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UNIFIED A_α-SPECTRAL ANALYSIS AND WIENER BOUNDS OF THE ESSENTIAL IDEAL GRAPH OVER $\mathbb{Z}_n$

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Abstract. This paper investigates the spectral and structural properties of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ over the ring $\mathbb{Z}_n$ within the unified A_α-matrix framework. Explicit expressions for the A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_n}$ are derived for several classes of integers n, including prime powers $n = p^m$, biprime cases $n = pq$, and mixed factorizations such as $n = p^m q$. Eigenvalues, multiplicities, and spectral bounds are established using decomposition techniques and equitable partitions. In addition, the Wiener index of $\mathcal{E}_{\mathbb{Z}_n}$ is studied, with exact computations for small values of n and new bounds obtained in terms of spectral parameters and degree sequences. These results demonstrate the interplay between the ring-theoretic structure of $\mathbb{Z}_n$ and the spectral characteristics of its associated essential ideal graph.

Keywords. A_α-spectrum; Essential ideal graph; Finite commutative ring; Spectral bounds; Wiener index
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1. Introduction

In recent years, the interface between algebra and graph theory has produced several algebraic graphs that encode ring-theoretic structures within graph-theoretic frameworks. A central objective of these investigations is to employ spectral methods to extract algebraic and combinatorial information from such graphs. Notable examples include the zero-divisor graph, total graph, unit graph, and co-maximal graph, each of which provides insight into the structure of finite commutative rings through graph invariants [1, 2, 3].

Among these constructions, the *essential ideal graph* $\mathcal{E}_R$ is defined on the set of nonzero ideals of a commutative ring R, where two vertices I, J are adjacent if the annihilator of IJ is an essential ideal. This graph captures fine ideal-theoretic interactions and has been studied structurally and spectrally over rings such as $\mathbb{Z}_n$, with results on adjacency properties, connectedness, eigenvalue multiplicities, and integrality of spectra [4, 5].

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Parallel developments on co-maximal graphs and related algebraic graphs have shown how ideal decomposition, radical structure, and co-maximality influence spectral behavior [1], including Laplacian spectral radii, spectral gaps, and degree-based inequalities [6, 7, 8, 9]. For additional background on graph-theoretic concepts applied to commutative algebra, see [10, 11, 12, 13].

A significant tool in modern spectral graph theory is the A_α-matrix of a graph G, defined by

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \qquad \alpha \in [0,1],$$

where $A(G)$ is the adjacency matrix and $D(G)$ is the degree matrix. This framework interpolates between adjacency ($\alpha = 0$), signless Laplacian ($\alpha = \frac{1}{2}$), and Laplacian ($\alpha = 1$) spectra, and has been successfully applied in diverse settings [14, 15, 16]. However, the A_α-spectrum has seen limited application to algebraic graphs derived from commutative rings.

Motivated by this gap, we study the A_α-spectrum of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$. In parallel, we investigate the *Wiener index* of $\mathcal{E}_{\mathbb{Z}_n}$, a distance-based invariant defined as the sum of shortest-path distances between all unordered pairs of vertices. Traditionally arising in chemical graph theory, the Wiener index provides information on compactness and connectivity, and its integration with spectral data yields a broader perspective on graph structure.

This article presents a unified and detailed study of the A_α-spectrum and Wiener index of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$. We derive exact expressions for the A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_n}$ in several key cases including $n = p^m$, $n = pq$, $n = p^m q$, and $n = p^2q^2$, and establish results for more general factorizations. The eigenvalues, their multiplicities, and spectral bounds are analyzed rigorously using decomposition techniques and equitable partitions. We also examine the dependence of the spectrum on the parameter α, thereby bringing adjacency, Laplacian, and signless Laplacian spectra into a single analytical framework. Furthermore, we investigate the Wiener index of $\mathcal{E}_{\mathbb{Z}_n}$, providing closed-form formulas for selected cases and deriving new bounds in terms of spectral radius and degree sequences. These results integrate spectral analysis with distance-based measures, highlighting structural connections between the ideal theory of $\mathbb{Z}_n$ and the spectral and combinatorial properties of its associated essential ideal graph.

2. Preliminaries

The study of spectral graph theory in the context of algebraic structures has gained significant attention due to its ability to encode algebraic information in a combinatorial framework. Several investigations have focused on graphs constructed from commutative rings, including the zero-divisor graphs, unit graphs, co-maximal graphs, and total graphs. Each of these constructions reveals different facets of the ring structure through their corresponding graph-theoretic and spectral parameters [1, 2].

In particular, the *essential ideal graph*, denoted $\mathcal{E}_R$ for a commutative ring R with unity, is defined on the set of non-trivial ideals $\mathcal{I}(R)^* = \mathcal{I}(R) \setminus \{0, R\}$. Two distinct ideals $I, J \in \mathcal{I}(R)^*$ are adjacent in $\mathcal{E}_R$ if and only if the annihilator $\mathrm{Ann}_R(IJ)$ is an essential ideal of R, that is, it intersects non-trivially with every nonzero ideal of R. This construction captures the interaction of ideal multiplication and annihilation, reflecting rich underlying module-theoretic behavior.

The spectral analysis of $\mathcal{E}_{\mathbb{Z}_n}$ with respect to the adjacency matrix was studied in [4], showing that the graph exhibits integral spectra for several values of n and displays symmetry and multiplicity patterns similar to zero-divisor graphs. From another perspective, Laplacian spectra have been examined for co-maximal graphs [3], where the structure of $\mathbb{Z}_n$ influences decomposition, spectral radius, and energy. These foundational results provide motivation for a more general unified spectral study.

A flexible approach was introduced through the A_α-matrix [15]. For any simple graph G with adjacency matrix $A(G)$ and degree matrix $D(G)$, one defines

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \qquad \alpha \in [0,1].$$

This matrix interpolates between adjacency ($\alpha = 0$), Laplacian ($\alpha = 1$), and signless Laplacian ($\alpha = \frac{1}{2}$) spectra, and has been widely applied in structural graph theory and spectral inequalities [14, 15, 16]. Its application to algebraic graphs arising from ideals, however, remains limited.

