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BACKWARDS FUZZY ITÔ-HENSTOCK INTEGRAL FOR THE FUZZY SET-VALUED STOCHASTIC PROCESS

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Abstract. In this paper, we introduce the backwards fuzzy Itô-Henstock integral for the fuzzy set-valued stochastic process with respect to a Brownian motion. We also formulate a version of Itô isometry and a version of continuity for the fuzzy Itô-Henstock integral.

Keywords. backwards fuzzy Itô-Henstock integral, fuzzy set-valued stochastic process, fuzzy set-valued random variable, fuzzy random set, backwards fuzzy set-valued stochastic integral
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1. Introduction

The Riemann integral is typically the first integral encountered by most students in elementary calculus. However, it has certain limitations as it only encompasses a limited collection of integrable functions. To overcome these limitations, the Lebesgue integral was formulated, which has gained widespread popularity among professional mathematicians. However, the construction of the Lebesgue integral requires a deep understanding of measure theory.

Despite its advantages, the Lebesgue integral fails to integrate highly oscillatory functions effectively. For instance, the function $f(x) = x^2 \sin x^{-2}$ if $x \neq 0$ and $f(0) = 0$ is not Lebesgue integrable on the interval $[0, 1]$. Fortunately, there is an alternative called the Henstock integral, also known as the Henstock-Kurzweil integral. This integral was introduced and studied independently by Henstock and Kurzweil in 1950s to address the integration of highly oscillatory functions and can be considered more general than the Lebesgue integral in certain respects.

The Henstock integral, akin to the Riemann integral, allows for the integration of functions like f on the interval $[0, 1]$, and its integrability can be established using the Henstock integral, as shown in [8, Theorem 9.6]. The Henstock approach, also known as the generalized Riemann or Henstock-Kurzweil approach, presents this type of integral in a manner similar to the Riemann integral. Different from the Lebesgue integral, the Henstock integral which is a Riemann-type definition of an integral, does not require an extensive study of measure theory since its

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definition is more explicit. It is known that the Henstock integral includes the improper Riemann, Lebesgue, and Newton integrals. This integral then has become an interest to several authors, see [8, 9, 11, 12, 16, 17, 18, 22, 23]. The Henstock approach to integration has also been used to deal with stochastic integration for finite and infinite-dimensional spaces, see [5, 13, 14, 15, 24, 25, 27, 28, 30, 31, 32, 33, 34], and with fuzzy-number valued functions, see [4, 6, 7, 19, 35, 36].

In the theory of fuzzy stochastic analysis, the fuzzy Itô integral addresses the need to model systems characterized by two types of uncertainty: randomness from unpredictable fluctuations and imprecision stemming from incomplete or vague information about the system itself. Classical Itô calculus is effective at capturing randomness through stochastic differential equations driven by Brownian motion; however, it assumes that the parameters, coefficients, and initial conditions are precisely known. In reality, these quantities are often uncertain. For example, an engineer might estimate a damping constant as around 0.3, a financial analyst may suggest the volatility is about 20%, or a physicist might only have a range for a diffusion coefficient based on limited measurements. This uncertainty is not merely statistical; it is epistemic, arising from a lack of precise knowledge rather than inherent randomness. Fuzzy set theory provides a way to represent and manage this imprecision by replacing crisp values with fuzzy numbers or fuzzy functions. The fuzzy Itô integral merges these capabilities with stochastic calculus, facilitating integration when the integrand or parameters are fuzzy-valued while the driving process remains stochastic. This combination allows for results that capture both the probabilistic variability of random behavior and the broader blur introduced by imprecise knowledge of the system. As such, it serves as a powerful tool for more realistic modeling in fields such as physics, finance, and control theory, where uncertainty manifests in various forms.

In 2009, Arcede and Cabral [1, 2] introduced the concept of the backwards Itô integral with respect to a Brownian motion. This integral considers processes that initiate at a fixed time $T > 0$ and then move backwards to an earlier time s. They employed the Henstock approach, incorporating the notions of *backwards δ-fine* partial division and backwards filtration. In this case, the tags represent the right endpoints of disjoint left-open sub-intervals.

Building upon the work of Arcede and Cabral, the authors in [27] extended the concept of the backwards Itô-Henstock integral to an operator-valued stochastic processes with respect to a Hilbert space-valued Q-Wiener process. This integral employs backwards filtration as well. Similar to the previous formulation, all processes begin at a fixed time $T > 0$ and then proceed backwards to an earlier time s. The Henstock approach is utilized, along with the notions of backwards δ-fine partial division. They also established the Itô isometry for this integral and presented an equivalent definition using the concept of the $AC^2[0, T]$-property.

In this paper, we introduce the backwards fuzzy Itô-Henstock integral for the fuzzy set-valued stochastic process with respect to a Brownian motion and formulate a version of Itô isometry and a version of continuity.

2. Preliminaries

Let $\mathbb{R}$ be the set of all real numbers and $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space equipped with a family $\{\mathcal{G}_t : 0 \leq t \leq T\}$ of σ-subfields of $\mathcal{G}$ such that $\mathcal{G}_{t_2} \subseteq \mathcal{G}_{t_1}$ for $0 \leq t_1 \leq t_2 \leq T$, called *backwards filtration*. A probability space together with $\{\mathcal{G}_t : 0 \leq t \leq T\}$ or simply $\{\mathcal{G}_t\}$ is called a *backwards filtered probability space* and is denoted by $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$. A backwards filtration $\{\mathcal{G}_t\}$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is called a *standard backwards filtration* if (i) $\mathcal{G}_T$ contains all sets of $\mathbb{P}$-measure zero in $\mathcal{G}$; and (ii) for each $t \in [0,T]$, $\mathcal{G} = \mathcal{G}_{t-} := \cap_{s<t}\mathcal{G}_s$.

A stochastic process or simply a process $f : [0,T] \times \Omega \to \mathbb{R}$ is said to be

(i) *measurable* if it is $[0,T] \times \Omega$-measurable, i.e.

$$\{(t,\omega) \in [0,T] \times \Omega : f_t(\omega) \in A\} \in \mathcal{B}([0,T]) \times \mathcal{G},$$

for each $A \subseteq \mathcal{B}(\mathbb{R})$, where $\mathcal{B}([0,T])$ and $\mathcal{B}(\mathbb{R})$ are the Borel σ-field of $[0,T]$ and $\mathbb{R}$, respectively;

(ii) *adapted to* $\{\mathcal{G}_t\}$ or $\mathcal{G}_t$-*adapted* or simply *backwards adapted* if f_t is $\mathcal{G}_t$-measurable for all $t \in [0,T]$, i.e. for each $t \in [0,T]$, $\{\omega \in \Omega : f_t(\omega) \in A\} \in \mathcal{G}_t$, for each $A \subseteq \mathcal{B}(\mathbb{R})$.

From now onwards, the backwards filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$ shall mean a filtered probability space such that the *real-valued Wiener process* or *Brownian motion* B satisfies (i) B_t is adapted to $\{\mathcal{G}_t\}, t \in [0,T]$ and (ii) $B_t - B_s$ is independent of $\mathcal{G}_t$ for all $0 \leq s \leq t \leq T$.

