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STABILITY, PERIODICITY AND OSCILLATION OF SOME SYSTEMS OF DIFFERENCE EQUATIONS DEFINED BY A PERMUTATION

YOUSSOUF AKROUR[1*] and AZZEDDINE BELLOUR[2]

Abstract. The present paper establishes results on the relationship between the stability characteristics, asymptotic behavior, periodicity, and oscillation of a difference equation and certain general systems of difference equations defined using permutation groups. We show that the study of the system reduces to the study of the equation. Our system includes many particular cases that have been studied in the literature.

Keywords. System of Difference Equations, Permutation Groups, Stability, Periodicity, Oscillation
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1. Introduction

Researchers are generally interested in the study of difference equations with respect to stability characteristics, asymptotic behavior, periodicity, oscillation, and solution formulas (see, e.g., [5, 6, 7, 10, 9, 12, 13]). Many works deal with considering conditions on the functions defining equations and systems. For example, Touafek et al. in [17, 18] extended the results of Moaaz et al. [11] and Abdelrahman et al. [1] by considering the condition of homogeneity of continuous real functions in the equations and systems. The same condition is also considered in the papers of Boulouh et al. [3, 4].

Stević in [14] gives a representation of the explicit solutions to the nonlinear difference equations

$$z_i = g^{-1}\left(ag(z_{i-1}) + bg(z_{i-2})\right), \quad i \geq 2,$$

by assuming that the function g is one-to-one. Later, this work was extended by Akrour et al. [2] by considering the same condition on the functions.

In [15], Stević conducts some analysis of the relationship between solvable equations and the Newton-Raphson iteration process

$$z_{i+1} = z_i - \frac{g(z_i)}{g'(z_i)}, \quad i \in \mathbb{N}_0,$$

where g is a quadratic function of the form

$$g(z) = z^2 = pz + q.$$

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* Corresponding author

Another condition, namely that g is a homeomorphism of $\mathbb{R}$, was assumed by Stević in [16] to establish the solvability of the following difference equation:

$$z_{i+1} = g^{-1}\left(g(z_i)\,\frac{\alpha g(z_{i-1}) + \beta g(z_{i-2})}{\gamma g(z_{i-1}) + \delta g(z_{i-2})}\right), \quad g(0) = 0, \quad i \in \mathbb{N}_0.$$

It is very important to point out that many systems are inspired by equations that were the subject of other research, or that the systems are constructed in a way that makes the equations of the system similar or somewhat symmetric (see, e.g., [9, 10, 11, 12, 13, 14]).

In this research, we shed light on a model of equations defined by a permutation group, in order to highlight the relationship between the behavior of the equation and that of the system. We were inspired by the aforementioned works, particularly those of [7, 8, 10].

Therefore, let us consider $p \in \mathbb{N}$, $\mathbb{N}_p = \{1, 2, \dots, p\}$, and $\mathcal{S}_p$ the group of permutations on $\mathbb{N}_p$. Let g be an arbitrary function from a nonempty set $I \subset \mathbb{R}$ into itself. Consider the following difference equation:

$$y_{n+1} = g(y_{n-k}), \quad n, k \in \mathbb{N}_0. \tag{1.1}$$

and the following system of p difference equations

$$z^{(1)}_{n+1} = g(z^{(\sigma(1))}_{n-k}),\ z^{(2)}_{n+1} = g(z^{(\sigma(2))}_{n-k}), \dots,\ z^{(p)}_{n+1} = g(z^{(\sigma(p))}_{n-k}), \tag{1.2}$$

where σ is a p cycle of $\mathcal{S}_p$.

