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NON-EXISTENCE OF PROPER WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS IN A LORENTZIAN NEARLY PARACOSYMPLECTIC MANIFOLD

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Abstract. We define the Lorentzian nearly cosymplectic condition and prove that proper warped product semi-slant submanifolds do not exist in a Lorentzian nearly paracosymplectic manifold. As a corollary, we also demonstrate the non-existence of proper warped product semi-invariant submanifolds in such a manifold.

Keywords. Warped products, Lorentzian almost paracontact structure, Lorentzian nearly paracosymplectic manifolds
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1. Introduction

Bishop and O'Neill introduced the notion of warped product manifolds as a natural generalization of the Riemannian product manifolds in [1]. The concept of warped products appeared in the mathematical and physical literature before Bishop-O'Neill, which was called semi-reducible spaces in [7]. A warped product manifold is a Riemannian or semi-Riemannian manifold whose metric tensor can be decomposed into a Cartesian product of the y geometry and the x geometry except that the x-part is warped, i.e. it is rescaled by a scalar function of the other coordinates y (cf. [3]). It is well known that the warped products play an important role in physics, especially in the theory of general relativity. The most famous examples, which appear in general relativity as exact solutions to the Einstein field equations, are the Schwarzschild metric and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The Schwarzschild metric is the spherically symmetric vacuum solution of the Einstein field equations and a Schwarzschild black hole is described by the Schwarzschild metric. The FLRW metric is known as the standard model of cosmology, and describes a homogeneous, isotropic expanding or contracting universe (cf. [8]). According to Nash embedding theorem, every Riemannian manifold can be isometrically embedded in an open subset of a Euclidean space $\mathbb{E}^m$ for sufficiently large dimension m (cf. [10]). Greene showed that Nash's theorem can be extended to any semi-Riemannian manifold, which can be smoothly isometrically embedded in an open subset of a semi-Euclidean space $\mathbb{E}^m_s$ for large enough dimension m and index s (cf. [3], [5], [9]). These embedding theorems

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show that every warped product can be embedded as a Riemannian or semi-Riemannian submanifold in a Euclidean, or semi-Euclidean space, respectively.

Perktaş, Kılıç and Keleş showed some non-existence results of warped product submanifolds in a Lorentzian paracosymplectic manifold (cf. [12]). In this paper, we generalize some of their results by defining a Lorentzian nearly paracosymplectic manifold, which can be seen as an analogue of a nearly Kähler manifold.

We recall some definitions and notations (cf. [12]). Let $\bar{M}$ be an m-dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\bar{M}$ such that

$$\eta(\xi) = -1, \ \ \varphi^2 = I + \eta \otimes \xi, \tag{1.1}$$

where I is the identity map. The equation (1.1) implies that we have

$$\eta \circ \varphi = 0, \ \ \varphi\xi = 0, \ \ \mathrm{rank}(\varphi) = m - 1. \tag{1.2}$$

Then, the manifold $\bar{M}$ admits a Lorentzian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{1.3}$$

for all vector fields $X, Y \in \Gamma(T\bar{M})$. Then, $\bar{M}$ is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). By applying (1.3) and $\eta(\xi) = -1$, we obtain that for all vector fields $X, Y \in \Gamma(T\bar{M})$,

$$g(X, \xi) = \eta(X), \tag{1.4}$$

and by combining (1.4) with (1.1)-(1.3), we get

$$g(X, \varphi Y) = g(\varphi X, Y). \tag{1.5}$$

Note that the Lorentzian metric g makes the vector field ξ a timelike unit vector field such that $g(\xi, \xi) = -1$ from (1.4). The manifold $\bar{M}$ with a Lorentzian almost paracontact structure on (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold.

Here, we note the following proposition.

Proposition 1.1. *(cf. [11]) On a semi-Riemannian manifold M, there is a unique affine connection D such that for any $X, Y, Z \in \Gamma(TM)$,*

(1) $[X, Y] = D_X Y - D_Y X$,
(2) $X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z)$.

The connection D is called the Levi-Civita connection of M, and is characterized by the Koszul formula:

$$\begin{aligned} 2g(D_Y Z, X) &= Y(g(Z, X)) + Z(g(X, Y)) - X(g(Y, Z)) \\ &\quad -g(Y, [Z, X]) + g(Z, [X, Y]) + g(X, [Y, Z]). \end{aligned}$$

With respect to the Levi-Civita connection, we define a Lorentzian nearly paracosymplectic manifold as follows.