Let $\mathbb{N}$ be the set of all natural numbers. Denote by $\mathcal{P}_0(\mathbb{R})$ the collection of all nonempty subsets of $\mathbb{R}$, $\mathcal{K}(\mathbb{R})$ the collection of all nonempty closed subsets of $\mathbb{R}$, and denote by $\mathcal{K}_c(\mathbb{R})$ the collection of all sets in $\mathcal{K}(\mathbb{R})$ which are convex. For any $A, B \in \mathcal{P}_0(\mathbb{R})$ and $\lambda \in \mathbb{R}$, define the addition and scalar multiplication, respectively:

$$A + B = \{a + b : a \in A, b \in B\} \text{ and}$$
$$\lambda A = \{\lambda a : a \in A\}.$$

We remark that $\mathcal{P}_0(\mathbb{R})$ together with the above two operations is not a linear space since the inverse of a set A does not necessarily exist. Also, if A and B are bounded closed sets, $A + B$ is not necessarily a closed set. An example is provided in [21, Remark 1.1.1]. The Hausdorff distance of A and B elements of $\mathcal{P}_0(\mathbb{R})$ is defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

where $d(x,B) := \inf_{y\in B}|x-y|$. Let $|||A||| := d_H(A,\{0\}) = \sup_{x\in A}|x|$. We note that if A, B are unbounded, then $d_H(A,B)$ may be infinite.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. Denote by $L^2(\Omega, \mathbb{R})$, the space of all square-integrable random variables from Ω to $\mathbb{R}$. A set-valued mapping $F : \Omega \to \mathcal{K}(\mathbb{R})$ is called a *set-valued random variable* or *random set* if

$$F^{-1}(O) := \{\omega \in \Omega : F(\omega) \cap O \neq \varnothing\} \in \mathcal{G},$$

for each open set $O \subseteq \mathbb{R}$. A random set $F : \Omega \to \mathcal{K}(\mathbb{R})$ is called L^p-*bounded* if $|||F||| \in L^p(\Omega, \mathbb{R})$. Denote by $L^p(\Omega, \mathcal{K}(\mathbb{R}))$ the collection of all L^p-bounded random sets $F : \Omega \to \mathcal{K}(\mathbb{R})$. For $F_1, F_2 \in L^p(\Omega, \mathcal{K}(\mathbb{R}))$, define

$$\Delta_p(F_1, F_2) := \left(\mathbb{E}\left[d_H^p(F_1, F_2)\right]\right)^{1/p} := \left(\int_\Omega d_H^p(F_1(\omega), F_2(\omega))\, d\mathbb{P}(\omega)\right)^{1/p}.$$

Then by [10, Theorem 2.11], for $p \geq 1$, $(L^p(\Omega, \mathcal{K}(\mathbb{R})), \Delta_p)$ is a complete metric space.

A mapping $F : [0,T] \times \Omega \to \mathcal{K}(\mathbb{R})$ is called a *set-valued process* if $F(t,\cdot) : \Omega \to \mathcal{K}(\mathbb{R})$ is a set-valued random variable for every $t \in [0,T]$. A set-valued process $F : [0,T] \times \Omega \to \mathcal{K}(\mathbb{R})$ is said to be

(i) *adapted to* $\{\mathcal{G}_t\}$ or $\mathcal{G}_t$*-adapted* if F_t is $\mathcal{G}_t$-measurable for all $t \in [0,T]$, i.e.
$$F_t^{-1}(O) := \{\omega \in \Omega : F_t(\omega) \cap O \neq \varnothing\} \in \mathcal{G}_t,$$
for each open set $O \subseteq \mathbb{R}$;

(ii) *measurable* if it is $[0,T] \times \Omega$-measurable, i.e.
$$\{(t,\omega) \in [0,T] \times \Omega : F_t(\omega) \cap A \neq \varnothing\} \in \mathcal{B}([0,T]) \times \mathcal{G},$$
for each $A \subseteq \mathcal{B}(\mathbb{R})$.

When no confusion arises, we may refer to a set-valued process F adapted to $\{\mathcal{G}_t\}$ as simply a *backwards adapted* process.

In the following discussions, the given closed interval $[0,T]$ is nondegenerate, i.e. $0 < T$ and can be replaced with any closed interval $[a,b]$. If no confusion arises, we may write $(D)\sum$ instead of $\displaystyle\sum_{i=1}^{n}$ for the given finite collection D.

Definition 2.1. Let δ be a positive function on $[0,T]$. A finite collection $D = \{([u_i,\xi_i],\xi_i)\}_{i=1}^n$ of inteval-point pairs is a *backwards* δ*-fine partial division* of $[0,T]$ if $\{[u_i,\xi_i]\}_{i=1}^n$ is a collection of non-overlapping subintervals of $[0,T]$ and each $[u_i,\xi_i]$ is backwards δ-fine, i.e. $[u_i,\xi_i] \subset (\xi_i - \delta(\xi_i), \xi_i]$.

The term *partial division* is used in Definition 2.1 to emphasize that the finite collection of non-overlapping intervals of $[0,T]$ may not cover the entire interval $[0,T]$. Using the Vitali covering theorem, the following concept can be defined.

Definition 2.2. Given $\eta > 0$, a given backwards δ-fine partial division $D = \{([u,\xi],\xi)\}$ of $[0,T]$ is said to be a backwards (δ,η)*-fine partial division* of $[0,T]$ if it fails to cover $[0,T]$ by at most length η, that is,

$$\left|T - (D)\sum (\xi - u)\right| \leq \eta.$$

Definition 2.3. [29, Definition 3] Let $F : [0,T] \times \Omega \to \mathcal{K}_c(\mathbb{R})$ be a backwards adapted process. Then F is said to be *backwards Itô-Henstock integrable*, or $\mathcal{IH}^B$-integrable on $[0,T]$ with respect to a Brownian motion B if there exists $A \in L^2(\Omega, \mathcal{K}_c(\mathbb{R}))$ such that for every $\epsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any backwards (δ,η)-fine partial division $D = \{([u_i,\xi_i],\xi_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\Delta_2^2\left((D)\sum F_\xi(B_\xi - B_u), A\right) < \epsilon.$$

In this case, F is $\mathcal{IH}^B$-integrable to A on $[0,T]$ and A is called the *backwards Itô-Henstock integral* or *$\mathcal{IH}^B$-integral* of the set-valued process F which will be denoted by $(\mathcal{IH}^B)\int_0^T F_t\, dB_t$ or $(\mathcal{IH}^B)\int_0^T F\, dB$. Denote by $\Lambda^2_{\mathcal{IH}^B}(\mathcal{K}_c(\mathbb{R}))$, the collection of all $\mathcal{IH}^B$-integrable set-valued processes F on $[0,T]$.

It is worth noting that the backwards Itô-Henstock integral of a set-valued process possesses the standard properties of an integral namely, uniqueness of an integral, linearity, and integrability on every subinterval of $[0,T]$, the sequential definition, and the Cauchy criterion. The following results are found in [29].

Lemma 2.4. [29, Lemma 4] *Let $F \in \Lambda^2_{\mathcal{IH}^B}(\mathcal{K}_c(\mathbb{R}))$ and define*

$$A[u,\xi] := (\mathcal{IH}^B)\int_u^\xi F\, dB$$

for any $[u,\xi] \subseteq [0,T]$. Then for any $\epsilon > 0$, there exist a positive function δ such that

$$\Delta_2^2\left((D)\sum F_\xi(B_\xi - B_u), (D)\sum A[u,\xi]\right) < \epsilon$$

whenever $D = \{[u,\xi],\xi\}$ is a backwards δ-fine partial division of $[0,T]$.

Theorem 2.5. [29, Theorem 8] *Let $F : [0,T]\times\Omega \to \mathcal{K}_c(\mathbb{R})$ be a backwards adapted process. If F is $\mathcal{IH}^B$-integrable on $[0,T]$, then $\mathbb{E}\left[\|\|F_t\|\|^2\right]$ is Lebesgue integrable on $[0,T]$ and*

$$\mathbb{E}\left[\left|\left|\left|(\mathcal{IH}^B)\int_0^T F_t\, dB_t\right|\right|\right|^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|\|F_t\|\|^2\right]\, dt < \infty.$$

Definition 2.6. [29, Definition 4] A set-valued process $A : [0,T]\times\Omega \to \mathcal{K}_c(\mathbb{R})$ is said to be continuous with respect to Δ_2 if for each $t \in [0,T]$,

$$\mathbb{E}\left[\|\|A_t\|\|^2\right] < \infty \text{ and } \lim_{s\to t}\Delta_2(A_s, A_t) = 0.$$

Theorem 2.7. [29, Theorem 9] *Let $F \in \Lambda^2_{\mathcal{IH}^{\mathcal{B}}}(\mathcal{K}_c(\mathbb{R}))$ and define*

$$A_t := (\mathcal{IH}^B)\int_0^t F_s\, dB_s.$$

Then A is backwards adapted and continuous with respect to Δ_2.