2. Stability results

The most important aim of this work is given in this section, which is to establish the stability of System (1.2) through that of Equation (1.1). Let us introduce the lemma below.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $(\alpha, \dots, \alpha) \in \mathbb{R}^p$.*

i) *$(\alpha, \dots, \alpha)$ is an equilibrium point of System (1.2) if and only if α is an equilibrium point of Equation (1.1).*

ii) *Suppose that α is the unique fixed point of g^p then, $(\alpha, \dots, \alpha)$ is the unique equilibrium point of System (1.2).*

Proof.

i) We have $(\alpha, \dots, \alpha)$ is an equilibrium point of System (1.2) if and only if $\alpha = g(\alpha)$ which is equivalent to α is an equilibrium point of Equation (1.1).

ii) Suppose that $(\alpha_1, \alpha_2, \dots, \alpha_p)$ is an equilibrium point of System (1.2). So,

$$\alpha_1 = g(\alpha_{\sigma(1)}),\ \alpha_2 = g(\alpha_{\sigma(2)}), \dots,\ \alpha_p = g(\alpha_{\sigma(p)}), \tag{2.1}$$

we obtain for every $j \in \{1, ..., p\}$,

$$\alpha_j = g^p(\alpha_{\sigma^p(j)}).$$

Thus, for every $j \in \{1, ..., p\}$

$$\alpha_j = g^p(\alpha_j).$$

Since, α is the unique fixed point of g^p, then $\alpha_j = \alpha$ for all $j \in \{1, ..., p\}$, which implies that $(\alpha, \dots, \alpha)$ is the unique equilibrium point of System (1.2). $\square$

The following lemma will be useful.

Lemma 2.2.

i) *For, $j=1,\ldots,p$, we have*

$$z_n^{(j)} = g^p\left(z_{n-p(k+1)}^{(j)}\right), \qquad n \geq p(k+1)-k. \tag{2.2}$$

ii) *For $j=1,\ldots,p$ and for $l=1,\ldots,p$, we have*

$$z_{rn+i}^{(j)} = g^{np+l}\left(z_{i-l(k+1)}^{\sigma^l(j)}\right), \text{ where } r=p(k+1) \text{ and } (l-1)(k+1)+1 \leq i \leq l(k+1). \tag{2.3}$$

Proof.

i) We have for all $j=1,\ldots,p$,

$$z_{n+1}^{(j)} = g\left(z_{n-k}^{(\sigma(j))}\right),$$

then

$$\begin{aligned} z_n^{(j)} &= g\left(z_{n-k-1}^{(\sigma(j))}\right), \\ &= g^2\left(z_{n-2(k+1)}^{(\sigma^2(j))}\right), \\ &\vdots \\ &= g^p\left(z_{n-p(k+1)}^{(\sigma^p(j))}\right). \end{aligned}$$

Hence, the result follows.

ii) Now, put $r=p(k+1)$ in (2.2), hence

$$z_n^{(j)} = g^p\left(z_{n-r}^{(j)}\right).$$

Therefore, for all $i=1,\ldots,r$,

$$z_{rn+i}^{(j)} = g^p\left(z_{r(n-1)+i}^{(j)}\right) = g^{np}\left(z_i^{(j)}\right). \tag{2.4}$$

On the other hand, from (1.2), for $j=1,\ldots,p$ and $l=1,\ldots,p$ with $(l-1)(k+1)+1 \leq i \leq l(k+1)$, we have

$$z_i^{(j)} = g\left(z_{i-(k+1)}^{(\sigma(j))}\right) = g^l\left(z_{i-l(k+1)}^{(\sigma^l(j))}\right),$$

substituting into (2.4), we obtain

$$z_{rn+i}^{(j)} = g^{np+l}\left(z_{i-l(k+1)}^{\sigma^l(j)}\right).$$

Thus, the result follows. □

The following theorem states that the behavior of System (1.2) around the equilibrium point is identical to that of Equation (1.1).

Theorem 2.3. *The following equivalences hold.*

a/ *System (1.2) is locally stable if and only if Equation (1.1) is locally stable.*

b/ *System (1.2) is asymptotically stable if and only if Equation (1.1) is asymptotically stable.*

c/ *System (1.2) is globally stable if and only if Equation (1.1) is globally stable.*

Proof.