Definition 1.2. A Lorentzian almost paracontact manifold $\bar{M}$ is called a Lorentzian nearly paracosymplectic manifold if

$$(\bar{\nabla}_X \varphi)Y + (\bar{\nabla}_Y \varphi)X = 0 \tag{1.6}$$

for any vector fields $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$.

We have the following on a Lorentzian nearly paracosymplectic manifold.

Lemma 1.3. *Let $\bar{M}$ be a Lorentzian nearly paracosymplectic manifold and let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then, we have*

$$\bar{\nabla}_\xi \xi = 0.$$

Proof. From (1.6), we have $(\bar{\nabla}_\xi \varphi)\xi = 0$, which implies that $\varphi \bar{\nabla}_\xi \xi = 0$ since $\varphi\xi = 0$. Then we obtain that

$$\begin{aligned} 0 &= \varphi^2 \bar{\nabla}_\xi \xi \\ &= \bar{\nabla}_\xi \xi + \eta(\bar{\nabla}_\xi \xi)\xi \\ &= \bar{\nabla}_\xi \xi, \end{aligned}$$

where we used $\eta(\bar{\nabla}_\xi \xi) = g(\xi, \bar{\nabla}_\xi \xi) = 0$ since $g(\xi, \bar{\nabla}_\xi \xi) = -g(\bar{\nabla}_\xi \xi, \xi)$. □

From the perspective of mathematical physics and the study of spacetime and black holes, the metric does not necessarily have to be positive definite. Thus, the geometry of warped product submanifolds with an indefinite metric has become a topic of investigation. In [4], Chen and Munteanu addressed the geometry of $\mathcal{PR}$-warped products in para-Kähler manifolds. Building on this work, the author studies warped products in a Lorentzian nearly paracosymplectic manifold, which can be viewed as a counterpart to a para-Kähler manifold.

Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $\bar{M}$. Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. The Gauss-Weingarten formulae are as follows: for $X, Y \in \Gamma(TM)$ and for $W \in \Gamma(T^\perp M)$,

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.7}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^\perp_X W, \tag{1.8}$$

where ∇ is the induced Levi-Civita connection on M, $\nabla^\perp$ is the connection in the normal bundle $T^\perp M$, h is the second fundamental form of M and A_W is the shape operator. The second fundamental form and the shape operator are related as follows:

$$g(A_W X, Y) = g(h(X, Y), W). \tag{1.9}$$

For $X \in \Gamma(TM)$ and for $W \in \Gamma(T^\perp M)$, we put

$$\varphi X = tX + nX, \;\; \varphi W = BW + CW, \tag{1.10}$$

where tX (resp. nX) is tangential (resp. normal) part of φX and BW (resp. CW) is tangential (resp. normal) part of φW.

Bishop, O'Neill introduced the notation of warped product manifolds in [1]. Let (N_1, g_{N1}) and (N_2, g_{N_2}) be two semi-Riemannian manifolds and $f : N_1 \to (0, \infty)$ be a smooth function. The warped product $M = N_1 \times_f N_2$ of N_1 and N_2 is the product manifold $N_1 \times N_2$ with the metric tensor $g = g_{N_1} \oplus f^2 g_{N_2}$, which is given by

$$g(X, Y) = g_{N_1}(d\pi(X), d\pi(Y)) + (f \circ \pi)^2 g_{N_2}(d\sigma(X), d\sigma(Y)),$$

where $X, Y \in \Gamma(T(N_1 \times N_2))$ and $\pi : N_1 \times N_2 \to N_1$ and $\sigma : N_1 \times N_2 \to N_2$ are the canonical projections. We say that a warped product manifold is *proper* when the warping function f is not constant.

Lemma 1.4. *(cf. [11]) Let $M = N_1 \times_f N_2$ be a warped product manifold. If $X, Y \in \Gamma(TN_1)$ and $U, V \in \Gamma(TN_2)$, then we have*

1. $\nabla_X Y \in \Gamma(TN_1)$,
2. $\nabla_X U = \nabla_U X = X(\ln f)U$,
3. $\nabla_U V = \nabla'_U V - g(U, V) grad(\ln f)$,

where ∇ and ∇' denote the Levi-Civita connections on M and N_2, respectively. Here, grad$(\ln f)$ is the gradient of $\ln f$ defined by $g(grad(\ln f), X) = X(\ln f)$ for $X \in \Gamma(TM)$.

Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to M. Denoting by $D : p \mapsto D_p \subset T_pM$ the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \langle\xi\rangle$. In this case, $g(X, X) > 0$ for any $X \neq 0 \in D$. For each $X \neq 0 \in \Gamma(T_pM)$ at $p \in M$ such that X is not proportional to ξ_p, we denote by $\theta(X)$ the angle between φX and T_pM. Since we have $\varphi\xi = 0$, θ agrees with the angle between φX and $D_p \subset T_pM$. Then M is called slant submanifold if the angle $\theta(X)$ is constant, which does not depend on the choice of $p \in M$ and $X \in T_pM \setminus \langle\xi_p\rangle$. The constant angle θ is called the slant angle of M in $\bar{M}$. The invariant and anti-invariant submanifolds of a Lorentzian almost paracontact manifold are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold, which is neither invariant and anti-invariant is said to be a *proper* slant submanifold.

Proposition 1.5. *(cf. [6, Theorem 3.2]) Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$t^2 = \lambda(I + \eta \otimes \xi).$$

Furthermore, in such case, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Corollary 1.6. *(cf. [6, Corollary 3.1]) Let M be a slant submanifold of a Lorentzian almost paracontact manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then*

$$g(tX, tY) = \cos^2\theta\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{1.11}$$

$$g(nX, nY) = \sin^2\theta\{g(X, Y) + \eta(X)\eta(Y)\} \tag{1.12}$$

for all $X, Y \in \Gamma(TM)$, where θ is the slant angle.

The following definition is a generalization of semi-invariant submanifolds.

Definition 1.7. Let M be an isometrically immersed submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. If there exists a pair of distributions D_T and D_θ on M such that $TM = D_T \oplus D_\theta \oplus \langle\xi\rangle$, where D_T is an invariant distribution, i.e., $\varphi D_T = D_T$, and D_θ is a slant distribution with the slant angle $\theta \neq 0$, then M is called semi-slant submanifold of $\bar{M}$. Note that if $\theta = \frac{\pi}{2}$, then it is a semi-invariant submanifold such that $TM = D_T \oplus D_\perp \oplus \langle\xi\rangle$, where $D_\perp$ is an anti-invariant distribution, i.e., $\varphi D_\perp \subseteq T^\perp M$.

The following lemmas are crucial for proving the main results of this paper.

Lemma 1.8. *(cf. [2, Proposition 2.1], [13, Proposition 2.1]) On a Lorentzian nearly paracosymplectic manifold $\bar{M}$, the vector field ξ is Killing.*

For the reader's convenience, we present a proof of Lemma 1.8.

Proof. We compute that for any $X \in \Gamma(T\bar{M})$, by applying the condition (1.6) and $\varphi\bar{\nabla}_X\xi = \bar{\nabla}_X\varphi\xi - (\bar{\nabla}_X\varphi)\xi = -(\bar{\nabla}_X\varphi)\xi$,

$$\begin{aligned}
g(\bar{\nabla}_X\xi, X) &= g(\varphi\bar{\nabla}_X\xi, \varphi X) - \eta(\bar{\nabla}_X\xi)\eta(X) \\
&= -g((\bar{\nabla}_X\varphi)\xi, \varphi X) \\
&= g((\bar{\nabla}_\xi\varphi)X, \varphi X) \\
&= g(\bar{\nabla}_\xi\varphi X, \varphi X) - g(\varphi\bar{\nabla}_\xi X, \varphi X) \\
&= \frac{1}{2}\xi(g(\varphi X, \varphi X)) - g(\bar{\nabla}_\xi X, X) - \eta(\bar{\nabla}_\xi X)\eta(X) \\
&= \frac{1}{2}\xi(g(X, X)) + \eta(\bar{\nabla}_\xi X)\eta(X) - \frac{1}{2}\xi(g(X, X)) - \eta(\bar{\nabla}_\xi X)\eta(X) \\
&= 0,
\end{aligned}$$

where we used that $\eta(\bar{\nabla}_X\xi) = g(\bar{\nabla}_X\xi, \xi) = -g(\xi, \bar{\nabla}_X\xi) = 0$ from $g(\xi, \xi) = -1$, and $\varphi\bar{\nabla}_X\xi = \bar{\nabla}_X\varphi\xi - (\bar{\nabla}_X\varphi)\xi = -(\bar{\nabla}_X\varphi)\xi$ from $\varphi\xi = 0$, and also used that