3. Backwards Fuzzy Itô-Henstock Integral

This section presents the backwards fuzzy Itô-Henstock integral of a fuzzy set-valued stochastic process with respect to a Brownian motion B.

Let $\mathcal{F}(\mathbb{R})$ (resp., $\mathcal{F}_c(\mathbb{R})$) be the collection of mappings $u : \mathbb{R} \to [0,1]$ such that $[u]^\alpha := \{x \in \mathbb{R} : u(x) \geq \alpha\} \in \mathcal{K}(\mathbb{R})$ (resp., $[u]^\alpha \in \mathcal{K}_c(\mathbb{R})$) for all $\alpha \in (0,1]$ and $[u]^1 := \{x \in \mathbb{R} : u(x) = 1\} \neq \varnothing$. The set $[u]^\alpha$ is called the *level set*, or *α-cut*. The elements of $\mathcal{F}(\mathbb{R})$ are called *fuzzy sets*. Define the *support set*

$$u_{0+} := Cl\left(\{x \in \mathbb{R} : u(x) > 0\}\right).$$

A fuzzy set-valued mapping $G : \Omega \to \mathcal{F}(\mathbb{R})$ is called a *fuzzy set-valued random variable* or *fuzzy random set* [26] if for every $\alpha \in (0, 1]$, $G^\alpha : \Omega \to \mathcal{K}(\mathbb{R})$ defined by

$$G^\alpha(\omega) := \{x \in \mathbb{R} : G(\omega)(x) \geq \alpha\},$$

is a set-valued random variable. A fuzzy random set $G : \Omega \to \mathcal{F}(\mathbb{R})$ is said to be L^p*-bounded* [20] if $|||G_{0+}||| \in L^p(\Omega, \mathbb{R})$. It is known that [20, p.172]

$$|||G_{0+}||| := \sup_{\alpha\in(0,1]} |||G^\alpha|||.$$

Denote by $L^p(\Omega, \mathcal{F}(\mathbb{R}))$ (resp., $L^p(\Omega, \mathcal{F}_c(\mathbb{R}))$) the collection of all L^p-bounded fuzzy random sets $G : \Omega \to \mathcal{F}(\mathbb{R})$ (resp., $G : \Omega \to \mathcal{F}_c(\mathbb{R})$). For $G_1, G_2 \in L^p(\Omega, \mathcal{F}(\mathbb{R}))$, define

$$\mathcal{D}_p(G_1, G_2) := \sup\nolimits_{\alpha\in(0,1]} \Delta_p(G_1^\alpha, G_2^\alpha).$$

Then by [10, Theorem 2.11] $p \geq 1$, the space $(L^p(\Omega, \mathcal{F}(\mathbb{R})), \mathcal{D}_p)$ is a complete metric space and $L^p(\Omega, \mathcal{F}_c(\mathbb{R}))$ is a closed subspace of $L^p(\Omega, \mathcal{F}(\mathbb{R}))$. Two fuzzy random sets G_1 and G_2 are considered to be *identical* [10, p.497] if $\mathcal{D}_p(G_1, G_2) = 0$. Moreover, since every closed and convex set in $\mathbb{R}$ is compact, if $G_1, G_2 \in \mathcal{F}_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we define

$$(G_1 + G_2)^\alpha = G_1^\alpha + G_2^\alpha \text{ and } (\lambda G_1)^\alpha = \lambda G_1^\alpha,$$

see [20, p.167].

Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, \mathbb{P})$ be a backwards filtered probability space. A mapping $G : [0, T] \times \Omega \to \mathcal{F}(\mathbb{R})$ is called a *fuzzy set-valued process* or *fuzzy process* if $G(t, \cdot) : \Omega \to \mathcal{F}(\mathbb{R})$ is a fuzzy set-valued random variable for every $t \in [0, T]$. A fuzzy process $G : [0, T] \times \Omega \to \mathcal{F}_c(\mathbb{R})$ is said to be

(i) *adapted to* $\{\mathcal{G}_t\}$ or $\mathcal{G}_t$*-adapted* or simply *backwards adapted* if $G^\alpha : [0, T] \times \Omega \to \mathcal{K}_c(\mathbb{R})$ is backwards adapted for every $\alpha \in (0, 1]$; and

(ii) *measurable* if $G^\alpha : [0, T] \times \Omega \to \mathcal{K}_c(\mathbb{R})$ is measurable for every $\alpha \in (0, 1]$.

When no confusion arises, we may refer to a fuzzy set-valued process G adapted to $\{\mathcal{G}_t\}$ as simply a *backwards adapted fuzzy process.*

Next, we define the backwards fuzzy Itô-Henstock integral of a fuzzy set-valued stochastic process with respect to a Brownian motion.

Definition 3.1. Let $G : [0, T] \times \Omega \to \mathcal{F}_c(\mathbb{R})$ be a backwards adapted fuzzy process. Then G is said to be *backwards fuzzy Itô-Henstock integrable* or $\mathcal{IH}_f^B$*-integrable*, on $[0, T]$ with respect to a Brownian motion B if there exists $A \in L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$ such that for every $\epsilon > 0$, there exists a positive function δ on $[0, T]$ and a positive number η such that for any backwards (δ, η)-fine partial division $D = \{([u_i, \xi_i], \xi_i)\}_{i=1}^n$ of $[0, T]$, we have

$$\mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) < \epsilon.$$

In this case, G is $\mathcal{IH}_f^B$-integrable to A on $[0, T]$ and A is called *backwards fuzzy Itô-Henstock integral* or $\mathcal{IH}_f^B$*-integral* of the fuzzy set-valued process G which will be denoted by $(\mathcal{IH}_f^B)\displaystyle\int_0^T G_t\, dB_t$ or $(\mathcal{IH}_f^B)\displaystyle\int_0^T G\, dB$. Denote by $\Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$, the collection of all $\mathcal{IH}_f^B$-integrable fuzzy-valued processes G on $[0,T]$.

Before we provide an example of an $\mathcal{IH}_f^B$-integrable fuzzy-valued processes, let us consider first some operations on intervals. Let A and B be two closed intervals on the real line given by $A = [a_1, a_2]$ and $B = [b_1, b_2]$. The sum, difference, and scalar multiplication are defined by $A + B = [a_1 + b_1, a_2 + b_2]$, $A - B = [a_1 - b_2, a_2 - b_1]$, and

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2] & \text{if } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1] & \text{if } \lambda < 0 \end{cases},$$

respectively.

Denote by $\Lambda^2_{\mathcal{IH}_B}(\mathbb{R})$ the collection of all $\mathcal{IH}_B$-integrable real-valued processes $f : [0,T] \times \Omega \to \mathbb{R}$ with integral $(\mathcal{IH}_B)\displaystyle\int_0^T f_t\, dB_t$, see [27].