We begin by noting that if System (1.2) is locally stable (asymptotically stable, globally stable, respectively), then the same holds for Equation (1.1), provided the initial values are equal, i.e.,

$$z_{-k}^{(1)} = z_{-k}^{(2)} = \cdots = z_{-k}^{(p)}, \quad \ldots, \quad z_0^{(1)} = z_0^{(2)} = \cdots = z_0^{(p)}.$$

a/ Let $\varepsilon > 0$. Since Equation (1.1) is locally stable, there exists $\delta > 0$ such that for all $y_{-k}, \dots, y_0 \in I$ with

$$\sum_{i=0}^{k} |y_{-i} - \alpha| < \delta,$$

we have

$$|y_n - \alpha| < \varepsilon, \quad \text{for all } n \geq -k. \tag{2.5}$$

By standard arguments, the condition (2.5) is equivalent to

$$|y_{(k+1)n+i} - \alpha| < \varepsilon, \quad \text{for all } n \geq -1 \text{ and } i = 1, \dots, k+1. \tag{2.6}$$

On the other hand, for all $n \geq -1$,

$$y_{(k+1)n+i} = g^{n+1}(y_{-k-1+i}), \quad \text{for } i = 1, \dots, k+1.$$

Putting $s = -k - 1 + i$ and $m = n + 1$, we obtain

$$y_{(k+1)m+s} = g^{m}(y_s), \quad \forall m \geq 0, \ \ s = -k, \dots, 0. \tag{2.7}$$

Then the condition (2.6) is equivalent to

$$|g^{n}(y_s) - \alpha| < \varepsilon, \quad \text{for all } n \geq 0 \text{ and } s = -k, \dots, 0. \tag{2.8}$$

Put $\delta' = \min(\varepsilon, \delta)$ and assume that

$$\sum_{j=1}^{p} \left(|z_{-k}^{(j)} - \alpha| + |z_{-k+1}^{(j)} - \alpha| + \cdots + |z_0^{(j)} - \alpha| \right) < \delta'. \tag{2.9}$$

Then, from (2.8), for all $n \geq 0$ and for all $j = 1, \dots, p$,

$$|g^{n}(z_s^{(j)}) - \alpha| < \varepsilon, \quad s = -k, \dots, 0.$$

Hence, for all $n \geq 0$, all $j = 1, \dots, p$ and all $l = 1, \dots, p$, we have

$$|g^{np+l}(z_s^{(j)}) - \alpha| < \varepsilon, \quad s = -k, \dots, 0, \tag{2.10}$$

Now, consider the subsequences defined in Lemma 2.2, for all $j = 1, \dots, p$ and $l = 1, \dots, p$:

$$z_{rn+i}^{(j)} = g^{np+l}\left(z_{i-l(k+1)}^{\sigma^l(j)} \right), \quad \text{where } r = p(k+1) \text{ and } (l-1)(k+1) + 1 \leq i \leq l(k+1).$$

Since, for all $j = 1, \dots, p$, $l = 1, \dots, p$ and $(l-1)(k+1) + 1 \leq i \leq l(k+1)$, we have

$$z_{i-l(k+1)}^{\sigma^l(j)} \in \{z_{-k}^{(r)}, z_{-k+1}^{(r)}, \dots, z_0^{(r)}, r = 1, \dots, p\},$$

then, from (2.10), we deduce that for all $(l-1)(k+1) + 1 \leq i \leq l(k+1)$,

$$|g^{np+l}(z_{i-l(k+1)}^{\sigma^l(j)}) - \alpha| < \varepsilon,$$

which implies that,

$$|z_{rn+i}^{(j)} - \alpha| < \varepsilon. \tag{2.11}$$

From (2.9) and (2.11), it follows that for all $n \geq -k$,

$$|z_n^{(j)} - \alpha| < \varepsilon.$$

Thus we obtain the local stability of System (1.2).

b/ Let $\varepsilon > 0$. Since Equation (1.1) is asymptotically stable, there exists $\gamma > 0$ such that if $y_{-k}, y_{-k+1}, \dots, y_0 \in I$ and

$$|y_{-k} - \alpha| + |y_{-k+1} - \alpha| + \cdots + |y_0 - \alpha| < \gamma,$$

then

$$\lim_{n\to\infty} y_n = \alpha.$$

By (2.7), this is equivalent to

$$\lim_{n\to\infty} g^n(y_s) = \alpha, \quad \forall s = -k, \dots, 0. \tag{2.12}$$

Now assume that

$$\sum_{j=1}^{p} \left(|z_{-k}^{(j)} - \alpha| + |z_{-k+1}^{(j)} - \alpha| + \cdots + |z_0^{(j)} - \alpha| \right) < \gamma. \tag{2.13}$$

