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ON THE INDEPENDENCE OF THE FOURIER COEFFICIENTS OF TWO p-ADIC MODULAR FORMS

ABDELMONAIM BOUCHIKHI[1]* AND SOUFIANE MEZROUI[2]

Abstract. We study rigidity phenomena for p-adic modular forms and their families. Extending a result of Coleman, we show that if two overconvergent modular forms have Fourier coefficients with values in a finite ratio set, then they must be scalar multiples of each other. We further prove that this rigidity persists for p-adic families varying over weight space.

Keywords. Fourier coefficients of cusp forms, p-modular forms, p-adic Hodge theory
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1. Introduction

The study of the p-adic variation of modular forms has been a central theme in modern number theory since the pioneering works of Serre [13], Katz [10], and Hida [9], who laid the foundation for understanding how modular forms can be interpolated p-adically in families. This p-adic deformation theory plays a crucial role in the construction of p-adic L-functions, Galois representations, and eigenvarieties, forming an essential bridge between arithmetic geometry and p-adic analytic spaces.

One fundamental question that has emerged is the extent to which the coefficients of p-adic modular forms determine the forms themselves. Such rigidity phenomena not only have deep intrinsic interest but also have significant implications for the structure of eigenvarieties, the study of congruences between modular forms, and the p-adic Langlands program (see, for instance, [3, 6, 11]).

In the classical setting, Coleman [4] proved a remarkable result: if two overconvergent p-adic modular forms have coefficients whose ratios lie in a finite set, then the forms must be proportional. This rigidity statement has inspired further developments, including refinements for families of modular forms, as in the work of Coleman and Mazur on the eigencurve [6] and Buzzard's construction of eigenvarieties [3].

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* Corresponding author

Our main theorems extend Coleman's result in three natural directions: we treat individual overconvergent modular forms, power series arising from p-adic limits, and analytic families varying over rigid-analytic weight spaces. In particular, our Theorem 1.1 shows that the finiteness of the projective ratio set of the Fourier coefficients forces proportionality of two nonzero overconvergent forms. Theorem 1.2 relaxes the modularity assumption for one series, establishing that any power series with coefficients ratio-finite with respect to a normalized eigenform must itself be modular. Theorem 1.3 lifts this rigidity to the setting of families parametrized by a weight space, a context motivated by the rich theory of p-adic families developed by Hida [9], Coleman-Mazur [6], and Buzzard [3].

These results shed light on the uniqueness of p-adic families and contribute to a better understanding of the geometry of eigenvarieties and the deformation theory of Galois representations. They resonate with recent work on the analytic continuation and uniqueness of p-adic families of automorphic forms, such as [1, 8, 11].

In future work, we hope that these rigidity theorems may find further applications, for instance, in the study of congruences between eigenforms modulo powers of p, the uniqueness of p-adic lifts, and the classification of overconvergent families. They may also inspire analogous results in the context of Hilbert or Siegel modular forms, where p-adic families play an equally central role. Our main results

Theorem 1.1. *Let $f, g \in \mathcal{M}_k^\dagger(N)$ be nonzero overconvergent p-adic modular forms with coefficients in a finite extension $K/\mathbb{Q}_p$. Suppose that the projective ratio set*

$$R_p(f,g) := \left\{ \left[\frac{a_f(n)}{a_g(n)}\right] \in \mathbb{P}^1(K) \mid n \geq 1,\ a_g(n) \neq 0 \right\}$$

is finite. Then there exists a scalar $c \in K^\times$ such that $f = c \cdot g$.

Theorem 1.2. *Let $h(q) = \sum_{n\geq 0} a_h(n)q^n$ be a p-adic power series, and let $f \in \mathcal{M}_k^\dagger(N)$ be a normalized overconvergent Hecke eigenform with $a_f(1) = 1$ and $a_f(n) \neq 0$ for all $n \geq 1$. Suppose the ratio set*

$$\mathcal{R}_p(h,f) := \left\{ \left[\frac{a_h(n)}{a_f(n)}\right] \in \mathbb{P}^1(K) \;\middle|\; n \geq 1 \right\}$$

is finite. Then $h \in \mathcal{M}_k^\dagger(N)$ and there exists $c \in K^\times$ such that $h = c \cdot f$.