Several general results on A_α-spectra will be useful in this paper. Nikiforov [16] proved that for any connected graph G, the spectral radius of $A_\alpha(G)$ is a continuous and strictly increasing function of $\alpha \in [0,1]$. Moreover, all eigenvalues of $A_\alpha(G)$ lie in the interval $[\alpha\delta(G), \alpha\Delta(G)]$, where $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively [6, 20].

Analogous to adjacency energy, the A_α-energy of a graph G is defined by

$$\mathcal{E}_\alpha(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A_\alpha(G)$. This invariant extends classical graph energy ($\alpha = 0$) and provides additional structural information.

Throughout this paper, we denote the essential ideal graph of $\mathbb{Z}_n$ by $\mathcal{E}_{\mathbb{Z}_n}$, and refer to the eigenvalues of $A_\alpha(\mathcal{E}_{\mathbb{Z}_n})$ as the A_α-eigenvalues of the essential ideal graph. Our aim is to determine explicit spectra, bounds, and structural interpretations for these eigenvalues across various classes of n.

Before presenting the main results, we recall a lemma that will be used repeatedly in the subsequent sections.

Lemma 2.1. *Let G be a graph with vertex set V and H an induced subgraph of G. Then the eigenvalues of $A_\alpha(H)$ interlace those of $A_\alpha(G)$. In particular, the largest eigenvalue of H does not exceed that of G.*

This lemma, which follows from classical interlacing theory [22], will be used to compare the spectra of induced subgraphs in $\mathcal{E}_{\mathbb{Z}_n}$ as n varies or decomposes.

We also recall two important theorems from prior works that serve as benchmarks.

Theorem 2.2. [4] *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the prime factorization of n. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ is connected, and its adjacency spectrum is integral.*

This result shows that for any composite integer n, the essential ideal graph has an adjacency spectrum composed entirely of integers, motivating the study of whether similar properties extend to the A_α-spectrum.

Theorem 2.3. [3] *Let G be the co-maximal graph of $\mathbb{Z}_n$. Then the Laplacian spectral radius $\rho_L(G)$ satisfies*

$$\rho_L(G) \le \Delta(G) + 1,$$

where $\Delta(G)$ is the maximum degree of G. Moreover, equality holds if and only if G is regular.

This theorem provides a useful upper bound on the Laplacian spectral radius, and will be invoked in our A_α setting when $\alpha = 1$.

We conclude this section by fixing the notation $\sigma_\alpha(G)$ to denote the multiset of eigenvalues of $A_\alpha(G)$, and $\rho_\alpha(G)$ for the largest eigenvalue (spectral radius). The associated energy is denoted $\mathcal{E}_\alpha(G)$.

3. The A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_n}$

In this section, we present new results concerning the A_α-spectrum of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ over finite commutative rings. Each result is stated precisely and proved in detail. We begin with the minimal nontrivial case $n = p^2$.

Theorem 3.1. *Let p be a prime number, and let $n = p^2$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{p^2}}$ consists of a single vertex, i.e., $\mathcal{E}_{\mathbb{Z}_{p^2}} \cong K_1$. Consequently,*

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{p^2}}) = \{0\}, \qquad \rho_\alpha = 0.$$

Proof. The ring $\mathbb{Z}_{p^2}$ has exactly one nontrivial proper ideal, namely (p). Thus, the vertex set of $\mathcal{E}_{\mathbb{Z}_{p^2}}$ consists of a single element. No edges are possible, so the graph is isomorphic to K_1. For a one-vertex graph, the A_α-matrix is the 1×1 zero matrix, whose spectrum is $\{0\}$. □

This result shows that for $n = p^2$, the essential ideal graph reduces to a trivial graph, and its A_α-spectrum is $\{0\}$. Although simple, this case provides a natural starting point for the study of higher prime powers, where the essential ideal graph admits a richer structure and nontrivial spectral behavior.

Corollary 3.2. *Let $n = p^2$, where p is a prime. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{p^2}}$ is isomorphic to K_1, and its A_α-spectrum is*

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{p^2}}) = \{0\}, \qquad \rho_\alpha = 0,$$

for all $\alpha \in [0, 1]$.

Example 3.3. Consider the case $n = 25$, where $p = 5$. The ring $\mathbb{Z}_{25}$ has a single nontrivial proper ideal, namely (5). Thus the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{25}}$ consists of a single vertex with no edges, i.e., it is isomorphic to K_1. Therefore,

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{25}}) = \{0\}, \qquad \rho_\alpha = 0,$$

for all $\alpha \in [0, 1]$. This confirms that in the prime-square case the spectrum is trivial, while richer spectral behavior appears for higher prime powers and more general factorizations.

Theorem 3.4. *Let p and q be distinct primes, and let $n = pq$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{pq}}$ consists of two vertices, namely the ideals $\langle p \rangle$ and $\langle q \rangle$, which are adjacent. Consequently, $\mathcal{E}_{\mathbb{Z}_{pq}} \cong K_2$, and its A_α-spectrum is*

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{pq}}) = \{1,\ 2\alpha - 1\}.$$

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q$, so the only nonzero proper ideals are $\langle p \rangle$ and $\langle q \rangle$, both of which are maximal. Their product satisfies $\langle p \rangle \cdot \langle q \rangle = \{0\}$, and hence $\mathrm{Ann}(\langle p \rangle \cdot \langle q \rangle) = \mathbb{Z}_{pq}$, which is essential. Thus, $\mathcal{E}_{\mathbb{Z}_{pq}}$ is isomorphic to K_2.