From (2.12) and (2.13), for all $j = 1, \dots, p$, we have

$$\lim_{n\to\infty} g^n(z_s^{(j)}) = \alpha, \quad s = -k, \dots, 0.$$

Hence, for all $j = 1, \dots, p$ and $l = 1, \dots, p$,

$$\lim_{n\to\infty} g^{np+l}(z_s^{(j)}) = \alpha, \quad s = -k, \dots, 0,$$

which implies, for all $(l-1)(k+1) + 1 \le i \le l(k+1)$,

$$\lim_{n\to\infty} z_{rn+i}^{(j)} = \lim_{n\to\infty} g^{np+l}(z_{i-l(k+1)}^{\sigma^l(j)}) = \alpha. \tag{2.14}$$

From (2.13) and (2.14), we obtain for all $j = 1, \dots, p$,

$$\lim_{n\to\infty} z_n^{(j)} = \alpha. \tag{2.15}$$

Thus, by the first part of Theorem 2.3 and (2.15), we obtain the asymptotic stability of System (1.2).

c/ Combining (a) and (b), we conclude that System (1.2) is globally stable.

□

3. Periodicity and Oscillatory

This section is devoted to obtaining results on periodicity and oscillation. We begin by recalling the notion of periodicity.

Definition 3.1 (Global periodicity)**.**

1) Equation (1.1) is called *globally periodic* with period $T \in \mathbb{N}^*$ if, for any initial values $y_{-k}, y_{-k+1}, \dots, y_0 \in I$, we have for all $n \ge -k$,

$$y_{n+T} = y_n.$$

2) System (1.2) is called *periodic* with period T if every solution $(z_n^{(1)}, z_n^{(2)}, \dots, z_n^{(p)})$ satisfies, for all $j = 1, \dots, p$ and all $n \ge -k$,

$$z_{n+T}^{(j)} = z_n^{(j)}.$$

Lemma 3.2. *Let T and m be positive integer numbers.*

i) *A sequence $\{z_n\}_{n\in\mathbb{N}}$ is periodic with period T, then all the subsequences $\{z_{ln+i}\}_{n\in\mathbb{N}}$, ($i = 1, \dots, l$) are periodic with period T for all $l \in \mathbb{N}^*$.*

ii) *The sequence $\{z_n\}_{n\in\mathbb{N}}$ is periodic with period mT if and only if the subsequences of the form $\{z_{mn+i}\}_{n\in\mathbb{N}}$, ($i = 1, \dots, m$) are periodic with period T.*

Proof.

i) Assume that sequence $\{z_n\}_{n\in\mathbb{N}}$ is periodic with period T, we have for $i = 1, \ldots, l$ and $n \in \mathbb{N}$

$$z_{l(n+T)+i} = z_{ln+lT+i} = z_{ln+i}, \tag{3.1}$$

which implies that the subsequence $\{z_{ln+i}\}$ is periodic with period T.

ii) If the sequence $\{z_n\}_{n\in\mathbb{N}}$ is periodic with period mT, then by (3.1) with $l = m$ we deduce that the subsequences $\{z_{mn+i}\}$ are periodic with period T.
Assume that for all $i = 1, \ldots, m$ the subsequence $\{z_{mn+i}\}$ is periodic with period T, we have for $n \in \mathbb{N}$, there exist $n' \in \mathbb{N}$ and $s \in \{1, \ldots, m\}$ such that $n = mn' + s$, hence

$$z_{n+mT} = z_{mn'+s+mT} = z_{m(n'+T)+s} = z_{mn'+s} = z_n.$$

which implies that sequence $\{z_n\}_{n\in\mathbb{N}}$ is periodic with period mT. □

Theorem 3.3.

a) *System (1.2) is periodic with period T, then Equation (1.1) is periodic with period T.*
b) *Equation (1.1) is periodic with period $T(k+1)$, then System (1.2) is periodic with period $Tp(k+1)$.*

Proof.