$$\xi(\eta(X)) = \xi(g(\xi, X)) = g(\xi, \bar{\nabla}_\xi X) = \eta(\bar{\nabla}_\xi X)$$

since we have $\bar{\nabla}_\xi\xi = 0$ from Lemma 1.3, and also used that

$$\begin{aligned}
& g(\bar{\nabla}_\xi\varphi X, \varphi X) = \xi(g(\varphi X, \varphi X)) - g(\varphi X, \bar{\nabla}_\xi\varphi X) \\
\Leftrightarrow \quad & g(\bar{\nabla}_\xi\varphi X, \varphi X) = \frac{1}{2}\xi(g(\varphi X, \varphi X)).
\end{aligned}$$

Hence, we obtain that for any $X \in \Gamma(T\bar{M})$,

$$g(\bar{\nabla}_X\xi, X) = 0. \tag{1.13}$$

Putting $Z := X + Y$ for arbitrary chosen $X, Y \in \Gamma(T\bar{M})$, and then we compute from (1.13), since $Z \in \Gamma(T\bar{M})$,

$$\begin{aligned}
0 &= g(\bar{\nabla}_Z\xi, Z) \\
&= g(\bar{\nabla}_X\xi + \bar{\nabla}_Y\xi, X + Y) \\
&= g(\bar{\nabla}_X\xi, X) + g(\bar{\nabla}_Y\xi, Y) + g(\bar{\nabla}_X\xi, Y) + g(\bar{\nabla}_Y\xi, X) \\
&= g(\bar{\nabla}_X\xi, Y) + g(\bar{\nabla}_Y\xi, X),
\end{aligned}$$

which implies that we have for any $X, Y \in \Gamma(T\bar{M})$,

$$g(\bar{\nabla}_X\xi, Y) + g(\bar{\nabla}_Y\xi, X) = 0. \tag{1.14}$$

Therefore, we conclude that ξ is a Killing vector field. □

Especially, on a Lorentzian nearly paracosymplectic manifold $\bar{M}$, by choosing $Y = X$ in (1.14), we have $g(\bar{\nabla}_X\xi, X) = 0$ for any $X \in \Gamma(T\bar{M})$. From the arbitrariness of $X \in \Gamma(T\bar{M})$, then we must have that for any $X \in \Gamma(T\bar{M})$,

$$\bar{\nabla}_X\xi = 0. \tag{1.15}$$

By combining (1.15) with (1.7), we have the following lemma.

Lemma 1.9. *Let M be an isometrically immersed submanifold of a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. Then we have*

$$\nabla_X\xi = 0, \tag{1.16}$$

$$h(X, \xi) = 0 \tag{1.17}$$

for all $X \in \Gamma(TM)$.

Then, we have the following lemma.

Lemma 1.10. *Let $N_1 \times_f N_2$ be a warped product submanifold of a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TN_1)$. Then, we have*

$$\xi(\ln f) = 0. \tag{1.18}$$

Proof. For any $Z \in \Gamma(TN_2)$, we compute by applying Lemma 1.4 (2) and (1.16),

$$0 = \nabla_Z \xi = \xi(\ln f)Z,$$

which implies that we have $\xi(\ln f) = 0$. □

We have the following result as in [12].

Lemma 1.11. *(cf. [12, Theorem 3.3 (i)]) Let $N_1 \times_f N_2$ be a warped product submanifold in a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$. If $\xi \in \Gamma(TN_2)$, then f is constant.*

Proof. For any $X \in \Gamma(TN_1)$, from Lemma 1.4 (2) and (1.16), we obtain

$$0 = \nabla_X \xi = X(\ln f)\xi,$$

which implies that f is constant if $\xi \in \Gamma(TN_2)$. □

The following immediate consequence is derived from the above lemma.

Proposition 1.12. *There does not exist a proper warped product submanifold $N_1 \times_f N_2$ in a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TN_2)$.*

We show that if a warped product submanifold $N_1 \times_f N_2$ is proper in a Lorentzian nearly paracosymplectic manifold, then $\xi \in \Gamma(TN_1)$.