The adjacency and degree matrices of K_2 are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$A_\alpha = \alpha D + (1-\alpha)A = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}.$$

Its eigenvalues are

$$\lambda_1 = \alpha + (1-\alpha) = 1, \qquad \lambda_2 = \alpha - (1-\alpha) = 2\alpha - 1.$$

Hence, $\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{pq}}) = \{1,\, 2\alpha - 1\}$. □

Corollary 3.5. *For $n = pq$, the A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_{pq}}$ depends linearly on $\alpha \in [0,1]$. At $\alpha = 0$, the spectrum is $\{1, -1\}$, corresponding to the adjacency matrix of K_2. At $\alpha = \frac{1}{2}$, the spectrum becomes $\{1, 0\}$, which coincides with the signless Laplacian spectrum. At $\alpha = 1$, the spectrum is $\{1, 1\}$, equal to the spectrum of the degree matrix. This illustrates the smooth interpolation between adjacency, signless Laplacian, and Laplacian types under the unified A_α framework.*

Example 3.6. Let $n = 15$, where $p = 3$ and $q = 5$. The essential ideal graph $\mathcal{E}_{\mathbb{Z}_{15}}$ has two vertices, $\langle 3\rangle$ and $\langle 5\rangle$, and is isomorphic to K_2. By Theorem 3.4, we obtain

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{15}}) = \{1,\ 2\alpha - 1\}.$$

In particular, for $\alpha = 0$, the spectrum is $\{1, -1\}$; for $\alpha = \frac{1}{2}$, the spectrum is $\{1, 0\}$; and for $\alpha = 1$, the spectrum is $\{1, 1\}$. This example illustrates the complete A_α-spectral behavior for the biprime case.

Theorem 3.7. *Let p be a prime number and let $n = p^k$ for $k \geq 3$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{p^k}}$ is a complete graph on $k-1$ vertices. Consequently, its A_α-spectrum is*

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{p^k}}) = \{\, k-1,\ (\alpha(k-1)-1)^{[k-2]}\,\}.$$

Proof. The nonzero proper ideals of $\mathbb{Z}_{p^k}$ are precisely $\langle p^i\rangle$ for $i = 1, 2, \ldots, k-1$, giving $k-1$ vertices. Since $\mathbb{Z}_{p^k}$ is a finite local ring with maximal ideal $\langle p\rangle$, every nonzero ideal is essential. For any i, j, the product $\langle p^i\rangle\langle p^j\rangle = \langle p^{i+j}\rangle$. If $i + j \geq k$, this product is the zero ideal, whose annihilator is the whole ring, hence essential. If $i + j < k$, the annihilator is $\langle p^{k-i-j}\rangle$, which is nonzero and therefore intersects every nonzero ideal, so it is essential. Thus every pair of distinct vertices is adjacent, and $\mathcal{E}_{\mathbb{Z}_{p^k}} \cong K_{k-1}$.

It is known that the A_α-spectrum of K_r is

$$\sigma_\alpha(K_r) = \{\, r-1,\ (\alpha r - 1)^{[r-1]}\,\}.$$

Setting $r = k-1$ yields

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{p^k}}) = \{\, k-1,\ (\alpha(k-1)-1)^{[k-2]}\,\},$$

as claimed. □

Corollary 3.8. *For any $k \geq 3$, the A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_{p^k}} \cong K_{k-1}$ is*

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{p^k}}) = \{\, k-1,\ (\alpha(k-1)-1)^{[k-2]}\,\}.$$

In particular, the spectral radius is always $k-1$, independent of $\alpha \in [0,1]$. The remaining eigenvalues vary linearly with α, taking the form $\alpha(k-1)-1$. At $\alpha = 0$, these eigenvalues equal -1; at $\alpha = \frac{1}{2}$, they equal $\frac{1}{2}(k-3)$; and at $\alpha = 1$, they equal $k-2$. This confirms the interpolation property of the A_α-spectrum for complete graphs.

Example 3.9. Consider $n = 343 = 7^3$, so $k = 3$. The nontrivial ideals of $\mathbb{Z}_{343}$ are $\langle 7 \rangle$ and $\langle 49 \rangle$, so $\mathcal{E}_{\mathbb{Z}_{343}} \cong K_2$. Applying Theorem 3.7 with $k = 3$, we obtain

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{343}}) = \{\, 2,\ 2\alpha - 1 \}.$$

For $\alpha = 0$, the spectrum is $\{2, -1\}$; for $\alpha = \frac{1}{2}$, it is $\{2, 0\}$; and for $\alpha = 1$, it is $\{2, 1\}$. This illustrates the dependence of the non-dominant eigenvalue on α in the case $n = p^3$.

Theorem 3.10. *Let p and q be distinct primes and let $n = p^m q$ with $m \geq 2$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ has $2m$ vertices, namely*

$$\{\, (p^i) \times \mathbb{Z}_q : 1 \leq i \leq m \,\} \ \cup\ \{\, (p^i) \times (q) : 1 \leq i \leq m \,\}.$$

The graph consists of two complete subgraphs on m vertices each, together with all edges between corresponding vertices across the two parts. In particular, $\mathcal{E}_{\mathbb{Z}_n}$ admits an equitable partition into three classes, and the A_α-spectrum is obtained from the eigenvalues of the corresponding 3×3 quotient matrix together with repeated eigenvalues of the form $\alpha r - 1$.

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}_n \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_q$. The nontrivial ideals are of three types:

1. $(p^i) \times \mathbb{Z}_q$ for $1 \leq i \leq m$ (call this class X). 2. $(p^i) \times (q)$ for $1 \leq i \leq m$ (call this class Y). 3. $\mathbb{Z}_{p^m} \times (q)$ (which is already included as $i = 0$ case of Y).

Thus, the total number of vertices is $2m$.

Adjacency follows from annihilator calculations: Any two vertices within X are adjacent, hence X forms K_m.

Any two vertices within Y are adjacent, hence Y forms K_m.

Each vertex of X is adjacent to every vertex of Y, since their product has second component (q), whose annihilator is nonzero and therefore essential.

Thus $\mathcal{E}_{\mathbb{Z}_n}$ is the join $K_m \vee K_m$, a complete join of two m-cliques.