a) Since Equation (1.1) is the uni-dimensional case of System (1.2), by taking the initial values equals, the periodicity of System (1.2) implies the periodicity of Equation (1.1).
b) Suppose that Equation (1.1) is periodic with period $T(k+1)$.
We have, the subsequences of $\{z_n^{(j)}\}$, $(j = 1, \ldots, p)$ as defined in Lemma 2.2 for all $l = 1, \ldots, p$ and $(l-1)(k+1)+1 \leq i \leq l(k+1)$ by

$$z^{(j)}_{p(k+1)n+i} = g^{np+l}\left(z_s^{\sigma^l(j)}\right), \text{ where } s = i - l(k+1) \in \{-k, \ldots, 0\},$$

are subsequences of $g^n\left(z_s^{\sigma^l(j)}\right)$. Since this sequence is a subsequence of the solution of Equation (1.1) with initial values $z_s^{\sigma^l(j)}$, $(s = -k, \ldots, 0)$, it follows from Lemma 3.2 that $g^n\left(z_s^{\sigma^l(j)}\right)$ is periodic with period T.
Again, by Lemma 3.2, all the subsequences $\{z^{(j)}_{p(k+1)n+i}\}$ are periodic with period T.
Therefore, by Lemma 3.2, the sequences $\{z_n^{(j)}\}$ are periodic with period $Tp(k+1)$. □

Proposition 3.4. *Let T be an non negative integer number,*

a) *If $g^{T+1} = f$, then System (1.2) is periodic with period $Tp(k+1)$.*
b) *If $g^T = I$ such that $p = Tr$ for some $r \in \mathbb{N}^*$, then System (1.2) is periodic with period $p(k+1)$.*

Proof.

a) If $g^{T+1} = f$, then for all $s = -k, \ldots, 0$, the sequence $\{g^n(y_s)\}$ is periodic with period T. It follows from Lemma 3.2 and (2.3) that the subsequences $\{z^{(j)}_{p(k+1)n+i}\}$ are also periodic with period T. Hence, System (1.2) is periodic with period $Tp(k+1)$.
b) If $g^T = I$, then $f^p = I$. From (2.3), the subsequences $\{z^{(j)}_{p(k+1)n+i}\}$ are constant. Therefore, by Lemma 3.2, System (1.2) is periodic with period $p(k+1)$. □

Definition 3.5. A sequence $\{z_n\}$ is said to oscillate about the equilibrium point α with a semi-cycle of length $l \in \mathbb{N}^*$ if $(z_i - \alpha)$ has the same sign for all $i = nl, \ldots, (n+1)l - 1$, and $(z_{nl} - \alpha)(z_{(n+1)l} - \alpha) < 0$ for all $n \in \mathbb{N}$.

Lemma 3.6. *Let ℓ and m be positive integer numbers. A sequence $\{z_n\}_{n\in\mathbb{N}}$ is oscillate about the equilibrium point α with a semi-cycle of length $m\ell$ if and only if all the subsequences of the form $\{z_{mn+i}\}_{n\in\mathbb{N}}$, $(i = 0, \ldots, m-1)$ are oscillate about the equilibrium point α with a semi-cycle of length ℓ and $(z_i - \alpha)$ are in the same sign for all $i = 0, \ldots, m-1$.*

Proof.

i) Assume that the sequence $\{z_n\}_{n\in\mathbb{N}}$ oscillates about the equilibrium point α with a semi-cycle of length $m\ell$. Then $(z_i - \alpha)$ has the same sign for all $i = nm\ell, \ldots, (n+1)m\ell - 1$, and

$$(z_{nm\ell} - \alpha)(z_{(n+1)m\ell} - \alpha) < 0 \quad \text{for all } n \in \mathbb{N}.$$

For $i = 0, \ldots, m-1$, $j = n\ell, \ldots, (n+1)\ell - 1$, and $n \in \mathbb{N}$, we have

$$nm\ell \le nm\ell + i \le jm + i \le m\ell(n+1) - m + i \le m\ell(n+1) - 1, \tag{3.2}$$

which implies

$$(n+1)m\ell \le (n+1)m\ell + i \le (n+2)m\ell - 1. \tag{3.3}$$

Hence, from (3.2) and (3.3), it follows that $(z_{mj+i} - \alpha)$ has the same sign for all $j = n\ell, \ldots, (n+1)\ell - 1$, and

$$(z_{nm\ell+i} - \alpha)(z_{(n+1)m\ell+i} - \alpha) < 0$$

for all $i = 0, \ldots, m-1$ and $n \in \mathbb{N}$. Thus, the subsequence $\{z_{mn+i}\}$ $(i = 0, \ldots, m-1)$ oscillates about the equilibrium point α with a semi-cycle of length ℓ.