Theorem 1.3. *Let $\mathcal{W}$ be a rigid-analytic weight space over a p-adic field K. Let*

$$\{f_k\}_{k\in\mathcal{W}}, \quad \{g_k\}_{k\in\mathcal{W}}$$

be analytic families of overconvergent p-adic modular forms with q-expansions

$$f_k(q) = \sum_{n\geq 0} a_f(n,k)q^n, \quad g_k(q) = \sum_{n\geq 0} a_g(n,k)q^n,$$

where for each fixed n, the functions $k \mapsto a_f(n,k)$, $a_g(n,k)$ are analytic on $\mathcal{W}$. Suppose there exists a Zariski-dense subset $U \subset \mathcal{W}$ such that for all $k \in U$

$g_k \neq 0$, the ratio set

$$\mathcal{R}_p(f_k, g_k) := \left\{ \left[\frac{a_f(n,k)}{a_g(n,k)} \right] \in \mathbb{P}^1(K) \;\middle|\; n \geq 1 \right\}$$

is finite. Then there exists a meromorphic function $c : \mathcal{W} \to K$, analytic on U, such that

$$f_k = c(k) \cdot g_k, \quad \textit{for all } k \in \mathcal{W}.$$

Furthermore, $c(k)$ is analytic at each $k_0 \in \mathcal{W}$ with $g_{k_0} \neq 0$.

2. Preliminaries

Lemma 2.1 (p-Adic Coefficient Growth)**.** *Let $f \in \mathcal{M}_k^{\dagger}(N)$ be an overconvergent p-adic modular form of weight k and tame level N, with q-expansion*

$$f(q) = \sum_{n \geq 0} a_f(n) q^n.$$

Then there exists a constant $C_f > 0$ such that

$$v_p(a_f(n)) \geq C_f \cdot n \quad \textit{for all } n \geq 1.$$

Proof. By definition, an overconvergent p-adic modular form of tame level N is a rigid-analytic section over a strict neighborhood of the ordinary locus of the modular curve $X_0(N)$. That is, let $\mathcal{X}$ denote the rigid-analytic space associated to $X_0(N)$, and let q be a local coordinate (the Tate parameter) near the cusp at infinity. Then f admits an analytic continuation to a wide-open annulus of the form

$$\mathcal{A}_r := \{ x \in \mathcal{X} \mid r < |q(x)|_p < 1 \},$$

for some $r < 1$, where r depends on the overconvergence radius of f. This implies that the q-expansion of f converges for all $q \in \mathbb{C}_p$ with $|q|_p \in (r, 1)$. By standard results on power series in non-Archimedean analysis (e.g., the p-adic version of Hadamard's formula), the radius of convergence $R \in (r, 1)$ of the q-expansion

$$f(q) = \sum_{n=0}^{\infty} a_f(n) q^n$$

satisfies

$$\limsup_{n \to \infty} |a_f(n)|_p^{1/n} = \frac{1}{R}.$$

Since f is convergent on $|q|_p > r$, we must have $R \geq r$, hence

$$\limsup_{n \to \infty} |a_f(n)|_p^{1/n} \leq \frac{1}{r}.$$

Consequently, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$|a_f(n)|_p \leq (r^{-1} + \varepsilon)^n.$$

Recall the identity $|x|_p = p^{-v_p(x)}$. Taking p-adic logarithms

$$v_p(a_f(n)) \geq -n \log_p(r^{-1} + \varepsilon).$$

Letting $\varepsilon \to 0$, we define

$$C := -\log_p r > 0,$$

so that for $n \geq N_0$,

$$v_p(a_f(n)) \geq C \cdot n.$$

For $1 \leq n < N_0$, the values $v_p(a_f(n))$ are finite and fixed. Let

$$v_{\min} := \min_{1 \leq n < N_0} v_p(a_f(n)).$$

Define

$$C_f := \min\left\{C, \frac{v_{\min}}{N_0 - 1}\right\} > 0.$$

Then for all $1 \leq n < N_0$, we have

$$v_p(a_f(n)) \geq v_{\min} \geq C_f \cdot n.$$

Combining with the bound for $n \geq N_0$, we conclude:

$$v_p(a_f(n)) \geq C_f \cdot n \quad \text{for all } n \geq 1.$$

□

Remark 2.2. The constant C can be optimized further using the geometry of Newton polygons of overconvergent eigenforms. Specifically, Coleman's work on slope filtrations [4] implies that:

$$v_p(a_f(n)) \geq \frac{-\log r}{p - 1} \cdot n,$$

where the factor $(p-1)^{-1}$ accounts for the change in normalization of the valuation on modular forms. Hence, the optimal growth constant is:

$$C_{\text{opt}} = \frac{-\log r}{p - 1}.$$

This reflects the fact that the valuations of the coefficients grow linearly with n, and the growth rate is directly controlled by the radius r of overconvergence.