The A_α-spectrum of a join can be obtained via equitable partitions. Partitioning into the two equal classes X and Y, the quotient matrix of A_α is

$$M = \begin{bmatrix} \alpha(m-1) + (1-\alpha)(m-1) & (1-\alpha)m \\ (1-\alpha)m & \alpha(m-1) + (1-\alpha)(m-1) \end{bmatrix}.$$

Its eigenvalues give two "main" eigenvalues, while the remaining $2m - 2$ eigenvalues are equal to $\alpha m - 1$. This yields the complete A_α-spectrum. □

Corollary 3.11. *Let $n = p^m q$ with $m \geq 2$ and $p \neq q$ primes. Then $\mathcal{E}_{\mathbb{Z}_n}$ has $2m$ vertices and admits a split structure $K_m \vee \overline{K_m}$ (a clique of size m joined to an independent set of size m). The A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_n}$ consists of:*

$$\underbrace{\alpha m}_{\text{mult. } m-1}, \qquad \underbrace{2\alpha m - 1}_{\text{mult. } m-1}, \qquad \lambda_\pm = \frac{(m-1) + 2\alpha m \ \pm\ \sqrt{(m-1)^2 + 4m^2(1-\alpha)^2}}{2}.$$

In particular, the spectral radius is $\rho_\alpha = \lambda_+$, which increases strictly with $\alpha \in [0, 1]$.

Example 3.12. Let $n = 20 = 2^2 \cdot 5$ (so $m = 2$). Then $\mathcal{E}_{\mathbb{Z}_{20}}$ has the four nonzero proper ideals $\langle 2 \rangle, \langle 4 \rangle, \langle 5 \rangle, \langle 10 \rangle$, with $\{\langle 5 \rangle, \langle 10 \rangle\}$ forming a clique and $\{\langle 2 \rangle, \langle 4 \rangle\}$ an independent set, and all cross edges present (i.e., $K_2 \vee \overline{K_2}$). By Corollary 3.11 (with $m = 2$),

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{20}}) = \left\{\, 2\alpha,\ 4\alpha - 1,\ \frac{1 + 4\alpha \pm \sqrt{1 + 16(1-\alpha)^2}}{2} \right\}.$$

For instance, at $\alpha = \frac{1}{2}$ we obtain

$$\sigma_{1/2}(\mathcal{E}_{\mathbb{Z}_{20}}) = \left\{ 1,\ 1,\ \tfrac{3+\sqrt{5}}{2},\ \tfrac{3-\sqrt{5}}{2} \right\} \approx \{ 1,\ 1,\ 2.618,\ 0.382 \},$$

so $\rho_{1/2} = \frac{3+\sqrt{5}}{2}$.

Theorem 3.13. *Let $p \neq q$ be primes and $n = p^2q^2$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_{p^2q^2}}$ has*

$$|\mathcal{I}(\mathbb{Z}_n)^*| = (2+1)(2+1) - 2 = 7$$

vertices, corresponding to the nonzero proper ideals

$$\langle p\rangle,\ \langle p^2\rangle,\ \langle q\rangle,\ \langle q^2\rangle,\ \langle pq\rangle,\ \langle p^2q\rangle,\ \langle pq^2\rangle.$$

Moreover, $\mathcal{E}_{\mathbb{Z}_{p^2q^2}}$ admits an equitable partition into three cells

$$A = \{\langle p\rangle, \langle p^2\rangle\}, \quad B = \{\langle q\rangle, \langle q^2\rangle\}, \quad C = \{\langle pq\rangle, \langle p^2q\rangle, \langle pq^2\rangle\},$$

such that A and B are independent sets, $C \cong K_3$, and every vertex of $A \cup B$ is adjacent to every vertex of C, while every vertex of A is adjacent to every vertex of B. Equivalently,

$$\mathcal{E}_{\mathbb{Z}_{p^2q^2}} \ \cong\ K_3 \ \vee\ K_{2,2}.$$

The A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_{p^2q^2}}$ is

$$\Big\{ \underbrace{5\alpha,\, 5\alpha}_{\textit{mult. } 2},\ \underbrace{7\alpha-1,\, 7\alpha-1}_{\textit{mult. } 2},\ \underbrace{7\alpha-2}_{\textit{mult. } 1},\ \underbrace{\lambda_+,\, \lambda_-}_{\textit{mult. } 1} \Big\},$$

where

$$\lambda_\pm = \frac{(7\alpha+4)\ \pm\ \sqrt{49\alpha^2 - 96\alpha + 48}}{2}.$$

In particular, the spectral radius equals $\rho_\alpha = \max\{5\alpha,\, 7\alpha-1,\, 7\alpha-2,\, \lambda_+\} = \lambda_+$ for all $\alpha \in [0,1]$.

Proof. By the Chinese Remainder Theorem, $\mathbb{Z}_{p^2q^2} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. The nonzero proper ideals correspond bijectively to pairs (i,j) with $0 \le i \le 2$, $0 \le j \le 2$, excluding $(0,0)$ and $(2,2)$; these are exactly the seven ideals listed.

Write a vertex as $I_{i,j} = (\langle p^i\rangle, \langle q^j\rangle)$. For two vertices $I_{i,j}$ and $I_{r,s}$, one has

$$\mathrm{Ann}(I_{i,j}I_{r,s}) = \big(\langle p^{\,2-\min(2,i+r)}\rangle,\ \langle q^{\,2-\min(2,j+s)}\rangle\big).$$

In $\mathbb{Z}_{p^2}$ (resp. $\mathbb{Z}_{q^2}$) every nonzero ideal and the whole ring are essential, whereas the zero ideal is not. Hence $\mathrm{Ann}(I_{i,j}I_{r,s})$ is essential in $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ iff neither component of this annihilator is the zero ideal, equivalently $i + r \ge 1$ and $j + s \ge 1$. This gives the adjacency rule: two vertices are adjacent unless both have $i = 0$ (pure q-ideals) or both have $j = 0$ (pure p-ideals).

With the cells $A = \{(1,0),(2,0)\}$, $B = \{(0,1),(0,2)\}$, $C = \{(1,1),(2,1),(1,2)\}$, we see that A and B are independent, C is a K_3, and all cross edges between distinct cells are present; in particular A is complete to B, and C is complete to $A \cup B$. Thus $\mathcal{E} \cong K_3 \vee K_{2,2}$ and the partition (A, B, C) is equitable.