ii) Conversely, suppose that all subsequences $\{z_{mn+i}\}_{n\in\mathbb{N}}$ $(i = 0, \ldots, m-1)$ oscillate about the equilibrium point α with a semi-cycle of length ℓ, and that $(z_i - \alpha)$ has the same sign for all $i = 0, \ldots, m-1$. Then $(z_{mj+i} - \alpha)$ has the same sign for all $j = n\ell, \ldots, (n+1)\ell - 1$, and

$$(z_{nm\ell+i} - \alpha)(z_{(n+1)m\ell+i} - \alpha) < 0$$

for all $i = 0, \ldots, m-1$ and $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $r \in \{nm\ell, \ldots, (n+1)m\ell - 1\}$. Then there exist $j \in \mathbb{N}$ and $i \in \{0, \ldots, m-1\}$ such that $r = mj + i$. We obtain

$$\begin{aligned} nm\ell \le mj + i \le (n+1)m\ell - 1 \quad &\Rightarrow \quad nm\ell - i \le mj \le (n+1)m\ell - i - 1,\ i \in \{0, \ldots, m-1\}, \\ &\Rightarrow \quad nm\ell + 1 - m \le mj \le (n+1)m\ell - 1, \\ &\Rightarrow \quad n\ell \le j \le (n+1)\ell - 1. \end{aligned}$$

Thus, $(z_r - \alpha)$ has the same sign for all $r = nm\ell, \ldots, (n+1)m\ell - 1$, and

$$(z_{nm\ell} - \alpha)(z_{(n+1)m\ell} - \alpha) < 0.$$

Therefore, the sequence $\{z_n\}_{n\in\mathbb{N}}$ oscillates about the equilibrium point α with a semi-cycle of length $m\ell$. □

Theorem 3.7. *System (1.2) oscillates about the equilibrium point $\overline{\alpha}$ with a semi-cycle of length $(k+1)p\ell$ if and only if Equation (1.1) oscillates about the equilibrium point α with a semi-cycle of length $(k+1)p\ell$.*

Proof.

a) Since Equation (1.1) can be regarded as a special case of System (1.2), by taking identical initial values, the oscillation of System (1.2) implies the oscillation of Equation (1.1) with the same semi-cycle length.

b) Suppose that Equation (1.1) oscillates about the equilibrium point α with a semi-cycle of length $(k+1)p\ell$.
Since $\left\{g^n\left(z_s^{\sigma^l(j)}\right)\right\}_n$, as defined in Lemma 2.2, is a subsequence of the solution of Equation (1.1) with initial values $z_s^{\sigma^l(j)}$, $(s=-k,\ \dots,\ 0)$, it follows from Lemma 3.6 that $\left\{g^n\left(z_s^{\sigma^l(j)}\right)\right\}_n$ oscillates about the equilibrium point α with a semi-cycle of length $p\ell$.
On the other hand, the subsequences $\{z^{(j)}_{p(k+1)n+i}\}_n$ of $\{z_n^{(j)}\}$, $(j=1,\ \dots,\ p)$, as defined in Lemma 2.2, are subsequences of $\{g^n\left(z_s^{\sigma^l(j)}\right)\}_n$.
Therefore, by Lemma 3.6, all subsequences $\{z^{(j)}_{p(k+1)n+i}\}$ oscillate about the equilibrium point α with a semi-cycle of length ℓ.
Thus, by Lemma 3.6, the sequences $\{z_n^{(j)}\}$ oscillate about the equilibrium point α with a semi-cycle of length $(k+1)p\ell$. □

4. Particular illustrative examples

Example 4.1. [10] Consider the following system of p difference equations

$$z_{n+1}^{(1)}=\frac{1}{A+z_{n-k}^{(\sigma(1))}},\ \dots,\ z_{n+1}^{(p)}=\frac{1}{A+z_{n-k}^{(\sigma(p))}},\qquad \sigma \text{ is a } p\text{-cycle in } \mathcal{S}_p. \tag{4.1}$$

The associated difference equation is

$$y_{n+1}=\frac{1}{A+y_{n-k}},\qquad n,k\in\mathbb{N}_0, \tag{4.2}$$

with equilibrium point $\alpha=\dfrac{A+\sqrt{A^2-4}}{2}$.