Lemma 1.13. *Let $M = N_1 \times_f N_2$ be a proper warped product submanifold in a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$ with $\xi \in \Gamma(TM)$. Then $\xi \in \Gamma(TN_1)$.*

Proof. From the assumption $\xi \in \Gamma(TM)$, we may write $\xi = \xi_1 + \xi_2$ for $\xi_i \in \Gamma(TN_i)$ $(i = 1, 2)$. From (1.16) and Lemma 1.4 (2), we have that for any $X \in \Gamma(TN_1)$,

$$0 = \nabla_X \xi = \nabla_X \xi_1 + \nabla_X \xi_2 = \nabla_X \xi_1 + X(\ln f)\xi_2. \tag{1.19}$$

From Lemma 1.4 (1), we have

$$\nabla_X \xi_1 \in \Gamma(TN_1), \quad X(\ln f)\xi_2 \in \Gamma(TN_2). \tag{1.20}$$

Combining (1.19) with (1.20), we have $\nabla_X \xi_1 = 0$ and $X(\ln f)\xi_2 = 0$, especially, we conclude that $\xi_2 = 0$ since M is assumed to be proper, i.e., $X(\ln f) \neq 0$. □

2. Non-existence of warped product semi-slant submanifolds

Let $(\bar{M}, \varphi, \xi, \eta, g)$ be a a Lorentzian nearly paracosymplectic manifold and let M be an isometrically immersed submanifold of $\bar{M}$. Here, we recall that we have $\varphi X = tX + nX$ for $X \in \Gamma(TM)$ (see (1.10)). The submanifold M is called an invariant submanifold if n is identically zero, that is, $\varphi X = tX \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is called an anti-invariant submanifold if t is identically zero, that is, $\varphi X = nX \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$. For later use, let N_T denote an invariant submanifold, $N_{\perp}$ denote an anti-invariant submanifold, and N_θ denote a slant submanifold with $\theta \neq 0$ of $\bar{M}$. In [12], Perktaş, Kılıç and Keleş showed the non-existence of proper warped product semi-slant submanifolds in a Lorentzian paracosymplectic manifold. In the case of a Lorentzian nearly paracosymplectic manifold, from Proposition 1.12, the remaining types are as follows:

(1) $N_T \times_f N_\theta$ (with $\xi \in \Gamma(TN_T)$),
(2) $N_\theta \times_f N_T$ (with $\xi \in \Gamma(TN_\theta)$).

In this paper, we show the non-existence of the type (1). The following theorem is our main result.

Theorem 2.1. *There do not exist proper warped product semi-slant submanifolds of the type $M = N_T \times_f N_\theta$ in a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$.*

Proof. Assume that M is a warped product semi-slant submanifolds of the type $N_T \times_f N_\theta$ in a Lorentzian nearly paracosymplectic manifold $\bar{M}$. Under this assumption of being a semi-slant submanifold of $\bar{M}$, we have $TM = D_T \oplus D_\theta \oplus \langle\xi\rangle$. We compute for any $X \in \Gamma(D_T)$ and $Z \in \Gamma(D_\theta)$, from the condition (1.6),

$$\begin{aligned}
\bar{\nabla}_X \varphi Z &= (\bar{\nabla}_X \varphi)Z + \varphi\bar{\nabla}_X Z \\
&= -(\bar{\nabla}_Z \varphi)X + \varphi\bar{\nabla}_X Z \\
&= -\bar{\nabla}_Z \varphi X + \varphi(\bar{\nabla}_X Z + \bar{\nabla}_Z X) \\
&= -\bar{\nabla}_Z tX + \varphi(\nabla_X Z + h(X,Z) + \nabla_Z X + h(X,Z)) \\
&= -\nabla_Z tX - h(tX,Z) + 2X(\ln f)\varphi Z + 2\varphi h(X,Z) \\
&= -tX(\ln f)Z - h(tX,Z) + 2X(\ln f)tZ \\
&\quad +2X(\ln f)nZ + 2Bh(X,Z) + 2Ch(X,Z). \qquad (2.1)
\end{aligned}$$

On the other hand, we compute using $\nabla_X tZ = X(\ln f)tZ$ from Lemma 1.4 (2),

$$\begin{aligned}
\bar{\nabla}_X \varphi Z &= \bar{\nabla}_X tZ + \bar{\nabla}_X nZ \\
&= X(\ln f)tZ + h(X,tZ) - A_{nZ}X + \nabla^{\perp}_X nZ. \qquad (2.2)
\end{aligned}$$