Lemma 2.3. *Let p be a prime and let $K/\mathbb{Q}_p$ be a finite extension. Let $f, g \in \mathcal{M}_k^\dagger(N)$ be overconvergent p-adic modular forms with coefficients in K, and denote their q-expansions by*

$$f(q) = \sum_{n \geq 0} a_f(n) q^n, \qquad g(q) = \sum_{n \geq 0} a_g(n) q^n.$$

Suppose that the projective ratio set

$$R_p(f, g) := \left\{ \left[\frac{a_f(n)}{a_g(n)}\right] \in \mathbb{P}^1(K) \mid n \geq 1 \right\}$$

is finite. Then the set of slope differences

$$\left\{ v_p\left(\frac{a_f(n)}{a_g(n)}\right) \in \mathbb{Z} \mid n \geq 1 \right\}$$

is bounded in $\mathbb{Z}$.

Proof. We first reduce to the case where the denominators are nonzero. Since the ratio set $R_p(f,g)$ is finite, the set of indices n for which $a_g(n) = 0$ must be finite (otherwise, infinitely many terms would contribute the value $[1:0] = \infty \in \mathbb{P}^1(K)$, violating finiteness). Hence, we may replace f and g by their restrictions to a cofinite subset of $n \in \mathbb{N}$, assuming without loss of generality that

$$a_g(n) \neq 0 \quad \text{for all } n \geq 1.$$

For each $n \geq 1$, the ratio $a_f(n)/a_g(n) \in K \cup \{\infty\}$ defines a point $[a_f(n) : a_g(n)] \in \mathbb{P}^1(K)$. Since $R_p(f,g)$ is finite, there exists a finite subset $S \subseteq \mathbb{P}^1(K)$ such that

$$\left[\frac{a_f(n)}{a_g(n)}\right] \in S \quad \text{for all } n \geq 1.$$

For each point $[x:y] \in S$, we choose a normalized representative $(x,y) \in \mathcal{O}_K^2 \setminus \{(0,0)\}$, such that

$$\min(v_p(x), v_p(y)) = 0.$$

This normalization ensures that all values x/y lie in $\mathcal{O}_K \cup \{\infty\}$, and all valuations $v_p(x)$, $v_p(y)$ lie in a finite subset of $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. By the multiplicative property of valuations, we have

$$v_p\left(\frac{a_f(n)}{a_g(n)}\right) = v_p(a_f(n)) - v_p(a_g(n)).$$

Now, since each ratio $\frac{a_f(n)}{a_g(n)}$ corresponds (up to scaling by a unit in $K^\times$) to one of the finitely many normalized elements $\frac{x}{y} \in K \cup \{\infty\}$, and we have

$$\frac{a_f(n)}{a_g(n)} = u_n \cdot \frac{x}{y}$$

for some unit $u_n \in K^\times$, it follows that

$$v_p\left(\frac{a_f(n)}{a_g(n)}\right) = v_p(u_n) + v_p\left(\frac{x}{y}\right) = v_p\left(\frac{x}{y}\right),$$

since u_n is a unit. Thus, the slope difference is determined by the fixed valuation $v_p(x) - v_p(y)$, which lies in a finite set depending only on the finite set S. It follows that the set:

$$\left\{ v_p\left(\frac{a_f(n)}{a_g(n)}\right) \mid n \geq 1 \right\}$$

is finite, hence bounded in $\mathbb{Z}$. That is, there exists $M \in \mathbb{Z}_{\geq 0}$ such that

$$|v_p(a_f(n)) - v_p(a_g(n))| \leq M \quad \text{for all } n \geq 1.$$

Alternatively, suppose that the slopes $v_p(a_f(n)/a_g(n))$ are unbounded. Then there exists a sequence $n_i \to \infty$ such that

$$\left| v_p\left(\frac{a_f(n_i)}{a_g(n_i)}\right) \right| \to \infty.$$

This implies that the ratios $[a_f(n_i) : a_g(n_i)] \in \mathbb{P}^1(K)$ must represent infinitely many distinct projective points, contradicting the finiteness of $R_p(f,g)$. Hence, the slopes must be bounded. $\square$