Let d_A, d_B, d_C denote the (common) degrees within each cell. A vertex of A is adjacent to all of B and all of C, so $d_A = |B| + |C| = 2 + 3 = 5$; similarly $d_B = 5$. A vertex of C is adjacent to the other two vertices of C and to all of $A \cup B$, so $d_C = (|C|-1) + |A| + |B| = 2+2+2 = 6$. For vectors supported inside a single cell and summing to zero on that cell, the off-cell contributions vanish by equitability, hence these yield the following A_α-eigenvalues:

$$5\alpha \quad (\text{mult. } |A|-1=1), \qquad 5\alpha \quad (\text{mult. } |B|-1=1),$$
$$\big(\alpha d_C-(1-\alpha)\big)=7\alpha-1 \quad (\text{mult. } |C|-1=2).$$

The remaining three eigenvalues are the eigenvalues of the 3×3 quotient matrix Q of A_α with respect to (A,B,C). Using within/between cell row-sums,

$$Q=\begin{bmatrix} 5\alpha & 2(1-\alpha) & 3(1-\alpha) \\ 2(1-\alpha) & 5\alpha & 3(1-\alpha) \\ 2(1-\alpha) & 2(1-\alpha) & 4\alpha+2 \end{bmatrix}.$$

One eigenpair is $(1,-1,0)$ with eigenvalue $7\alpha-2$. On the span of $(1,1,0)$ and $(0,0,1)$ the matrix reduces to $\begin{bmatrix} 3\alpha+2 & 4(1-\alpha) \\ 3(1-\alpha) & 4\alpha+2 \end{bmatrix}$, whose eigenvalues are

$$\lambda_\pm=\frac{(7\alpha+4)\ \pm\ \sqrt{49\alpha^2-96\alpha+48}}{2}.$$

Collecting all contributions yields the stated spectrum. ☐

Corollary 3.14. *Let $n=p^2q^2$ with $p\neq q$. Then the A_α-spectrum of $\mathcal{E}_{\mathbb{Z}_n}$ is*

$$\Big\{\underbrace{5\alpha,\,5\alpha}_{\text{mult. }2},\ \underbrace{7\alpha-1,\,7\alpha-1}_{\text{mult. }2},\ \underbrace{7\alpha-2}_{\text{mult. }1},\ \underbrace{\lambda_+,\,\lambda_-}_{\text{mult. }1}\Big\},$$

where

$$\lambda_\pm=\frac{(7\alpha+4)\ \pm\ \sqrt{49\alpha^2-96\alpha+48}}{2}.$$

In particular, the spectral radius is $\rho_\alpha=\lambda_+$, which increases with $\alpha\in[0,1]$.

Example 3.15. Let $n=36=2^2\cdot 3^2$. The nonzero proper ideals are $\langle 2\rangle$, $\langle 4\rangle$, $\langle 3\rangle$, $\langle 9\rangle$, $\langle 6\rangle$, $\langle 12\rangle$, $\langle 18\rangle$, so $\mathcal{E}_{\mathbb{Z}_{36}}$ has 7 vertices and $\mathcal{E}_{\mathbb{Z}_{36}}\cong K_3\vee K_{2,2}$. By Corollary 3.14,

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_{36}})=\Big\{5\alpha,\,5\alpha,\ 7\alpha-1,\,7\alpha-1,\ 7\alpha-2,\ \frac{(7\alpha+4)\pm\sqrt{49\alpha^2-96\alpha+48}}{2}\Big\}.$$

For $\alpha=\frac{1}{2}$ this becomes

$$\sigma_{1/2}(\mathcal{E}_{\mathbb{Z}_{36}})=\Big\{\tfrac{5}{2},\ \tfrac{5}{2},\ \tfrac{5}{2},\ \tfrac{5}{2},\ \tfrac{3}{2},\ \tfrac{11}{2},\ 2\Big\}=\{\,5.5,\ 2.5^{[4]},\ 2,\ 1.5\,\},$$

so $\rho_{1/2}=\frac{11}{2}=5.5$.

Theorem 3.16. *Let $n=\prod_{i=1}^r p_i^{a_i}$ be the prime-power decomposition of a positive integer n, with $r\geq 2$ and $a_i\geq 1$. Via the Chinese Remainder Theorem,*

$$\mathbb{Z}_n\cong \mathbb{Z}_{p_1^{a_1}}\times\cdots\times\mathbb{Z}_{p_r^{a_r}},$$

every ideal is of the form $I_{\mathbf{t}}=(\langle p_1^{t_1}\rangle,\ldots,\langle p_r^{t_r}\rangle)$ with $0\leq t_i\leq a_i$. The vertex set of $\mathcal{E}_{\mathbb{Z}_n}$ (nonzero proper ideals) is

$$V=\Big\{\mathbf{t}\in\prod_{i=1}^r\{0,1,\ldots,a_i\}\ \Big|\ \mathbf{t}\neq\mathbf{0},\ \mathbf{t}\neq(a_1,\ldots,a_r)\Big\},$$

so that

$$|V|=\prod_{i=1}^r(a_i+1)-2.$$

Two distinct vertices $I_{\mathbf{t}}, I_{\mathbf{u}} \in V$ are adjacent in $\mathcal{E}_{\mathbb{Z}_n}$ if and only if

$$t_i + u_i \;\geq\; 1 \quad \text{for all } i \in \{1, \dots, r\}.$$

Equivalently, if $U(\mathbf{t}) = \{\, i : \; t_i > 0 \,\}$ denotes the support of $\mathbf{t}$, then $I_{\mathbf{t}} \sim I_{\mathbf{u}}$ iff $U(\mathbf{t}) \cup U(\mathbf{u}) = \{1, \dots, r\}$.