By combining (2.1) with (2.2) and taking the tangential part, for any $X \in \Gamma(D_T)$ and for any $Z \in \Gamma(D_\theta)$, we obtain

$$A_{nZ}X = -X(\ln f)tZ + tX(\ln f)Z - 2Bh(X,Z). \qquad (2.3)$$

Then, by applying (2.3), we get that from (1.5) and (1.9),

$$\begin{aligned}
& g(Bh(X,Z),Z) \\
=\ & g(\varphi h(X,Z),Z) \\
=\ & g(h(X,Z),\varphi Z) \\
=\ & g(h(X,Z),nZ) \\
=\ & g(A_{nZ}X,Z) \\
=\ & -X(\ln f)g(tZ,Z) + tX(\ln f)g(Z,Z) - 2g(Bh(X,Z),Z), \qquad (2.4)
\end{aligned}$$

which gives us that

$$3g(Bh(X,Z),Z) = tX(\ln f)g(Z,Z) - X(\ln f)g(Z,tZ). \qquad (2.5)$$

Note that for any $X \in \Gamma(D_T)$ and $Z \in \Gamma(D_\theta)$, we have

$$\eta(\bar{\nabla}_Z Z) = Z(g(\xi,Z)) - g(\bar{\nabla}_Z \xi, Z) = 0, \qquad (2.6)$$

where we used $\eta(Z) = g(\xi,Z) = 0$ since we have $TM = D_T \oplus D_\theta \oplus \langle\xi\rangle$, and we used that $\bar{\nabla}\xi = 0$ from (1.15).

Next, by using (1.3), (1.7), (1.9), (1.10) and Lemma 1.4 (2), we compute

$$\begin{aligned}
X(\ln f)g(Z,Z) &= g(\nabla_Z X, Z)\\
&= g(\nabla_Z X, Z) + g(h(Z,X),Z)\\
&= g(\bar{\nabla}_Z X, Z)\\
&= Z(g(X,Z)) - g(X, \bar{\nabla}_Z Z)\\
&= -g(X, \bar{\nabla}_Z Z)\\
&= -g(\varphi X, \varphi\bar{\nabla}_Z Z) + \eta(X)\eta(\bar{\nabla}_Z Z)\\
&= -g(tX, \bar{\nabla}_Z \varphi Z)\\
&= -g(tX, \bar{\nabla}_Z tZ) - g(tX, \bar{\nabla}_Z nZ)\\
&= -Z(g(tX,tZ)) + g(\bar{\nabla}_Z tX, tZ) + g(tX, A_{nZ}Z)\\
&= g(\nabla_Z tX, tZ) + g(h(tX,Z), nZ)\\
&= tX(\ln f)g(Z,tZ) + g(Bh(tX,Z),Z), \qquad (2.7)
\end{aligned}$$

where we used $g(X,Z) = 0$ since the distributions D_T and D_θ are orthogonal each other, and since we have $(\bar{\nabla}_Z\varphi)Z = 0$ from (1.6), we used that

$$\varphi\bar{\nabla}_Z Z = \bar{\nabla}_Z\varphi Z - (\bar{\nabla}_Z\varphi)Z = \bar{\nabla}_Z\varphi Z,$$

and since $\eta(X) = \eta(Z) = 0$, we also used that

$$g(tX,tZ) = g(\varphi X, \varphi Z) = g(X,Z) + \eta(X)\eta(Z) = 0,$$

and

$$g(h(tX,Z),nZ) = g(h(tX,Z),\varphi Z) = g(\varphi h(tX,Z),Z) = g(Bh(tX,Z),Z).$$

From (2.7), we obtain that for any $X \in \Gamma(D_T)$ and any $Z \in \Gamma(D_\theta)$,

$$g(Bh(tX,Z),Z) = X(\ln f)g(Z,Z) - tX(\ln f)g(Z,tZ). \qquad (2.8)$$

Since we have $\varphi X = tX \in \Gamma(D_T)$, that is $\varphi D_T = D_T$, we may replace X with tX in the equation (2.8), and then we obtain

$$g(Bh(t^2X,Z),Z) = tX(\ln f)g(Z,Z) - t^2X(\ln f)g(Z,tZ). \qquad (2.9)$$

Since we have $t^2X = \varphi tX = \varphi^2 X = X + \eta(X)\xi$ for $X \in \Gamma(D_T)$, $h(\xi, Z) = 0$ from (1.17) and $\xi(\ln f) = 0$ from (1.18),