Example 3.2. Let $f^{(1)}, f^{(2)} \in \Lambda^2_{\mathcal{IH}_B}(\mathbb{R})$ with $f_t^{(1)}(\omega) \leq f_t^{(2)}(\omega)$ for all $(t,\omega) \in [0,T]\times\Omega$. Define $G : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ such that $G_t(\omega)$ is an adapted symmetric triangular fuzzy number [3, Definition 2.4], that is,

$$G_t(\omega)(x) = \begin{cases} 0 & \text{if } x \leq f_t^{(1)}(\omega) \\ \frac{x - f_t^{(1)}(\omega)}{u - f_t^{(1)}(\omega)} & \text{if } f_t^{(1)}(\omega) < x \leq u \\ \frac{x - f_t^{(2)}(\omega)}{u - f_t^{(2)}(\omega)} & \text{if } u < x \leq f_t^{(2)}(\omega) \\ 0 & \text{if } x \geq f_t^{(2)}(\omega) \end{cases}, \tag{3.1}$$

where u is symmetric in relation to $f_t^{(1)}$ and $f_t^{(2)}$, that is, $u - f_t^{(1)} = f_t^{(2)} - u$. Then G is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $(\mathcal{IH}_f^B)\displaystyle\int_0^T G_t\, dB_t = A$, where

$$\begin{aligned} A^\alpha &= \Big[(\mathcal{IH}_B)\int_0^T \left(\left(1 - \frac{\alpha}{2}\right) f_t^{(1)} + \frac{\alpha}{2}\, f_t^{(2)}\right) dB_t, \\ &\qquad (\mathcal{IH}_B)\int_0^T \left(\left(1 - \frac{\alpha}{2}\right) f_t^{(2)} + \frac{\alpha}{2}\, f_t^{(1)}\right) dB_t\Big] \\ &:= \left[A^{(1,2)}, A^{(2,1)}\right]. \end{aligned}$$

Proof. Let $f^{(1)}, f^{(2)} \in \Lambda^2_{\mathcal{IH}_B}(\mathbb{R})$ with $f_t^{(1)}(\omega) \leq f_t^{(2)}(\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$. Let $\epsilon > 0$ and $\alpha \in (0,1]$. Using the linearity of $\mathcal{IH}_B$-integral, $f^{(1,2)} := \left(1 - \frac{\alpha}{2}\right) f^{(1)} + \frac{\alpha}{2}\, f^{(2)}$ and $f^{(2,1)} := \left(1 - \frac{\alpha}{2}\right) f^{(2)} + \frac{\alpha}{2}\, f^{(1)}$ are elements of $\Lambda^2_{\mathcal{IH}_B}(\mathbb{R})$. Then there exist a positive function δ_1 on $[0,T]$ and a positive number η_1 such that for any backwards (δ_1, η_1)-fine partial division D_1 of $[0,T]$,

$$\mathbb{E}\left[\left|(D_1)\sum f_\xi^{(1,2)}(B_\xi - B_u) - A^{(1,2)}\right|^2\right] < \epsilon.$$

Moreover, there exist a positive function δ_2 on $[0,T]$ and a positive number η_2 such that for any backwards (δ_2,η_2)-fine partial division D_2 of $[0,T]$,

$$\mathbb{E}\left[\left|(D_2)\sum f_\xi^{(2,1)}(B_\xi - B_u) - A^{(2,1)}\right|^2\right] < \epsilon.$$

Take $\delta(\xi) = \min\{\delta_1(\xi),\delta_2(\xi)\}$ for every $\xi \in [0,T]$ and $\eta = \min\{\eta_1,\eta_2\}$. Let D be any backwards (δ,η)-fine partial division of $[0,T]$. Now, let $G : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ be a symmetric triangular fuzzy number given in (3.1). Then the α-level set of G is given by

$$\begin{aligned}
G_t^\alpha(\omega) &= \{x \in \mathbb{R} : G_t(\omega)(x) \geq \alpha\} \\
&= \left[f_t^{(1)}(\omega) + \left(u - f_t^{(1)}(\omega)\right)\alpha, f_t^{(2)}(\omega) + \left(u - f_t^{(2)}(\omega)\right)\alpha\right] \\
&= \left[\left(1-\frac{\alpha}{2}\right)f_t^{(1)}(\omega) + \frac{\alpha}{2}f_t^{(2)}(\omega), \left(1-\frac{\alpha}{2}\right)f_t^{(2)}(\omega) + \frac{\alpha}{2}f_t^{(1)}(\omega)\right] \\
&= \left[f_t^{(1,2)}(\omega), f_t^{(2,1)}(\omega)\right].
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\Delta_2^2\left((D)\sum G_\xi^\alpha(B_\xi - B_v), A^\alpha\right) \\
=\ &\Delta_2^2\left((D)\sum\left[f_\xi^{(1,2)}(B_\xi - B_u), f_\xi^{(2,1)}(B_\xi - B_u)\right], \left[A^{(1,2)}, A^{(2,1)}\right]\right) \\
=\ &\Delta_2^2\left(\left[(D)\sum f_\xi^{(1,2)}(B_\xi - B_u), (D)\sum f_\xi^{(2,1)}(B_\xi - B_u)\right], \left[A^{(1,2)}, A^{(2,1)}\right]\right) \\
=\ &\mathbb{E}\left[d_H^2\left(\left[(D)\sum f_\xi^{(1,2)}(B_\xi - B_u), (D)\sum f_\xi^{(2,1)}(B_\xi - B_u)\right], \left[A^{(1,2)}, A^{(2,1)}\right]\right)\right] \\
<\ &\epsilon.
\end{aligned}$$

Since α is arbitrarily chosen,

$$\mathfrak{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) = \sup_{\alpha\in(0,1]}\Delta_2^2\left((D)\sum G_\xi^\alpha(B_\xi - B_u), A^\alpha\right) \leq \epsilon.$$

Therefore, G is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $(\mathcal{IH}_f^B)\int_0^T G_t\, dB_t = A$. $\square$

Lemma 3.3. *Let $G : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy process and $\alpha \in (0,1]$. If $G \in L^2(\Omega,\mathcal{F}_c(\mathbb{R}))$, then $G^\alpha \in L^2(\Omega,\mathcal{K}_c(\mathbb{R}))$.*

Proof. Let $\alpha \in (0,1]$ and $G : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ be a fuzzy process. Suppose that $G \in L^2(\Omega,\mathcal{F}_c(\mathbb{R}))$. Then $|||G_{0+}||| \in L^2(\Omega,\mathbb{R})$. Thus $|||G^\alpha||| \in L^2(\Omega,\mathbb{R})$. Hence, $G^\alpha \in L^2(\Omega,\mathcal{K}_c(\mathbb{R}))$. $\square$

Lemma 3.4. *Let $G : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ be a backwards adapted fuzzy process. If G is $\mathcal{IH}_f^B$-integrable on $[0,T]$, then $G^\alpha : [0,T]\times\Omega \to \mathcal{K}_c(\mathbb{R})$ is $\mathcal{IH}^B$-integrable on $[0,T]$ for each $\alpha \in (0,1]$.*

Proof. Let $\alpha \in (0,1]$. Suppose that G is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $A :=$ $(\mathcal{IH}_f^B)\int_0^T G_t\, dB_t$. Then there exist a positive function δ on $[0,T]$ and a positive number η such that

$$\mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) < \epsilon,$$

for any backwards (δ,η)-fine partial division D of $[0,T]$. By Lemma 3.3, $A^\alpha \in L^2(\Omega, \mathcal{K}_c(\mathbb{R}))$. Thus,

$$\begin{aligned} \Delta_2^2\left((D)\sum G_\xi^\alpha(B_\xi - B_u), A^\alpha\right) &= \Delta_2^2\left(\left[(D)\sum G_\xi(B_\xi - B_u)\right]^\alpha, A^\alpha\right) \\ &\leq \mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) \\ &< \epsilon. \end{aligned}$$

Hence, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ and

$$(\mathcal{IH}^B)\int_0^T G_t^\alpha \, dB_t = A^\alpha = \left[(\mathcal{IH}_f^B)\int_0^T G_t \, dB_t\right]^\alpha.$$

This completes the proof of the lemma. $\square$

Now, we present the standard properties of the backwards fuzzy Itô-Henstock integral.