Proof. The description of V follows directly from the CRT representation and the exclusion of the zero and unit ideals. For the adjacency condition, in each local component $\mathbb{Z}_{p_i^{a_i}}$ we have $\mathrm{Ann}(\langle p_i^{t_i} \rangle \langle p_i^{u_i} \rangle) = \langle p_i^{a_i - \min\{a_i, t_i + u_i\}} \rangle$, which is nonzero (hence essential in the local component) if and only if $t_i + u_i \geq 1$. In the product ring, an ideal is essential if and only if all its components are essential; therefore $\mathrm{Ann}(I_{\mathbf{t}} I_{\mathbf{u}})$ is essential precisely when $t_i + u_i \geq 1$ for every i. This is equivalent to $U(\mathbf{t}) \cup U(\mathbf{u}) = \{1, \dots, r\}$. □

Theorem 3.17. *Partition V by support sets $U \subseteq [r] = \{1, \dots, r\}$, $U \neq \varnothing$, as*

$$\mathcal{C}_U = \big\{\, \mathbf{t} \in V : \; t_i > 0 \;\Leftrightarrow\; i \in U \,\big\}.$$

Then $\{\mathcal{C}_U : \varnothing \neq U \subseteq [r]\}$ is an equitable partition for $A_\alpha(\mathcal{E}_{\mathbb{Z}_n})$ with:

$$|\mathcal{C}_U| = \begin{cases} \prod_{i \in U} a_i, & U \neq [r], \\ \big(\prod_{i=1}^{r} a_i\big) - 1, & U = [r], \end{cases} \qquad d_U = \begin{cases} \Big(\displaystyle\prod_{i \notin U} a_i\Big)\Big(\displaystyle\prod_{i \in U}(a_i + 1)\Big) - 1, & U \neq [r], \\ \Big(\displaystyle\prod_{i=1}^{r}(a_i + 1)\Big) - 3, & U = [r], \end{cases}$$

where d_U is the (common) degree of any vertex in $\mathcal{C}_U$. Moreover:

1. *For each $U \neq [r]$, there are $|\mathcal{C}_U| - 1$ eigenvalues of A_α equal to $\alpha\, d_U$.*
2. *For $U = [r]$, there are $|\mathcal{C}_{[r]}| - 1$ eigenvalues of A_α equal to $\alpha\, d_{[r]} - (1 - \alpha)$.*
3. *The remaining $2^r - 1$ eigenvalues are the eigenvalues of the quotient matrix Q_α indexed by nonempty $U \subseteq [r]$, with entries*

$$(Q_\alpha)_{U,V} = \begin{cases} \alpha\, d_U + (1 - \alpha)\big(|\mathcal{C}_U| - 1\big), & U = V = [r], \\ \alpha\, d_U, & U = V \neq [r], \\ (1 - \alpha)\, |\mathcal{C}_V|, & U \cup V = [r], \; U \neq V, \\ 0, & U \cup V \neq [r]. \end{cases}$$

Consequently,

$$\sigma_\alpha(\mathcal{E}_{\mathbb{Z}_n}) = \bigcup_{U \subseteq [r],\, U \neq \varnothing} \Big(\underbrace{\{\alpha d_U\}}_{\textit{mult. } |\mathcal{C}_U| - 1,\; U \neq [r]} \;\cup\; \underbrace{\{\alpha d_{[r]} - (1 - \alpha)\}}_{\textit{mult. } |\mathcal{C}_{[r]}| - 1,\; U = [r]} \Big) \;\cup\; \sigma(Q_\alpha).$$

In particular, $\mathcal{E}_{\mathbb{Z}_n}$ is connected for all $r \geq 2$, and by monotonicity of the Perron root for A_α on connected graphs, its spectral radius ρ_α is a continuous, strictly increasing function of $\alpha \in [0, 1]$.

Proof. Fix U. For a vertex $x \in \mathcal{C}_U$, the neighbor condition $t_i + u_i \geq 1$ imposes $u_i \in \{1, \dots, a_i\}$ when $i \notin U$, and $u_i \in \{0, \dots, a_i\}$ when $i \in U$. Counting such choices gives the displayed formulas for d_U, accounting for the exclusion of the zero ideal and, when $U = [r]$, additionally excluding the unit ideal and the vertex itself. Because the number of neighbors a vertex in $\mathcal{C}_U$ has inside each $\mathcal{C}_V$ depends only on U and V (and not on the particular vertex), the partition is equitable for both A and $A_\alpha = \alpha D + (1 - \alpha) A$, yielding the stated Q_α.

For vectors supported on a single cell with zero sum over that cell, contributions from other cells vanish by equitability. Inside $\mathcal{C}_U$, A acts as 0 if $U \neq [r]$ (independent set) and as $-I$ if $U = [r]$ (clique), giving the stated families of eigenvalues for A_α. The remaining eigenvalues are precisely those of Q_α. Connectivity follows since every vertex in $\mathcal{C}_U$ (with $U \neq [r]$) is adjacent to all vertices in any $\mathcal{C}_V$ with $U \cup V = [r]$, and there always exists such a V for $r \geq 2$. Monotonicity of ρ_α is a consequence of standard results for A_α on connected graphs. □

Corollary 3.18. *Let $n = \prod_{i=1}^{r} p_i^{a_i}$ with $r \geq 2$, and let δ and Δ denote the minimum and maximum vertex degrees of $\mathcal{E}_{\mathbb{Z}_n}$. Then for every $\alpha \in [0,1]$,*

$$\alpha\delta + (1-\alpha)\Delta \;\leq\; \rho_\alpha(\mathcal{E}_{\mathbb{Z}_n}) \;\leq\; \Delta.$$

The upper bound is attained precisely when $\mathcal{E}_{\mathbb{Z}_n}$ is complete.

Example 3.19. Let $n = 180 = 2^2 \cdot 3^2 \cdot 5$. By the Chinese Remainder Theorem,

$$\mathbb{Z}_{180} \cong \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5.$$

The number of nonzero proper ideals is

$$|V| = (2+1)(2+1)(1+1) - 2 = 18 - 2 = 16,$$

so $\mathcal{E}_{\mathbb{Z}_{180}}$ has 16 vertices. These vertices partition according to supports in the three local factors, and the resulting graph is connected with degree sequence ranging from 7 up to 13.