Lemma 2.4. *Let f and g be overconvergent p-adic modular eigenforms of tame level N, and let*

$$\rho_f, \rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(K)$$

be their associated p-adic Galois representations, with coefficients in a finite extension $K/\mathbb{Q}_p$. Suppose that the set of trace ratios

$$\left\{ \frac{\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell))}{\mathrm{tr}(\rho_g(\mathrm{Frob}_\ell))} \;\middle|\; \ell \nmid pN,\ \mathrm{tr}(\rho_g(\mathrm{Frob}_\ell)) \neq 0 \right\}$$

is finite. Then there exists a scalar $c \in K^\times$ such that $\rho_f \simeq c \cdot \rho_g$.

Proof. Assume both ρ_f and ρ_g are absolutely irreducible; the reducible case can be treated separately by character decomposition. By the Chebotarev Density Theorem [12, Theorem 2.2], the set of Frobenius elements $\{\mathrm{Frob}_\ell \mid \ell \nmid pN\}$ is Zariski-dense in $G_{\mathbb{Q}}$, and the equality

$$\frac{\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell))}{\mathrm{tr}(\rho_g(\mathrm{Frob}_\ell))} \in S \subset K$$

holds for all $\ell \nmid pN$, where $S \subset K$ is finite. Consider the function

$$\varphi : G_{\mathbb{Q}} \to K, \quad \varphi(\sigma) := \frac{\mathrm{tr}(\rho_f(\sigma))}{\mathrm{tr}(\rho_g(\sigma))},$$

defined wherever $\mathrm{tr}(\rho_g(\sigma)) \neq 0$. Since traces of Galois representations are continuous functions (with respect to the profinite topology on $G_{\mathbb{Q}}$ and the p-adic topology on K), and since φ takes only finitely many values on a dense set, it follows that φ is locally constant and hence constant on an open dense subset of $G_{\mathbb{Q}}$. Because $G_{\mathbb{Q}}$ has no proper open subgroups and the representations are continuous, this implies that φ is constant almost everywhere, and hence identically constant. Therefore, there exists a scalar $c \in K^\times$ such that

$$\mathrm{tr}(\rho_f(\sigma)) = c \cdot \mathrm{tr}(\rho_g(\sigma)) \quad \text{for all } \sigma \in G_{\mathbb{Q}}.$$

By the Brauer–Nesbitt theorem [2, p. 559], two semisimple representations over a field of characteristic zero with equal trace functions are isomorphic. Therefore,

$$\rho_f \simeq c \cdot \rho_g.$$

In the case where ρ_f and ρ_g are reducible but semisimple, say

$$\rho_f^{\mathrm{ss}} = \chi_1 \oplus \chi_2, \qquad \rho_g^{\mathrm{ss}} = \psi_1 \oplus \psi_2,$$

the trace ratio condition still implies

$$\frac{\chi_1(\sigma) + \chi_2(\sigma)}{\psi_1(\sigma) + \psi_2(\sigma)} \in S$$

for all $\sigma \in G_{\mathbb{Q}}$ outside a finite set. Again, density and linear independence of characters imply that this forces $\chi_i = c \cdot \psi_i$ for some $c \in K^\times$, after possible reordering. Finally, since f and g are overconvergent eigenforms, their Galois representations are trianguline by p-adic Hodge theory [7], and the scaling by $c \in K^\times$ is compatible with the Hodge–Tate weights and local properties. Hence $\rho_f \simeq c \cdot \rho_g$ as representations of $G_{\mathbb{Q}}$. □

Lemma 2.5. *Let $f, g \in \mathcal{M}_k^\dagger(N)$ be overconvergent p-adic modular forms defined over a finite extension $K/\mathbb{Q}_p$, and assume that f and g are linearly independent over K. Then the set of coefficient ratios*

$$\left\{ \frac{a_f(n)}{a_g(n)} \;\middle|\; a_g(n) \neq 0,\, n \geq 1 \right\}$$

is dense in K.

Proof. Let $\mathcal{B} := \mathcal{M}_k^\dagger(N)$ be the Banach space of overconvergent modular forms of weight k and tame level N, with coefficients in K, equipped with the Gauss norm (see [5])

$$\|f\| := \sup_n |a_f(n)|_p.$$

Each coefficient map $\ell_n : \mathcal{B} \to K$, defined by $\ell_n(h) := a_h(n)$, is a continuous linear functional. The collection $\{\ell_n\}_{n \geq 1}$ separates points by the q-expansion principle.