$$h(t^2X,Z) = h(X,Z) + \eta(X)h(\xi,Z) = h(X,Z), \qquad (2.10)$$

$$t^2X(\ln f) = X(\ln f) + \eta(X)\xi(\ln f) = X(\ln f), \qquad (2.11)$$

by combining (2.10)-(2.11) with (2.9), we obtain

$$g(Bh(X,Z),Z) = tX(\ln f)g(Z,Z) - X(\ln f)g(Z,tZ). \qquad (2.12)$$

Combining (2.5) with (2.12), we have

$$3g(Bh(X,Z),Z) = g(Bh(X,Z),Z), \qquad (2.13)$$

which gives us that for any $X \in \Gamma(D_T)$ and for any $Z \in \Gamma(D_\theta)$,

$$g(Bh(X,Z),Z) = 0 \qquad (2.14)$$

and

$$tX(\ln f)g(Z,Z) = X(\ln f)g(Z,tZ). \qquad (2.15)$$

By replacing X with tX in (2.15), we obtain that by using $t^2X(\ln f) = X(\ln f)$ from (2.11),

$$X(\ln f)g(Z,Z) = tX(\ln f)g(Z,tZ). \qquad (2.16)$$

Since the distributions D_T and D_θ are orthogonal each other and $\varphi D_T = D_T$, we have for any $X \in \Gamma(D_T)$ and any $Z \in \Gamma(D_\theta)$,

$$g(tZ, X) = g(\varphi Z, X) = g(Z, \varphi X) = 0,$$

which implies that

$$tD_\theta \perp D_T \tag{2.17}$$

and then since we also have $g(tZ, \xi) = g(\varphi Z, \xi) = g(Z, \varphi\xi) = 0$,

$$tD_\theta \perp (D_T \oplus \langle\xi\rangle). \tag{2.18}$$

On the other hand, since we have $tD_\theta \subset TM$, by considering (2.18), then we obtain

$$tD_\theta \subseteq D_\theta. \tag{2.19}$$

Especially, we have from (2.19),

$$tZ \in \Gamma(D_\theta). \tag{2.20}$$

Since we may replace Z with $tZ \in \Gamma(D_\theta)$ from (2.20) in (2.14), we obtain that

$$g(Bh(X, tZ), tZ) = 0. \tag{2.21}$$

Since we have that $\tilde{Z} := Z + tZ \in \Gamma(D_\theta)$ from (2.20), by polarization identity, we obtain by applying (2.14) and (2.21),

$$\begin{aligned} 0 &= g(Bh(X, \tilde{Z}), \tilde{Z}) \\ &= g(Bh(X, Z), Z) + g(Bh(X, Z), tZ) + g(Bh(X, tZ), Z) + g(Bh(X, tZ), tZ) \\ &= g(Bh(X, Z), tZ) + g(Bh(X, tZ), Z), \end{aligned} \tag{2.22}$$

which implies that

$$g(Bh(X, Z), tZ) = -g(Bh(X, tZ), Z). \tag{2.23}$$

From (2.20), we replace Z with tZ in the equation (2.3), and then we obtain

$$A_{ntZ}X = -X(\ln f)t^2 Z + tX(\ln f)tZ - 2Bh(X, tZ). \tag{2.24}$$

Note that from Proposition 1.5 and $\eta(Z) = 0$, we have

$$t^2 Z = \cos^2\theta\{Z + \eta(Z)\xi\} = \cos^2\theta \cdot Z. \tag{2.25}$$

Then, by using (1.3), (1.9), (1.10), (2.16), (2.24) and (2.25), we get

$$\begin{aligned} g(Bh(X, Z), tZ) &= g(\varphi h(X, Z), tZ) \\ &= g(h(X, Z), \varphi tZ) \\ &= g(h(X, Z), ntZ) \\ &= g(A_{ntZ}X, Z) \\ &= -X(\ln f)g(t^2 Z, Z) + tX(\ln f)g(tZ, Z) - 2g(Bh(X, tZ)Z) \\ &= -X(\ln f)\cos^2\theta g(Z, Z) + X(\ln f)g(Z, Z) - 2g(Bh(X, tZ)Z) \\ &= X(\ln f)\sin^2\theta g(Z, Z) - 2g(Bh(X, tZ), Z), \end{aligned} \tag{2.26}$$

which gives us that

$$X(\ln f)\sin^2\theta g(Z, Z) = g(Bh(X, Z), tZ) + 2g(Bh(X, tZ), Z). \tag{2.27}$$