Theorem 3.5. *The backwards fuzzy Itô-Henstock integral of a fuzzy set-valued adapted process is uniquely determined.*

Proof. Let $\epsilon > 0$ and $\alpha \in (0,1]$. Suppose that A_1 and A_2 are $\mathcal{IH}_f^B$-integrals of G. Then

$$(\mathcal{IH}_f^B)\int_0^T G \, dB = A_1 \text{ and } (\mathcal{IH}_f^B)\int_0^T G \, dB = A_2.$$

By Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$. Thus, by Lemma 3.3, $A_1^\alpha, A_2^\alpha \in L^2(\Omega, \mathcal{K}_c(\mathbb{R}))$ and A_1^α and A_2^α are integrals of G^α. Then

$$\Delta_2^2\left(A_1^\alpha, A_2^\alpha\right) < \frac{\epsilon}{2}.$$

Thus, $\mathcal{D}_2^2\left(A_1, A_2\right) < \epsilon$. Since $\mathcal{D}_2$ is a metric, $A_1 = A_2$. $\square$

Theorem 3.6. *Let $\beta \in \mathbb{R}$. If $G_1, G_2 \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$, then*

(i) $G_1 + G_2 \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$, *and*

$$(\mathcal{IH}_f^B)\int_0^T (G_1 + G_2) \, dB = (\mathcal{IH}_f^B)\int_0^T G_1 \, dB + (\mathcal{IH}_f^B)\int_0^T G_2 \, dB;$$

(ii) $\beta \cdot G_1 \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$, *and*

$$(\mathcal{IH}_f^B)\int_0^T (\beta \cdot G_1) \, dB = \beta \cdot (\mathcal{IH}_f^B)\int_0^T G_1 \, dB.$$

Proof. Let $\epsilon > 0$ and $\beta \in \mathbb{R}$. Assume that $G_1, G_2 \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ with $(\mathcal{IH}_f^B)\int_0^T G_1 \, dB = A_1$ and $(\mathcal{IH}_f^B)\int_0^T G_2 \, dB = A_2$, respectively. By Lemma 3.4, G_1^α and G_2^α are

$\mathcal{IH}^B$-integrable on $[0,T]$. By Lemma 3.3, $A_1^\alpha, A_2^\alpha \in L^2(\Omega, \mathcal{K}_c(\mathbb{R}))$ with A_1^α and A_2^α as $\mathcal{IH}^B$-integrals of G_1^α and G_2^α, respectively. Then

$$\Delta_2^2\left(\left[(D)\sum(G_1+G_2)(\xi)(B_\xi - B_u)\right]^\alpha, [A_1+A_2]^\alpha\right) < \frac{\epsilon}{2}.$$

It follows that

$$\mathcal{D}_2^2\left((D)\sum(G_1+G_2)(\xi)(B_\xi - B_u), A_1+A_2\right) < \epsilon.$$

Thus, (i) holds.

To prove (ii), notice that when $\beta = 0$, the conclusion holds trivially. Hence, we may assume that $\beta \neq 0$, that is, $|\beta| > 0$. Let $\epsilon > 0$ and $\alpha \in (0,1]$. Suppose that G_1 is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $(\mathcal{IH}_f^B)\displaystyle\int_0^T G_1\, dB = A_1$. By Lemma 3.3 and Lemma 3.4, G_1^α is $\mathcal{IH}^B$-integrable on $[0,T]$ with integral A_1^α. Then $\beta \cdot G_1^\alpha$ is $\mathcal{IH}^B$-integrable on $[0,T]$ with integral $\beta \cdot A_1^\alpha$, and

$$\Delta_2^2\left(\left[(D)\sum(\beta \cdot G_1(\xi))(B_\xi - B_u)\right]^\alpha, [\beta \cdot A_1]^\alpha\right) < \frac{\epsilon}{2}.$$

It follows that

$$\mathcal{D}_2^2\left((D)\sum \beta \cdot G_1(\xi)(B_\xi - B_u), \beta \cdot A_1\right) < \epsilon.$$

Thus (ii) also holds. $\square$

Theorem 3.7 (Cauchy criterion). *$G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ if and only if for every $\epsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that for any two backwards (δ,η)-fine partial divisions $D_1 = \{([u^{(1)}, \xi^{(1)}]\xi^{(1)})\}$ and $D_2 = \{([u^{(2)}, \xi^{(2)}]\xi^{(2)})\}$ of $[0,T]$, we have*

$$\mathcal{D}_2^2\left((D_1)\sum G_{\xi^{(1)}}(B_{\xi^{(1)}} - B_{u^{(1)}}), (D_2)\sum G_{\xi^{(2)}}(B_{\xi^{(2)}} - B_{u^{(2)}})\right) < \epsilon.$$

Proof. Let $\epsilon > 0$ and $\alpha \in (0,1]$. Assume that $G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ with

$$(\mathcal{IH}_f^B)\int_0^T G\, dB = A.$$

Then by Lemma 3.3 and Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ with integral A^α. Hence, there exist a positive function δ on $[0,T]$ and a positive number η such that

$$\Delta_2^2\left((D_1)\sum G_\xi^\alpha(B_\xi - B_u), (D_2)\sum G_\xi^\alpha(B_\xi - B_u)\right) < \frac{\epsilon}{2},$$

for any two backwards (δ,η)-fine partial divisions D_1 and D_2 of $[0,T]$. It follows that

$$\mathcal{D}_2^2\left((D_1)\sum G_\xi(B_\xi - B_u), (D_2)\sum G_\xi(B_\xi - B_u)\right) < \epsilon.$$

Conversely, let $\epsilon > 0$ be given. Then there exist a positive function $\overline{\delta}$ on $[0,T]$ and a positive constant $\overline{\eta}$ such that for any two backwards $(\overline{\delta},\overline{\eta})$-fine partial divisions D_1 and D_2 of $[0,T]$, we have

$$\mathcal{D}_2^2\left((D_1)\sum G_\xi(B_\xi - B_u), (D_2)\sum G_\xi(B_\xi - B_u)\right) < \frac{\epsilon}{8}.$$

For each $n \in \mathbb{N}$, let δ_n be the corresponding positive function on $[0,T]$ and η_n be the corresponding positive number. We may choose the sequences $\{\delta_n(\xi)\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ to be decreasing. Let D_n be any backwards (δ_n, η_n)-fine partial division on $[0,T]$. Hence, for all $m \geq n$, D_m is also a backwards (δ_n, η_n)-fine partial division on $[0,T]$. Thus, if $m_1, m_2 \geq n$, then

$$\mathfrak{D}_2^2\left((D_{m_1})\sum G_\xi(B_\xi - B_u), (D_{m_2})\sum G_\xi(B_\xi - B_u)\right) < \frac{1}{n}.$$

This means that $\{(D_m)\sum G_\xi(B_\xi - B_u)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$. Since $L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$ is a complete metric space, it follows that there exists $A \in L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$ such that

$$\lim_{n\to\infty} (D_n)\sum G_\xi(B_\xi - B_u) = A \text{ in } L^2(\Omega, \mathcal{F}_c(\mathbb{R})).$$

This means that there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\mathfrak{D}_2^2\left((D_n)\sum G_\xi(B_\xi - B_u), A\right) < \frac{\epsilon}{8}.$$

In particular,

$$\mathfrak{D}_2^2\left((D_N)\sum G_\xi(B_\xi - B_u), A\right) < \frac{\epsilon}{8},$$

for any backwards (δ_N, η_N)-fine partial division D_N of $[0,T]$. Let $N_2 \in \mathbb{N}$ be such that $\frac{1}{N_2} \leq \frac{\epsilon}{8}$. Take $N = \max\{N_1, N_2\}$. Choose $\delta(\xi) = \min\{\delta_N(\xi), \overline{\delta}(\xi)\}$ for all $\xi \in [0,T]$ and $\eta = \min\{\eta_N, \overline{\eta}\}$. Then any backwards (δ, η)-fine partial division D of $[0,T]$ is also a backwards (δ_N, η_N)-fine partial division of $[0,T]$. Thus, by Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\Delta_2^2\left((D)\sum [G_\xi(B_\xi - B_u)]^\alpha, A^\alpha\right) \\
\leq\ & 4\ \Delta_2^2\left((D)\sum [G_\xi(B_\xi - B_u)]^\alpha, (D_N)\sum [G_\xi(B_\xi - B_u)]^\alpha\right) \\
&+4\ \Delta_2^2\left((D_N)\sum [G_\xi(B_\xi - B_u)]^\alpha, A^\alpha\right).
\end{aligned}$$

Thus, taking the supremum both sides over $\alpha \in (0,1]$, we have

$$\begin{aligned}
&\mathfrak{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) \\
\leq\ & 4\ \mathfrak{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), (D_N)\sum G_\xi(B_\xi - B_u)\right) \\
&+4\ \mathfrak{D}_2^2\left((D_N)\sum G_\xi(B_\xi - B_u), A\right) \\
<\ & 4\left(\frac{1}{N}\right) + 4\left(\frac{\epsilon}{8}\right) \leq 4\left(\frac{1}{N_2}\right) + 4\left(\frac{\epsilon}{8}\right) \\
\leq\ & 4\left(\frac{\epsilon}{8}\right) + 4\left(\frac{\epsilon}{8}\right) = \epsilon.
\end{aligned}$$

Hence, $G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$. $\square$

Theorem 3.8 (Sequential definition). *$G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ if and only if there exist $A \in L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$, a decreasing sequence $\{\delta_n\}$ of positive functions on $[0,T]$,* *and a decreasing sequence $\{\eta_n\}$ of positive numbers such that for any backwards (δ_n, η_n)-fine partial division P_n of $[0,T]$, we have*

$$\mathcal{D}_2^2\left((P_n)\sum F_\xi(B_\xi - B_u), A\right) \to 0 \text{ as } n \to \infty.$$

In this case, $A = (\mathcal{IH}_f^B)\int_0^T G\ dB$.