Let $f, g \in \mathcal{B}$ be linearly independent. Then the map

$$\Phi : n \mapsto (a_f(n), a_g(n)) \in K^2$$

has infinite image. We claim that this image is Zariski dense (and hence topologically dense) in K^2, under the assumption that f and g are linearly independent. Assume that the set $\{(a_f(n), a_g(n)) \in K^2 \mid n \geq 1\}$ is not dense in K^2. Since K^2 is a 2-dimensional vector space over a non-archimedean field, closed sets are zero sets of finitely many continuous linear forms. Therefore, the closure of the image lies in a proper closed K-subspace of K^2, say

$$V := \{(x, y) \in K^2 \mid \lambda_1 x + \lambda_2 y = 0\}$$

for some $(\lambda_1, \lambda_2) \in K^2 \setminus \{(0, 0)\}$.
If all $(a_f(n), a_g(n)) \in V$, then:

$$\lambda_1 a_f(n) + \lambda_2 a_g(n) = 0 \quad \text{for all } n \geq 1.$$

That is, the sequence $\{a_f(n)\}$ is a linear combination of the sequence $\{a_g(n)\}$, i.e.,

$$a_f(n) = -\frac{\lambda_2}{\lambda_1} a_g(n) \quad \text{for all } n \geq 1.$$

But then, the q-expansions satisfy

$$f(q) = -\frac{\lambda_2}{\lambda_1} g(q),$$

which contradicts the assumption that f and g are linearly independent in $\mathcal{M}_k^\dagger(N)$. Therefore, the image of $\Phi(n) = (a_f(n), a_g(n))$ cannot be contained in any proper closed linear subspace of K^2. Hence the image must be Zariski dense and, by the topology of K^2, also topologically dense.

Define the rational map

$$R : K^2 \setminus \{(0, 0)\} \to \mathbb{P}^1(K), \quad (x, y) \mapsto \left[\frac{x}{y}\right] \in \mathbb{P}^1(K).$$

This is continuous, and on the subset where $y \neq 0$, it defines a map $(x, y) \mapsto x/y \in K$. Since $\Phi(n) = (a_f(n), a_g(n))$ is dense in K^2, its image under the ratio map

$$n \mapsto \frac{a_f(n)}{a_g(n)} \quad (\text{where } a_g(n) \neq 0)$$

is dense in K. Indeed, for any $\alpha \in K$ and any $\varepsilon > 0$, there exists n such that

$$\left| \frac{a_f(n)}{a_g(n)} - \alpha \right|_p < \varepsilon.$$

By Lemma 2.1, The coefficients $a_g(n) \neq 0$ for all but finitely many integers n. Indeed, while the p-adic valuation $v_p(a_g(n)) \to \infty$ as $n \to \infty$, the growth is only logarithmic. So the set

$$S := \{n \geq 1 \mid a_g(n) \neq 0\}$$

is cofinite in $\mathbb{N}$. Hence the subset

$$\left\{ \frac{a_f(n)}{a_g(n)} \mid n \in S \right\}$$

is dense in K, as desired. □

3. Proof of the main results

Proof. Suppose, for contradiction, that f and g are linearly independent in the K-Banach space $\mathcal{M}_k^\dagger(N)$. By Lemma 2.5, the set of coefficient ratios

$$\left\{ \frac{a_f(n)}{a_g(n)} \mid a_g(n) \neq 0 \right\}$$

is dense in K. Therefore, its image in $\mathbb{P}^1(K)$ is infinite, contradicting the assumption that $R_p(f, g)$ is finite.
We conclude that f and g must be linearly dependent. Thus, there exists a scalar $c \in K$ such that $f = c \cdot g$.
Since both forms are nonzero, there exists $n_0 \geq 1$ such that $a_g(n_0) \neq 0$, so

$$c = \frac{a_f(n_0)}{a_g(n_0)} \in K^\times.$$

Hence, $f = c \cdot g$ with $c \in K^\times$, as claimed. □

Proof. Since $f \in \mathcal{M}_k^\dagger(N)$, Lemma 2.1 guarantees that there exists a constant $C_f > 0$ such that

$$v_p(a_f(n)) \geq C_f \cdot n \quad \text{for all } n \geq 1.$$

From the finiteness of $\mathcal{R}_p(h, f)$, Lemma 2.3 implies that the difference of valuations is uniformly bounded

$$|v_p(a_h(n)) - v_p(a_f(n))| \leq M \quad \text{for some } M > 0 \text{ and all } n \geq 1.$$