Kaouache et al. [10] proved the global stability of System (4.1) about (α,α) in the case $p=2$. By Theorem 2.3, this implies that Equation (4.2) is globally stable about α, and consequently System (4.1) is globally stable about $(\alpha,\dots,\alpha)\in\mathbb{R}^p$ for all $p\geq 2$. This result answers the open problem raised in [10].

A particular case with $A=6$ and $p=2$ was studied by Halim et al. [6]. Figure 1 illustrates our stability result for System (4.1) with $A=6$, $p=3$, $k=2$, and $\sigma=(1\ 2\ 3)$, using the initial values $z_{-2}^{(1)}=\frac{2}{3}$, $z_{-1}^{(1)}=\frac{1}{2}$, $z_0^{(1)}=\frac{1}{10}$, $z_{-2}^{(2)}=\frac{3}{2}$, $z_{-1}^{(2)}=1$, $z_0^{(2)}=\frac{1}{5}$, $z_{-2}^{(3)}=2$, $z_{-1}^{(3)}=3$, $z_0^{(3)}=\frac{1}{3}$.

Example 4.2. [7] Consider the following system of p difference equations

$$z_{n+1}^{(1)}=\frac{z_{n-k}^{(\sigma(1))}+a}{\alpha z_{n-k}^{(\sigma(1))}-1},\ \dots,\ z_{n+1}^{(p)}=\frac{z_{n-k}^{(\sigma(p))}+a}{\alpha z_{n-k}^{(\sigma(p))}-1},\qquad \sigma \text{ is a } p\text{-cycle in } \mathcal{S}_p. \tag{4.3}$$

The associated difference equation is

$$y_{n+1}=g(y_{n-k})=\frac{y_{n-k}+a}{\alpha y_{n-k}-1},\qquad n,k\in\mathbb{N}_0. \tag{4.4}$$

Since $g^2=I$, Proposition 3.4(a) implies that System (4.3) is periodic with period $2p(k+1)$ if p is odd, while Proposition 3.4(b) gives period $p(k+1)$ if p is even.

Hamza et al. [7] proved that System (4.3) is periodic with period $6(k+1)$ for $p=3$, with period $2(k+1)$ for $p=2$, and with period 4 when $p=4$ and $k=0$. These results agree with our conclusions.

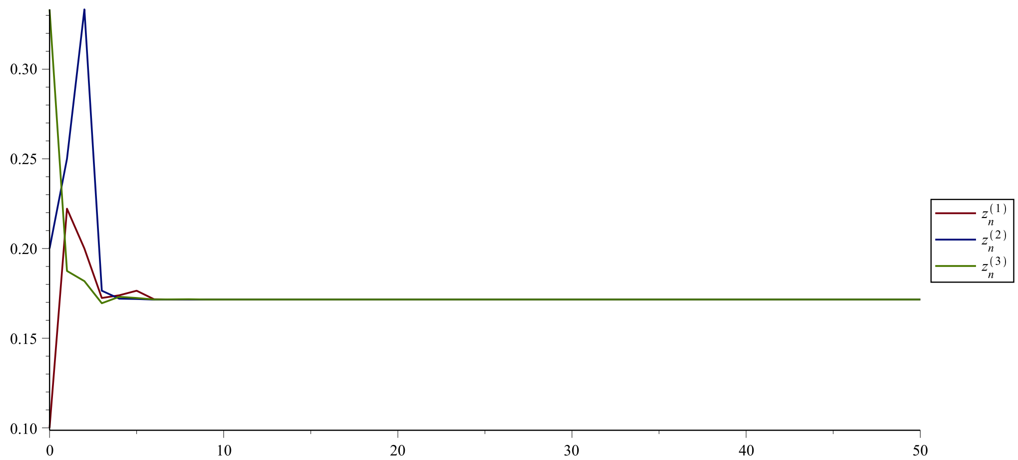

FIGURE 1. The figure shows that System (4.1) is stable.