By replacing X with tX in the formula (2.3), we obtain

$$\begin{aligned} A_{nZ}tX &= -tX(\ln f)tZ + t^2 X(\ln f)Z - 2Bh(tX, Z) \\ &= -tX(\ln f)tZ + X(\ln f)Z - 2Bh(tX, Z), \end{aligned} \tag{2.28}$$

where we used that since we have $t^2X(\ln f) = X(\ln f)$ from (2.11). Then, we get that by applying (1.3), (1.9) and (2.28),

$$\begin{aligned} & g(Bh(tX,tZ),Z) \\ = \; & g(\varphi h(tX,tZ),Z) \\ = \; & g(h(tX,tZ),\varphi Z) \\ = \; & g(h(tX,tZ),nZ) \\ = \; & g(A_{nZ}tX,tZ) \\ = \; & -tX(\ln f)g(tZ,tZ) + X(\ln f)g(Z,tZ) - 2g(Bh(tX,Z),tZ). \end{aligned} \tag{2.29}$$

By replacing X with tX in (2.29),

$$g(Bh(t^2X,tZ),Z) = -t^2X(\ln f)g(tZ,tZ) + tX(\ln f)g(Z,tZ) - 2g(Bh(t^2X,Z),tZ), \tag{2.30}$$

which implies that we have from (2.10)-(2.11),

$$g(Bh(X,tZ),Z) + 2g(Bh(X,Z),tZ) = -X(\ln f)g(tZ,tZ) + tX(\ln f)g(Z,tZ). \tag{2.31}$$

By applying (2.16) to the formula (2.31), and by using (1.10) and $\eta(Z) = 0$, we obtain that

$$\begin{aligned} & g(Bh(X,tZ),Z) + 2g(Bh(X,Z),tZ) \\ = \; & tX(\ln f)g(Z,tZ) - X(\ln f)g(tZ,tZ) \\ = \; & X(\ln f)g(Z,Z) - X(\ln f)\cos^2\theta\{g(Z,Z) + \eta(Z)\eta(Z)\} \\ = \; & X(\ln f)(1-\cos^2\theta)g(Z,Z) \\ = \; & X(\ln f)\sin^2\theta g(Z,Z). \end{aligned} \tag{2.32}$$

By combining (2.27) with (2.32), we obtian

$$g(Bh(X,tZ),Z) + 2g(Bh(X,Z),tZ) = g(Bh(X,Z),tZ) + 2g(Bh(X,tZ),Z), \tag{2.33}$$

which gives us that

$$g(Bh(X,Z),tZ) = g(Bh(X,tZ),Z). \tag{2.34}$$

By combining (2.23) with (2.34), we obtain

$$g(Bh(X,Z),tZ) = g(Bh(X,tZ),Z) = 0. \tag{2.35}$$

By combining (2.35) with (2.27) (or (2.32)), we finally have that

$$X(\ln f)\sin^2\theta g(Z,Z) = 0, \tag{2.36}$$

which implies that either $\theta = 0$ or $X(\ln f) = 0$. Since N_θ has been assumed to be with $\theta \neq 0$, we obtain that $X(\ln f) = 0$ for all $X \in \Gamma(TN_T)$. Therefore, f is constant on N_T. □

Perktaş, Kılıç and Keleş showed the non-existence of proper warped product semi-invariant submanifolds $N_T \times_f N_\perp$ in a Lorentzian paracosymplectic manifold in [12]. Note that the result of Theorem 2.1 implies the non-existence result in the case of proper warped product semi-invariant submanifolds $M = N_T \times_f N_\perp$ in a Lorentzian nearly paracosymplectic manifold because Theorem 2.1 includes the case of $\theta = \frac{\pi}{2}$, which can be regarded as a generalized result of [12]. Note that an isometrically immersed submanifold M in a Lorentzian nearly paracosymplectic manifoldis is called semi-invariant if it is endowed with the pair of orthogonal distribution $(D_T, D_\perp)$ satisfying the condition $TM = D_T \oplus D_\perp \oplus \langle\xi\rangle$ (cf. [12]).

Corollary 2.2. *There do not exist proper warped product semi-invariant submanifolds $N_T \times_f N_\perp$ in a Lorentzian nearly paracosymplectic manifold $(\bar{M}, \varphi, \xi, \eta, g)$.*

We proved the non-existence of proper warped product semi-slant type (1) in a Lorentzian nearly paracosymplectic manifold. However, we were unable to thoroughly study the existence or non-existence of proper warped product semi-slant type (2), and also of proper warped product anti-slant submanifolds in a Lorentzian nearly paracosymplectic manifold. We expect these to be thoroughly studied in the near future.

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