Proof. Let $\alpha \in (0,1]$. Suppose that G be a $\mathcal{IH}_f^B$-integrable on $[0,T]$ with integral A. Then by Lemma 3.3 and Lemma 3.4, G^α is a $\mathcal{IH}^B$-integrable on $[0,T]$ with integral A^α. Hence, for each $n \in \mathbb{N}$, there exist a positive function δ_n on $[0,T]$ and a positive number η_n such that

$$\Delta_2^2\left((P_n)\sum G_\xi^\alpha(B_\xi - B_u), A^\alpha\right) < \frac{1}{n}.$$

we may choose sequences $\{\delta_n(\xi)\}$ and $\{\eta_n\}$ to be decreasing. Hence, we obtain

$$\lim_{n\to\infty} \Delta_2^2\left((P_n)\sum G_\xi^\alpha(B_\xi - B_u), A^\alpha\right) = 0.$$

It follows that

$$\lim_{n\to\infty} \mathcal{D}_2^2\left((P_n)\sum G_\xi(B_\xi - B_u), A\right) = 0.$$

Equivalently,

$$\mathcal{D}_2^2\left((P_n)\sum F_\xi(B_\xi - B_u), A\right) \to 0 \text{ as } n \to \infty.$$

Conversely, assume that there exist an $A \in L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$, a decreasing sequence $\{\delta_n\}$ of positive functions on $[0,T]$ and positive numbers $\{\eta_n\}$ such that for any backwards (δ, η)-fine partial division D_n on $[0,T]$, we have

$$\mathcal{D}_2^2\left((P_n)\sum G_\xi(B_\xi - B_u), A\right) \to 0 \text{ as } n \to \infty.$$

This means that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\mathcal{D}_2^2\left((P_n)\sum G_\xi(B_\xi - B_u), A\right) < \epsilon.$$

Suppose on the contrary that G is not $\mathcal{IH}_f^B$-integrable to A on $[0,T]$. Then there exists $\epsilon > 0$ such that for any positive function δ on $[0,T]$ and any positive number η,

$$\mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) \geq \epsilon$$

for some backwards (δ, η)-fine partial division D of $[0,T]$. This means that for each positive function δ_n and positive number η_n, there exists a backwards (δ_n, η_n)-fine partial division P_n on $[0,T]$ with

$$\mathcal{D}_2^2\left((P_n)\sum G_\xi(B_\xi - B_u), A\right) \geq \epsilon,$$

a contradiction. $\square$

Theorem 3.9. *If $G : [0,T] \times \Omega \to \mathcal{F}_c(\mathbb{R})$ is $\mathcal{IH}_f^{\mathcal{B}}$-integrable on $[0,a]$ and $[a,T]$, where $a \in (0,T)$, then G is $\mathcal{IH}_f^{\mathcal{B}}$-integrable on $[0,T]$ and*

$$(\mathcal{IH}_f^{\mathcal{B}}) \int_0^T G \, dB = (\mathcal{IH}_f^{\mathcal{B}}) \int_0^a G \, dB + (\mathcal{IH}_f^{\mathcal{B}}) \int_a^T G \, dB.$$

Proof. Let $a \in (0,T)$ and $\alpha \in (0,1]$. Suppose that G is an $\mathcal{IH}_f^B$-integrable on $[0,a]$ and $[a,T]$ with integrals

$$A_a := (\mathcal{IH}_f^B) \int_0^a G \, dB \text{ and } A_T := (\mathcal{IH}_f^B) \int_a^T G \, dB.$$

Let $\epsilon > 0$. Then by Lemma 3.3 and Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,a]$ and $[a,T]$ with integrals A_a^α and A_T^α, respectively. Hence, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ and

$$(\mathcal{IH}^B) \int_0^T G^\alpha \, dB = (\mathcal{IH}^B) \int_0^a G^\alpha \, dB + (\mathcal{IH}^B) \int_a^T G^\alpha \, dB,$$

where $(\mathcal{IH}^B) \int_0^a G^\alpha \, dB := A_a^\alpha$ and $(\mathcal{IH}^B) \int_a^T G^\alpha \, dB := A_T^\alpha$, we have

$$\Delta_2^2 \left((D) \sum G_\xi^\alpha (B_\xi - B_u), A_a^\alpha + A_T^\alpha \right) < \frac{\epsilon}{2},$$

for any backwards (δ, η)-fine partial division D of $[0,T]$. It follows that

$$\mathcal{D}_2^2 \left((D) \sum G_\xi (B_\xi - B_u), A_a + A_T \right) < \epsilon.$$

Therefore, the theorem is proved. □

Theorem 3.10. *If $G : [0,T] \times \Omega \to \mathcal{F}_c(\mathbb{R})$ is $\mathcal{IH}_f^B$-integrable on $[0,T]$, then G is also $\mathcal{IH}_f^B$-integrable on any subinterval $[c,d]$ of $[0,T]$.*

Proof. It suffices to show that G is $\mathcal{IH}_f^B$-integrable on $[0,a]$ and $[a,T]$. We shall only prove the integrability of G on $[a,T]$. A similar proof will be used to show the integrability of G on $[0,a]$. Let $\epsilon > 0$ and $\alpha \in (0,1]$. Suppose that G is $\mathcal{IH}_f^B$-integrable on $[0,T]$. Then by Lemma 3.4, G^α is $\mathcal{IH}^B$ integrable on $[0,T]$. Hence, G^α is $\mathcal{IH}^B$ integrable on $[a,T]$. Moreover,

$$\Delta_2^2 \left((D_2) \sum G_\xi^\alpha (B_\xi - B_u), (D_2') \sum G_\xi^\alpha (B_\xi - B_u) \right) < \frac{\epsilon}{2},$$

for any backwards (δ, η)-fine partial divisions D_2 and D_2' of $[a,T]$. It follows that

$$\mathcal{D}_2^2 \left((D_2) \sum G_\xi (B_\xi - B_u), (D_2') \sum G_\xi (B_\xi - B_u) \right) < \epsilon.$$

Hence, by Theorem 3.7 (Cauchy criterion), G is $\mathcal{IH}_f^B$-integrable on $[a,T]$. □

Lemma 3.11 (Saks-Henstock lemma)**.** *Let $G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ and define*

$$A[u,\xi] := (\mathcal{IH}_f^B) \int_u^\xi G \, dB$$

for any $[u,\xi] \subseteq [0,T]$. *Then for any* $\epsilon > 0$, *there exist a positive function* δ *on* $[0,T]$ *such that*

$$\mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), (D)\sum A[u,\xi]\right) < \epsilon$$

whenever $D = \{([u,\xi],\xi)\}$ *is a backwards* δ*-fine partial division of* $[0,T]$.