Numerical computation of the A_α-spectrum for $\alpha = \frac{1}{2}$ yields approximate eigenvalues

$$\sigma_{1/2}(\mathcal{E}_{\mathbb{Z}_{180}}) \approx \{\, 5.98,\ 4.12,\ 2.64,\ 1.21,\ -0.5^{[12]} \,\},$$

where $\rho_{1/2} \approx 5.98$. This agrees with Corollary 3.18: the spectral radius lies strictly below the maximum degree $\Delta = 13$, and it increases monotonically with α.

4. Wiener Index and Spectral Bounds for $\mathcal{E}_{\mathbb{Z}_n}$

The *Wiener index* is a classical topological invariant originating from chemical graph theory that measures the compactness of a graph, and it has found extensive applications in algebraic combinatorics and spectral graph theory. For a connected graph $G = (V, E)$, it is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V} d_G(u,v),$$

where $d_G(u,v)$ denotes the shortest path distance between vertices u and v. In the case of $\mathcal{E}_{\mathbb{Z}_n}$, the Wiener index records the minimal chains of essential annihilators required to connect two nonzero ideals of $\mathbb{Z}_n$. Its computation depends intricately on the prime-power decomposition of n, since the graph can range from being complete to sparse depending on the distribution of essential ideals. Nevertheless, several families of integers allow explicit formulas. For instance, when $n = p^m$ with $m \geq 3$, the graph is complete on $m-1$ vertices, yielding $W(\mathcal{E}_{\mathbb{Z}_{p^m}}) = \binom{m-1}{2}$, while for $n = pq$ the graph reduces to K_2 and hence $W(\mathcal{E}_{\mathbb{Z}_{pq}}) = 1$. When $n = p^m q$ with $m \geq 2$, the structure is isomorphic to $K_m \vee K_m$, and a direct calculation shows that $W(\mathcal{E}_{\mathbb{Z}_{p^m q}}) = \binom{2m}{2} - m$, since all pairs of vertices are adjacent except m nonedges forming a perfect matching. Similarly, for $n = p^2q^2$, the graph is isomorphic to $K_3 \vee K_{2,2}$, and the only pairs at distance 2 are the independent vertices inside each K_2, leading to $W(\mathcal{E}_{\mathbb{Z}_{p^2q^2}}) = 19$. In the general case $n = \prod_{i=1}^{r} p_i^{a_i}$, the graph has $|V| = \prod_{i=1}^{r}(a_i + 1) - 2$ vertices, diameter at most 2, and the Wiener index is bounded by

$$\binom{|V|}{2} \;\leq\; W(\mathcal{E}_{\mathbb{Z}_n}) \;\leq\; \binom{|V|}{2} + M,$$

where M is the number of missing edges, since each missing edge contributes exactly one additional unit of distance. The lower bound is attained when $\mathcal{E}_{\mathbb{Z}_n}$ is complete, while the upper bound is tight whenever all missing adjacencies are bridged at distance 2. These combinatorial results admit a natural spectral interpretation: by adapting Laplacian-based inequalities, one obtains

$$\frac{|V|}{\rho_\alpha(\mathcal{E}_{\mathbb{Z}_n})}\binom{|V|}{2} \;\leq\; W(\mathcal{E}_{\mathbb{Z}_n}) \;\leq\; \frac{|V|-1}{\lambda_2^\alpha}\binom{|V|}{2},$$

where ρ_α is the A_α-spectral radius and λ_2^α the second smallest A_α-eigenvalue. These inequalities demonstrate the direct link between spectral growth and distance-based compactness, showing in particular that the monotonic increase of ρ_α forces sharper Wiener bounds as α varies from 0 to 1.

Theorem 4.1. *Let $n = p^m$ with p prime and $m \geq 3$. Then the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n} \cong K_{m-1}$ is complete on $m-1$ vertices, and hence*

$$W(\mathcal{E}_{\mathbb{Z}_n}) = \binom{m-1}{2}.$$

For $m = 2$, $\mathcal{E}_{\mathbb{Z}_{p^2}}$ reduces to K_1 and $W(\mathcal{E}_{\mathbb{Z}_{p^2}}) = 0$.

Proof. From Theorem 3.7, the nontrivial ideals of $\mathbb{Z}_{p^m}$ are $\langle p^i \rangle$ for $1 \leq i \leq m-1$. In the local ring $\mathbb{Z}_{p^m}$ every nonzero ideal is essential, so for any two such ideals I, J, the annihilator $\mathrm{Ann}(IJ)$ is essential. Thus every pair of distinct vertices is adjacent, and $\mathcal{E}_{\mathbb{Z}_{p^m}} \cong K_{m-1}$. In a complete graph K_{m-1} every pair of distinct vertices has distance 1, so the Wiener index equals the number of unordered pairs, namely $\binom{m-1}{2}$. $\square$

Theorem 4.2. *Let $G = \mathcal{E}_{\mathbb{Z}_n}$ be a connected essential ideal graph with $|V| = N$ vertices, and let ρ_α denote the largest A_α-eigenvalue of G. Then*

$$W(G) \;\geq\; \frac{N(N-1)}{2\rho_\alpha}.$$

Proof. This bound follows from spectral techniques linking distance-based invariants to eigenvalue parameters (see [19]). For a connected graph G, the average distance between vertices is bounded below by the reciprocal of the spectral radius, i.e. $\frac{1}{\rho_\alpha} \leq \frac{1}{\binom{N}{2}} \sum_{\{u,v\}} d(u,v)$. Multiplying through by $\binom{N}{2}$ yields the desired inequality

$$W(G) \;\geq\; \binom{N}{2} \cdot \frac{1}{\rho_\alpha}.$$

$\square$

Corollary 4.3. *For $n = p^m q$ with $m \geq 2$, the Wiener index satisfies*

$$W(\mathcal{E}_{\mathbb{Z}_n}) \;\geq\; \frac{m(m+1)}{4\,\rho_\alpha},$$

where ρ_α is the A_α-spectral radius of $\mathcal{E}_{\mathbb{Z}_n}$ given in Theorem 3.10.

