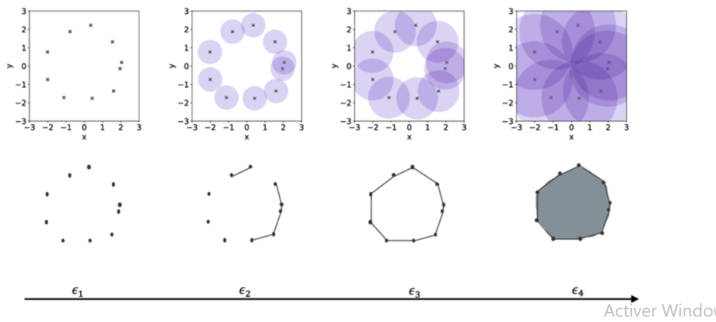

FIGURE 2. Evolution of a Vietoris–Rips complex: as the distance threshold ϵ increases from ϵ_1 to ϵ_2 , simplices form between nearby points, gradually revealing the underlying topological structure—from isolated points to edges, loops, and finally filled surfaces.

Definition 4.2. The Vietoris-Rips complex at scale $r > 0$, denoted $\mathrm{VR}_r(X)$, is the abstract simplicial complex whose p-simplices correspond to finite subsets $\{x_0, \ldots, x_p\} \subset X$ of diameter less than r, i.e.,

$$\mathrm{diam}(\{x_0, \ldots, x_p\}) < r.$$

This construction allows one to build a filtration $\mathrm{VR}_r(X) \subseteq \mathrm{VR}_s(X)$ for $r < s$, which is then analyzed using persistent homology.

The use of Vietoris-Rips complexes is particularly suited to studying the shape of high-dimensional and noisy data.

4.2. **Numerical Framework.** We consider the nonlinear biharmonic problem:

$$\Delta^2 u - a(x,y)\Delta u = (\mu f(x,y) + \lambda g(x,y)|u|^{p-1})u + 0.1, \quad \text{in } \Omega = (0,1)^2, \quad (4.3)$$

with Navier boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega, \qquad \Delta u = 0 \quad \text{on } \partial\Omega. \quad (4.4)$$

We set the parameters as follows to ensure consistency with the hypotheses in Section 2:

$$\begin{aligned} \lambda = -2.0, \quad \mu = 1, \quad p = 1.5 \quad (1 < p < 2), \\ a(x,y) &= 1 + 0.5\sin(\pi x)\cos(\pi y), \\ f(x,y) &= 1 - (x^2 + y^2), \\ g(x,y) &= x^2 + y^2. \end{aligned}$$

Discretization and Numerical Scheme. The biharmonic operator is discretized using a five-point finite difference stencil for the Laplacian and a nine-point approximation for the biharmonic term $\Delta^2 u$. The problem is then formulated as a nonlinear system of the form:

$$Au + \lambda B(u) = F,$$

where A is the stiffness matrix incorporating the linear terms and boundary conditions, $B(u)$ represents the nonlinear term $g(x, y)|u|^{p-1}u$, and F includes the source terms and boundary values.

We solve the system using a fixed-point iteration scheme with damping:

$$u^{(k+1)} = u^{(k)} - \alpha \cdot (Au^{(k)} + \lambda B(u^{(k)}) - F),$$

where $\alpha > 0$ is chosen to ensure convergence. Convergence is checked via the norm $\|u^{(k+1)} - u^{(k)}\| < 10^{-6}$, and the iteration is stopped after 100 iterations or upon convergence.

Extraction of Critical Points. From the discrete solution u_{ij}, we extract the local minima, maxima, and saddle points using finite-difference approximations of the gradient and Hessian. A point (i, j) is labeled as a critical point if the discrete gradient vanishes and the Hessian has the appropriate eigenvalue signature (positive-definite, negative-definite, or indefinite).

The resulting set of critical points $X = \{x_1, \ldots, x_n\}$ is treated as a point cloud in $\mathbb{R}^2$, which serves as input for topological data analysis.

4.3. **Persistent Homology via Vietoris-Rips Filtration.** We apply persistent homology to the point cloud X using the Vietoris–Rips complex $VR_r(X)$ at varying filtration radii $r > 0$. We compute persistence diagram (see Figure 3) and its corresponding persistence barcode (see Figure 4) in dimensions H_0 (connected components) and H_1 (loops), which provide a scale-dependent summary of the topological features.

Interpretation of the Persistence Diagram and Barcode:
Dimension 0 (H_0 – Connected Components):
The red dots illustrated in Figure 3 and bars shown in Figure 4, represent the evolution of connected components across the filtration. Most red bars start at filtration value $b = 0$ and die around $d \approx 1.0$, indicating that the critical points (likely local minima or maxima of the PDE solution) begin as many isolated components, gradually merging as the filtration progresses.

Two red square points are located relatively far from the diagonal, with one of them exhibiting infinite persistence. This suggests the presence of *two significant connected components* in the underlying data. One component persists over a considerable range of scales before merging, and the other persists indefinitely, corresponding to the dominant, globally connected structure that spans the entire dataset.

This behavior implies that the spatial distribution of critical points initially contains clusters or separate regions, which merge around scale ~ 1.0. The persistence of these two components reveals their topological relevance, distinguishing them from noise.