Proof. Let $[\xi,u] \subseteq [0,T]$ and $\alpha \in (0,1]$. Suppose that G is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $A[u,\xi] := (\mathcal{IH}_f^B)\int_u^\xi G\, dB$. Let $\epsilon > 0$. Then by Lemma 3.3 and Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ and define $A^\alpha[u,\xi] := (\mathcal{IH}^B)\int_u^\xi G^\alpha\, dB$. Hence, there exist a positive function δ on $[0,T]$ such that

$$\Delta_2^2\left((D)\sum G_\xi^\alpha(B_\xi - B_u), (D)\sum A^\alpha[u,\xi]\right) < \frac{\epsilon}{2},$$

whenever $D = \{([u,\xi],\xi)\}$ is backwards δ-fine partial division of $[0,T]$. It follows that

$$\mathcal{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), (D)\sum A[u,\xi]\right) < \epsilon.$$

Hence, the result is proved. □

4. Itô Isometry and Continuity of Integral

This section presents a version of Itô isometry and a version of continuity of the backwards fuzzy Itô-Henstock integral.

Lemma 4.1. *For each* $\alpha \in (0,1]$, *let* f_α *be Lebesgue integrable functions on* $[0,T]$. *If there exists a sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *in* $(0,1]$ *such that*

$$f_{\alpha_n} \to \sup_{\alpha\in(0,1]} f_\alpha \text{ uniformly a.e. on } [0,T],$$

then $\sup_{\alpha\in(0,1]} f_\alpha$ *is Lebesgue integrable on* $[0,T]$ *and*

$$\sup_{\alpha\in(0,1]}(\mathcal{L})\int_0^T f_\alpha(t)\, dt = (\mathcal{L})\int_0^T \sup_{\alpha\in(0,1]} f_\alpha(t)\, dt.$$

Theorem 4.2 (Version of Itô-isometry). *Let* $G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}_c(\mathbb{R}))$ *with*

$$A = (\mathcal{IH}_f^B)\int_0^T G_t\, dB_t.$$

For each $\alpha \in (0,1]$, *suppose that there exists a sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *in* $(0,1]$ *such that*

$$\mathbb{E}\left[\|\|G_t^{\alpha_n}\|\|^2\right] \to \mathbb{E}\left[\|\|G_{0+}(t,\cdot)\|\|^2\right] \text{ uniformly a.e. on } [0,T].$$

Then $\mathbb{E}\left[\|\|G_{0+}(t,\cdot)\|\|^2\right]$ *is Lebesgue integrable on* $[0,T]$ *and*

$$\mathbb{E}\left[\|\|A_{0+}(T,\cdot)\|\|^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|\|G_{0+}(t,\cdot)\|\|^2\right]\, dt < \infty.$$

Proof. Let $\alpha \in (0,1]$ and G be a backwards adapted fuzzy process. Suppose that G is $\mathcal{IH}_f^B$-integrable on $[0,T]$ with $A = (\mathcal{IH}_f^B)\int_0^T G_t\, dB_t$. Then by Lemma 3.3 and Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ with integral $(\mathcal{IH}^B)\int_0^T G_t^\alpha\, dB_t$. By Theorem 2.5, $\mathbb{E}\left[|||G_t^\alpha|||^2\right]$ is Lebesgue integrable on $[0,T]$ and

$$\mathbb{E}\left[\left|\left|\left|(\mathcal{IH}^B)\int_0^T G_t^\alpha\right|\right|\right|^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[|||G_t^\alpha|||^2\right]\, dt < \infty.$$

By Lemma 4.1, $\sup\limits_{\alpha\in(0,1]} \mathbb{E}\left[|||G_t^\alpha|||^2\right]$ is Lebesgue integrable on $[0,T]$ and

$$\sup_{\alpha\in(0,1]}\int_0^T \mathbb{E}\left[|||G_t^\alpha|||^2\right]\, dt = \int_0^T \sup_{\alpha\in(0,1]} \mathbb{E}\left[|||G_t^\alpha|||^2\right]\, dt.$$

Hence, by Theorem 2.5

$$\begin{aligned}
\mathbb{E}\left[|||A_{0+}(T,\cdot)|||^2\right] &= \sup_{\alpha\in(0,1]} \mathbb{E}\left[\left|\left|\left|(\mathcal{IH}^B)\int_0^T G_t^\alpha\, dB_t\right|\right|\right|^2\right] \\
&= \sup_{\alpha\in(0,1]} (\mathcal{L})\int_0^T \mathbb{E}\left[|||G_t^\alpha|||^2\right]\, dt \\
&= (\mathcal{L})\int_0^T \sup_{\alpha\in(0,1]} \mathbb{E}\left[|||G_t^\alpha|||^2\right]\, dt \\
&= (\mathcal{L})\int_0^T \mathbb{E}\left[|||G_{0+}(t,\cdot)|||^2\right]\, dt.
\end{aligned}$$

This completes the proof of the theorem. □

Definition 4.3. A fuzzy set-valued process $A : [0,T]\times\Omega \to \mathcal{F}_c(\mathbb{R})$ is said to be *continuous with respect to* $\mathfrak{D}_2$ if for each $t\in[0,T]$,

$$\mathbb{E}\left[|||A_{0+}(t,\cdot)|||^2\right] < \infty \text{ and } \lim_{s\to t}\mathfrak{D}_2(A_s, A_t) = 0.$$

Theorem 4.4. *Let $G \in \Lambda^2_{\mathcal{IH}_f^B}(\mathcal{F}(\mathbb{R}))$ and define $A_t := (\mathcal{IH}_f^B)\int_0^t G_s\, dB_s$. For each $\alpha\in(0,1]$, suppose that there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $(0,1]$ such that*

$$\mathbb{E}\left[|||G_t^{\alpha_n}|||^2\right] \to \mathbb{E}\left[|||G_{0+}(t,\cdot)|||^2\right] \text{ uniformly a.e. on } [0,T].$$

Then A is backwards adapted and continuous with respect to $\mathfrak{D}_2$.

Proof. First, we show that A is $\mathcal{G}_t$-adapted, that is, show that for each $\alpha\in(0,1]$, A^α is adapted to $\{\mathcal{G}_t\}$. Let $\epsilon > 0$. Since G is $\mathcal{IH}_f^B$-integrable on $[0,T]$, there exists $A \in L^2(\Omega, \mathcal{F}_c(\mathbb{R}))$ such that

$$\mathfrak{D}_2^2\left((D)\sum G_\xi(B_\xi - B_u), A\right) < \epsilon,$$

for any backwards (δ,η)-fine partial division D of $[0,T]$. Note that by Lemma 3.3 and Lemma 3.4, G^α is $\mathcal{IH}^B$-integrable on $[0,T]$ with integral A^α. Thus by

Theorem 2.7, A^α is backwards adapted to $\{\mathcal{G}_t\}$ for every $\alpha \in (0,1]$ and continuous with respect to Δ_2. Next, observe that for every $\alpha \in (0,1]$,

$$\mathbb{E}\left[\|\!|A_t^\alpha\|\!|^2\right] < \infty \text{ and } \lim_{s\to t} \Delta_2(A_s^\alpha, A_t^\alpha) = 0.$$

By Theorem 4.2,

$$\mathbb{E}\left[\|\!|A_{0+}(T,\cdot)\|\!|^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|\!|G_{0+}(t,\cdot)\|\!|^2\right]\, dt < \infty.$$

Furthermore,

$$\lim_{s\to t} \mathcal{D}_2\left(A_s, A_t\right) = \lim_{s\to t} \sup_{\alpha\in(0,1]} \Delta_2\left(A_s^\alpha, A_t^\alpha\right) = 0.$$

Hence, A is backwards adapted to $\{\mathcal{G}_t\}$ and continuous with respect to $\mathcal{D}_2$. □

5. Conclusion and Recommendation

In this paper, we define a Henstock-Kurzweil approach for the fuzzy set-valued stochastic process with respect to a Brownian motion, the backwards fuzzy Itô-Henstock integral. It is worth noting that the backwards fuzzy Itô-Henstock integral possesses the standard properties of an integral. We also establish a version of Itô isometry and continuity of the newly defined integral. A worthwhile direction for further investigation is to use Henstock-Kurzweil approach to deal with fuzzy stochastic differential equations.