To illustrate, consider System (4.3) with $p = 4$, $k = 0$, $a = 2$, $\alpha = 1$, and $\sigma = (1\ 3\ 4\ 2)$, with initial values $z_0^{(1)} = -1$, $z_0^{(2)} = \frac{1}{2}$, $z_0^{(3)} = 2$, $z_0^{(4)} = \frac{1}{4}$.

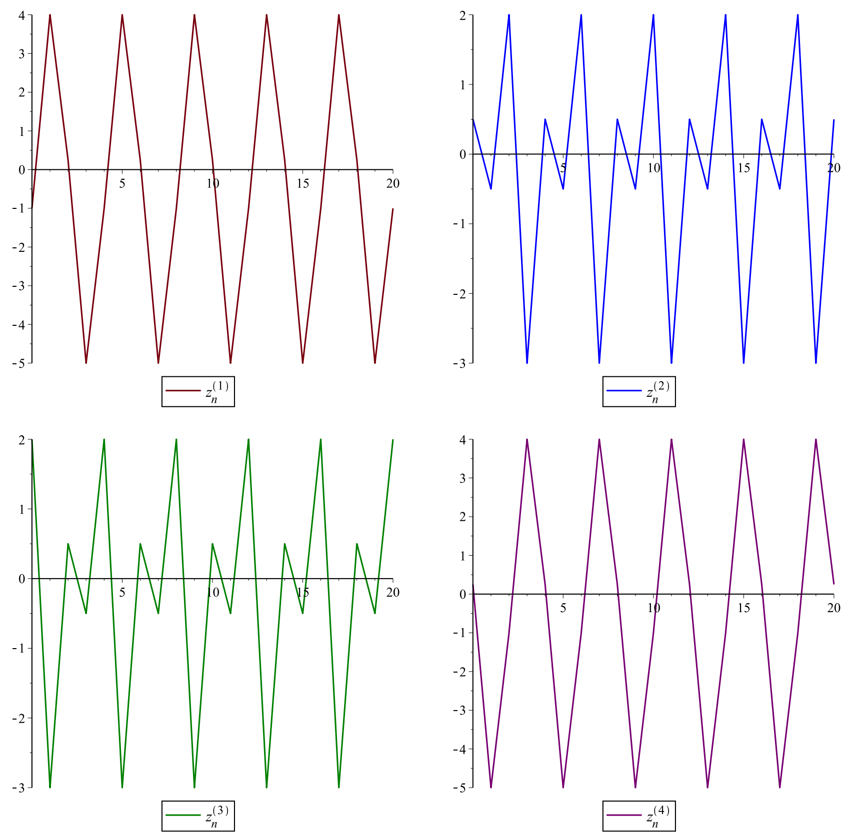

FIGURE 2. The figure shows that System (4.3) is periodic with period 4.

5. Conclusion

Generally, every equation or system has its own properties concerning stability, periodicity, oscillation, and the form of the solutions. These properties depend on the choice of the functions defining the equations and systems. However, one can observe that in the case of systems obtained by generalizing equations through symmetry or permutation, there is a similarity in their properties. The objective of this work is to establish some results regarding this observation.

We show that all studies of the properties of System (1.2) reduce to the study of Equation (1.1). Moreover, System (1.2) includes many particular cases that have been studied in the literature (see, for example, [6, 10]).

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Conflicts of Interest

The authors declare no conflicts of interest.

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[1] LMAM Laboratory, Department of Mathematics, University of Mohamed Seddik BenYahia, Jijel, Algeria. And, Department of Computer Sciences and Mathematics, Higher Normal School Assia Djebar, Constantine, Algeria.

Email address: youssouf.akrour@gmail.com, akrour.youssouf@ensc.dz

[2] MAD Laboratory, Department of Computer Sciences and Mathematics, Higher Normal School Assia Djebar, Constantine, Algeria.

Email address: bellourazze123@yahoo.com