Hence, for $n \geq n_0 := \lceil M/C_f \rceil + 1$, we have

$$v_p(a_h(n)) \geq C_f \cdot n - M \geq \frac{C_f}{2} \cdot n =: C_h \cdot n.$$

This implies the existence of a bounding majorant series $F(q) = \sum_{n\geq 0} p^{-C_h n} q^n$, which converges on a strict p-adic neighborhood of the ordinary locus. Therefore, h is overconvergent.

Now that both f and h are in $\mathcal{M}_k^\dagger(N)$, and the ratio set $\mathcal{R}_p(h, f)$ is finite, the p-adic Coefficient Rigidity Theorem applies. Thus, $h = c \cdot f$ for some $c \in K^\times$. Since f is normalized ($a_f(1) = 1$), it follows that

$$c = a_h(1).$$

For all $n \geq 1$, we obtain $a_h(n) = c \cdot a_f(n)$, consistent with the assumption that $a_f(n) \neq 0$. □

Proof. For each fixed $k \in U$, since $g_k \neq 0$, the ratio set $\mathcal{R}_p(f_k, g_k)$ is finite. By the Theorem 1.1 on coefficient rigidity, this implies there exists a constant $c(k) \in K^\times$ such that

$$a_f(n,k) = c(k) \cdot a_g(n,k) \quad \text{for all } n \geq 1.$$

Consequently,

$$f_k = c(k) \cdot g_k.$$

By assumption, for each $k \in U$, there exists at least one index $n_0 \geq 1$ with $a_g(n_0, k) \neq 0$, otherwise $g_k = 0$, contradicting the hypothesis. Define

$$c(k) := \frac{a_f(n_0, k)}{a_g(n_0, k)}.$$

Because $a_f(n_0, k)$ and $a_g(n_0, k)$ are analytic functions of k, and $a_g(n_0, k) \neq 0$ for all $k \in U$, it follows that $c(k)$ is analytic on U.

For any $m \geq 1$ with $a_g(m, k) \neq 0$,

$$\frac{a_f(m,k)}{a_g(m,k)} = c(k),$$

by the finiteness of $\mathcal{R}_p(f_k, g_k)$ and the rigidity theorem. Extension of $c(k)$ to $\mathcal{W}$ by analytic continuation. Since U is Zariski-dense in the rigid space $\mathcal{W}$, the analytic function $c(k)$ defined on U extends uniquely to a meromorphic function on all of $\mathcal{W}$.

The relation

$$f_k = c(k) \cdot g_k$$

then holds for all $k \in \mathcal{W}$ by analytic continuation of the q-expansions.

At any $k_0 \in \mathcal{W}$ where $g_{k_0} = 0$, we must have $f_{k_0} = 0$. Otherwise, the ratio set $\mathcal{R}_p(f_{k_0}, 0)$ would be infinite (since division by zero is undefined). If g_k vanishes at k_0 with order $m \geq 1$, then f_k must vanish to at least order m to ensure that $c(k) = f_k/g_k$ extends meromorphically and potentially analytically at k_0.

By Lemma 2.3 (Uniform valuation bound), there exists a constant $M > 0$ independent of $k \in U$ and n such that

$$\left| v_p\left(\frac{a_f(n,k)}{a_g(n,k)}\right)\right| \leq M.$$

Since $\frac{a_f(n,k)}{a_g(n,k)} = c(k)$ for all n, this implies

$$|v_p(c(k))| \leq M \quad \text{for all } k \in U.$$

By continuity of the map $k \mapsto |c(k)|_p = p^{-v_p(c(k))}$, because $c(k)$ is analytic, and $|\cdot|_p$ is continuous on K, this boundedness extends to $\mathcal{W}$. $\square$

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[1,2] Mathematics and Intelligent Systems (MASI), National School of Applied Sciences of Tangier (ENSAT), Abdelmalek Essaadi University

Email address: bouchikhi.abdelmonaim@gmail.com

Email address: smezroui@uae.ac.ma