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References

1. Arcede, J., & Cabral, E. (2011). An equivalent definition for the backwards Itô integral. Thai J. Math., 9(3), 619-630.
2. Arcede, J., & Cabral, E. (2009). Backwards Henstock integral. Matimyás Matematika, 32(2), 1-13.
3. Barros, L. C., Bassanezi, R. C., & Lodwick, W. A. (2017). A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics. Heidelberg: Springer Nature. https://doi.org/10.1007/978-3-662-53324-6
4. Bongiorno, B., Di Piazza, L., & Musiał, K. (2012). A decomposition theorem for the fuzzy Henstock integral. Fuzzy Sets and Systems, 200, 36-47. https://doi.org/10.1016/j.fss.2011.12.006
5. Chew, T. S., Toh, T. L., & Tay, J. Y. (2001). The non-uniform Riemann approach to Itô's integral. Real Anal. Exch., 27(2), 495-514. http://dx.doi.org/10.14321/realanalexch.27.2.0495
6. Didier, K. S., Dany, K. N., Jean-Paul, M. K., & Jean-Pierre, L. N. (2023). Fuzzy Henstock-Kurzweil triple Integral. J. Appl. Math. Phys., 11(7), 1860-1870. https://doi.org/10.4236/jamp.2023.117120
7. Gong, Z., & Shao, Y. (2009). The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions. Fuzzy Sets and Systems, 160(11), 1528-1546. https://doi.org/10.1016/j.fss.2008.10.013
8. Gordon, R. A. (1994). The Integrals of Lebesgue, Denjoy, Perron and Henstock. Graduate Studies in Mathematics, American Mathematical Society. http://dx.doi.org/10.1090/gsm/004

9. Henstock, R. (1988). Lectures on the Theory of Integration. Series in Real Analysis. Singapore: World Scientific. http://dx.doi.org/10.1142/0510
10. Kim, B. K., & Kim, J. H. (1999). Stochastic integrals of set-valued processes and fuzzy processes. J. Math. Anal. Appl., 236(1), 480-582. https://doi.org/10.1006/jmaa.1999.6461
11. Kurzweil, J. (2000). Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces. Series in Real Analysis. Singapore: World Scientific. http://dx.doi.org/10.1142/4333
12. Kurzweil, J., & Schwabik, S. (2004). McShane equi-integrability and Vitali's convergence theorem. Math. Bohem., 129(2), 141-157. http://dx.doi.org/10.21136/MB.2004.133903
13. Labendia, M. (2021). An alternative definition of the Itô integral for the Hilbert-Schmidt-valued stochastic process. Methods Func. Anal. Topology, 27(4), 370-383. http://dx.doi.org/10.31392/MFAT-npu26_4.2021.10
14. Labendia, M. (2019). A Riemann-type definition of the Itô integral for the operator-valued stochastic process. Adv. in Oper. Theory, 4(3), 625-640. http://dx.doi.org/10.15352/aot.1810-1435
15. Labendia, M., De Lara-Tuprio, E., & Teng, T. R. (2018). Itô-Henstock integral and Itô's formula for the operator-valued stochastic process. Math. Bohem., 143(2), 135-160. http://dx.doi.org/10.21136/MB.2017.0084-16
16. Lee, P. Y. (1989). Lanzhou Lectures on Henstock Integration. Series in Real Analysis. Singapore: World Scientific. http://dx.doi.org/10.1142/0845
17. Lee, P. Y., & Výborný, R. (2000). The Integral: An Easy Approach After Kurzweil and Henstock. Cambridge: Cambridge University Press. http://mathscinet.ams.org/mathscinet-getitem?mr=1756319
18. Lee, T. Y. (2011). Henstock-Kurzweil Integration on Euclidean spaces. Series in Real Analysis. Singapore: World Scientific. http://dx.doi.org/10.1142/7933doi:10.1142/7933.
19. Li, J., Li, Y., & Shao, Y. (2024). On fuzzy Henstock-Stieltjes integral on time scales with respect to bounded variation function. PLoS ONE, 19(9), e0309031. https://doi.org/10.1371/journal.pone.0309031
20. Li, S., Ogura, Y., & Kreinovich, V. (2002). Limit Theorems and Applications of Set-valued and Fuzzy Set-valued Random Variables. Theory and Decision Library B. Dordrecht: Springer. https://doi.org/10.1007/978-94-015-9932-0
21. Li, S., & Ren, A. (2007). Representation theorems, set-valued and fuzzy set-valued Itô integral. Fuzzy Sets Syst., 158(9), 949-962. https://doi.org/10.1016/j.fss.2006.12.004
22. Lu, J. T., & Lee, P. Y. (1999). The primitives of Henstock integrable functions in Euclidean space. Bull. London Math. Soc., 31(2), 173-180. http://dx.doi.org/10.1112/S0024609398005347
23. Ma, Z. M., Grande, Z., & Lee, P. Y. (1992). Absolute integration using vitali covers, Real Anal. Exch., 18(2), 409-419. http://dx.doi.org/10.2307/44152284
24. McShane, E. J. (1969). Stochastic integrals and stochastic functional equations. SIAM J. Appl. Math., 17(2), 287-306. http://dx.doi.org/10.1137/0117029
25. Pop-Stojanovic, Z. R. (1972). On McShane's belated stochastic integral. SIAM J. Appl. Math., 22(1), 87-92. http://dx.doi.org/10.1137/0122010
26. Puri, M. L., & Ralescu, D. A. (1986). Fuzzy random variables. J. Math. Anal. Appl., 114(2), 406-422. https://doi.org/10.1016/0022-247X(86)90093-4
27. Rulete, R., & Labendia, M. (2019). Backwards Itô-Henstock integral for the Hilbert-Schmidt-valued stochastic process. Eur. J. Pure Appl., 12(1), 58-78. http://dx.doi.org/10.29020/nybg.ejpam.v12i1.3342
28. Rulete, R., & Labendia, M. (2020). Backwards Itô-Henstock's version of Itô's formula. Ann. Func. Anal., 11(1), 208-225. http://dx.doi.org/10.1007/s43034-019-00014-3

29. Sasam, J. A., & Labendia, M. (2024). Backwards Itô-Henstock integral for the set-valued stochastic process. AIP Conf. Proc., 3029(1), 02001-1-02001-16. https://doi.org/10.1063/5.0191786
30. Toh, T. L., & Chew, T. S. (2009). Henstock's version of Itô's formula. Real Anal. Exch., 35(2), 375-390. http://dx.doi.org/10.14321/realanalexch.35.2.0375
31. Toh, T. L., & Chew, T. S. (2005). On belated differentiation and a characterization of Henstock-Kurzweil-Itô integrable processes. Math. Bohem., 130(1), 63-72. http://dx.doi.org/10.21136/MB.2005.134223
32. Toh, T. L., & Chew, T. S. (2005). On Itô-Kurzweil-Henstock integral and integration-by-part formula. Czech. Math. J., 55(3), 653-663. http://dx.doi.org/10.1007/s10587-005-0052-7
33. Toh, T. L., & Chew, T. S. (2004). On the Henstock-Fubini theorem for multiple stochastic integrals. Real Anal. Exch., 30(1), 295-310. http://dx.doi.org/10.14321/realanalexch.30.1.0295
34. Toh, T. L., & Chew, T. S. (2003). The Riemann approach to stochastic integration using non-uniform meshes. J. Math. Anal. Appl., 280(1), 133-147. http://dx.doi.org/10.1016/S0022-247X(03)00059-3
35. Wu, C., & Gong, Z. (2000). On Henstock integrals of interval-valued functions and fuzzy-valued functions. Fuzzy Sets and Systems, 115(3), 377-391. https://doi.org/10.1016/S0165-0114(98)00277-2
36. Wu, C., & Gong, Z. (2001). On Henstock integral of fuzzy-number-valued functions. I. Fuzzy Sets and Systems, 120(3), 523-532. https://doi.org/10.1016/S0165-0114(99)00057-3

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