Dimension 1 (H_1 – Loops):
The blue dots depicted in Figure 3 and bars illustrated in Figure 4, represent the birth and death of loops (1-dimensional cycles). The diagram reveals *two*

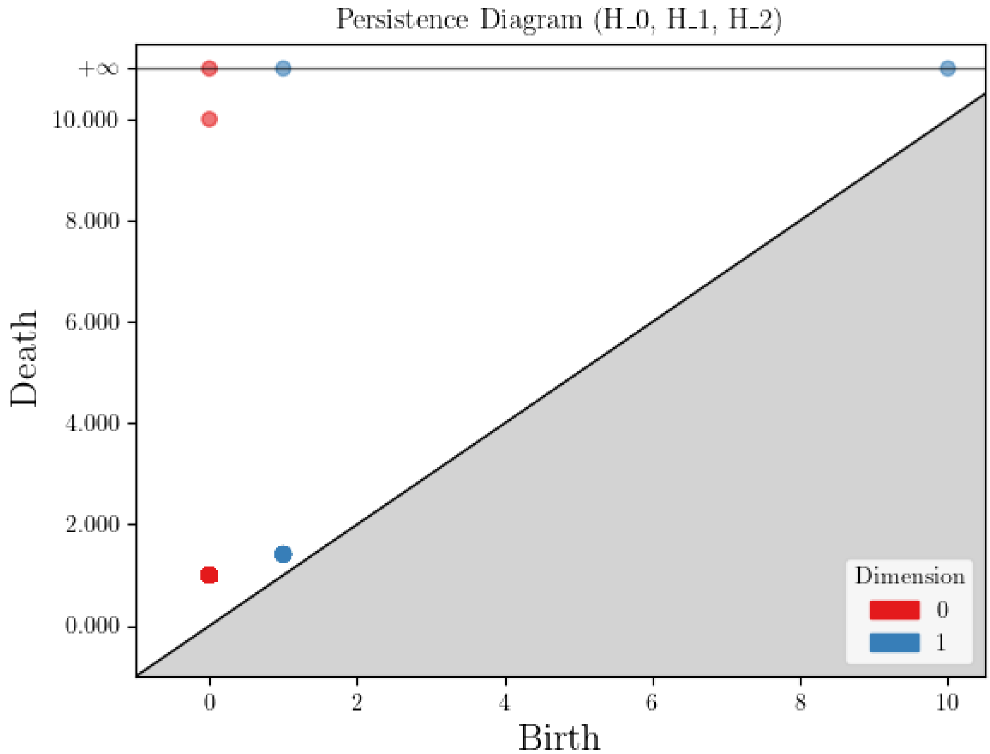

Figure 3. Persistence diagram computed from the critical points of $u(x, y)$.

prominent blue bars—one begins at $b \approx 1.2$ and persists significantly, suggesting a robust topological loop, while the other also has a moderate persistence, indicating a secondary but meaningful loop structure.

Topological Insight. The persistence diagram (see Figure 3) and barcode (see Figure 4) jointly indicate that:

a. Two H_0 features are significantly persistent: one merges at a moderate scale, the other persists infinitely, highlighting two topologically meaningful connected components.
b. Two H_1 features have non-trivial persistence, indicating two distinct loop-like structures in the data.
c. Features near the diagonal in both dimensions likely represent topological noise or minor artifacts.
d. The absence of H_2 features implies that the space has no higher-dimensional voids or cavities (like hollow spheres or bubbles).

These observations support the conclusion that the critical points of the PDE solution form a structure with:

- Initial multiplicity of components that merge into a globally connected structure,
- Two robust loop structures revealing rich 1-dimensional topology,

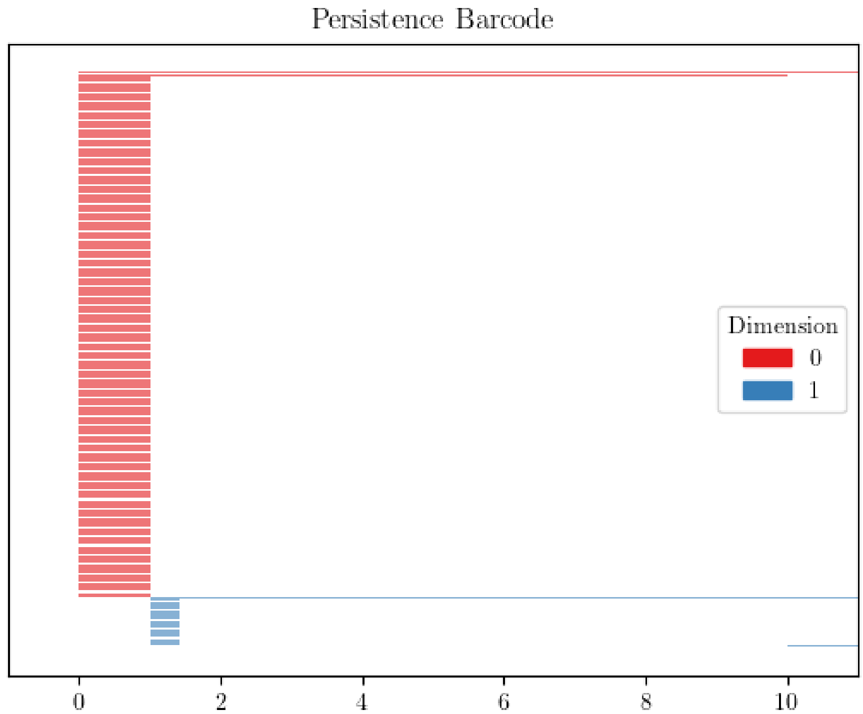

FIGURE 4. Corresponding persistence barcode showing lifespans of topological features.

- Minor features close to the diagonal that can be ignored.

CONCLUSION

In conclusion, this work introduces a new class of Lane–Emden equations grounded in their original formulations. We have rigorously formulated the model and proved the existence and uniqueness of its weak solution. By employing persistent homology, the persistence diagram and barcode derived from the PDE solution show that its critical points form a structure with two significant connected components and two prominent homological cycles. While features with limited persistence are attributed to noise, the persistent components underscore a robust underlying topological organization, suggestive of a connected, looped geometry such as an annulus or a deformed circle.

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