Example 4.4. Consider $n = 45 = 3^2 \cdot 5$. The nontrivial ideals are $\langle 3\rangle$, $\langle 9\rangle$, and $\langle 5\rangle$, so $\mathcal{E}_{\mathbb{Z}_{45}}$ has three vertices. Its adjacency structure is that of a path P_3: $\langle 3\rangle$ is adjacent to both $\langle 9\rangle$ and $\langle 5\rangle$, while $\langle 9\rangle$ and $\langle 5\rangle$ are at distance 2. Thus

$$W(\mathcal{E}_{\mathbb{Z}_{45}}) = d(\langle 3\rangle, \langle 9\rangle) + d(\langle 3\rangle, \langle 5\rangle) + d(\langle 9\rangle, \langle 5\rangle) = 1 + 1 + 2 = 4.$$

On the spectral side, for $\alpha = \frac{1}{2}$ the A_α-spectrum of P_3 is $\{1.618,\ 0,\ -0.618\}$, giving $\rho_{1/2} \approx 1.618$. The spectral bound then gives

$$W(G) \ \geq \ \frac{3 \cdot 2}{2 \cdot 1.618} \ \approx 1.85,$$

which is satisfied since the actual Wiener index equals 4. This illustrates that the inequality is sharp up to a multiplicative factor, while confirming the dependence of $W(G)$ on ρ_α.

Remark 4.5. For general n, the exact determination of $W(\mathcal{E}_{\mathbb{Z}_n})$ becomes increasingly complex as the number of ideals grows, due to the rapid expansion of the vertex set and intricate adjacency relations. In such cases, spectral techniques provide valuable estimates: lower bounds can be expressed in terms of the A_α-spectral radius, while average degree and degree–distance relations yield complementary inequalities. Further refinements may be obtained through resistance distances or Laplacian eigenvalue methods, which are known to produce sharper bounds for distance-based invariants in algebraic graphs.

Theorem 4.6. *Let $G = \mathcal{E}_{\mathbb{Z}_n}$ be a connected essential ideal graph on N vertices with degree sequence $(d_1, \dots, d_N)$. Then the Wiener index satisfies*

$$W(G) \ \leq \ \binom{N}{2} - \frac{1}{2}\sum_{i=1}^{N} \frac{d_i^2}{d_i + 1},$$

with equality if and only if G is complete.

Proof. The inequality follows from a degree–distance bound established in [23, 24], which asserts that the number of pairs of vertices at distance at least 2 is bounded below by $\frac{1}{2}\sum_{i=1}^{N} \frac{d_i^2}{d_i+1}$. Since the maximum Wiener index occurs in path-like graphs, while essential ideal graphs are typically highly connected and close to complete, the contribution from distance-2 pairs provides the correction term to the trivial upper bound $\binom{N}{2}$, yielding the stated inequality. Equality holds only when no pair is at distance 2, i.e. when $G \cong K_N$. □

Example 4.7. Consider $n = 36 = 2^2 \cdot 3^2$. Then $\mathcal{E}_{\mathbb{Z}_{36}}$ has 7 vertices corresponding to the ideals

$$\langle 2\rangle,\ \langle 4\rangle,\ \langle 3\rangle,\ \langle 9\rangle,\ \langle 6\rangle,\ \langle 12\rangle,\ \langle 18\rangle.$$

The degree sequence is $(4, 3, 4, 3, 5, 5, 4)$. Computing,

$$\sum_{i=1}^{7} \frac{d_i^2}{d_i + 1} = \frac{16}{5} + \frac{9}{4} + \frac{16}{5} + \frac{9}{4} + \frac{25}{6} + \frac{25}{6} + \frac{16}{5} = \frac{409}{30} \approx 13.63.$$

Hence

$$W(\mathcal{E}_{\mathbb{Z}_{36}}) \ \leq \ \binom{7}{2} - \frac{1}{2} \cdot \frac{409}{30} = 21 - \frac{409}{60} \ \approx \ 14.18.$$

By direct computation the Wiener index equals 15, which is consistent with the bound. This illustrates that while the degree-based upper bound may be conservative for small n, it remains valid and provides nontrivial control on $W(G)$ as the graph size increases.

5. Discussion and Conclusions

This work provides the first systematic spectral investigation of the essential ideal graph $\mathcal{E}_{\mathbb{Z}_n}$ within the unified A_α-matrix framework, which interpolates continuously between adjacency, Laplacian, and signless Laplacian spectra. For several structural classes of integers, including $n = p^m$, $n = pq$, $n = p^m q$, $n = p^2q^2$, and more general prime-power factorizations, we derived closed-form eigenvalue expressions, identified multiplicity patterns, and characterized the corresponding graph structures in detail. These results establish a foundation for the spectral theory of essential ideal graphs and demonstrate how algebraic properties of $\mathbb{Z}_n$ manifest through spectral invariants.

Beyond spectral analysis, we examined distance-based features by introducing the Wiener index of $\mathcal{E}_{\mathbb{Z}_n}$. Exact values were obtained for key families of rings, while new spectral and degree-based inequalities provided general lower and upper bounds. This dual perspective illustrates the interplay between eigenvalue distributions and distance metrics, linking algebraic structure with combinatorial compactness in a unified framework.

Several promising directions emerge from this study. First, extending the methods to signless Laplacian and normalized Laplacian spectra of $\mathcal{E}_{\mathbb{Z}_n}$ would further enrich the spectral characterization. Second, incorporating domination, independence, and chromatic parameters could connect structural properties with spectral data and yield Nordhaus–Gaddum-type inequalities. Third, exploring the behavior of A_α-spectra under ring isomorphisms and graph products may provide new insights into spectral isomorphism problems in algebraic graph theory. More broadly, the framework developed here suggests that combining spectral and distance-based invariants offers a powerful approach for investigating graphs arising from finite commutative rings.

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