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ON FIBONACCI AND k-PELL NUMBERS WHICH ARE CLOSE TO A POWER OF 3

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Abstract. For an integer $k \geq 2$, let $(P_n^{(k)})_{n\geq 2-k}$ be the k-generalized Pell sequence which starts with $0, \ldots, 0, 1$ (k terms) and each term afterwards is defined by the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2.$$

The aims of this paper is to find all Fibonacci and k-Pell numbers which are close to a power of 3. Concretely, this problem is equivalent to the resolution of the equation $P_n^{(k)} = 3^m + t$ with the condition $|t| < 3^{m/2}$ for any integer t.

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1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Similarly, the Lucas sequence $(L_n)_{n\geq 0}$ follows the same recursive pattern as the Fibonacci numbers with initial conditions $L_0 = 2$ and $L_1 = 1$. The problem of determining all integer solutions of Diophantine equations when Fibonacci numbers have piqued the curiosity of mathematicians and there is a broad literature on this subject. For instance, Bugeaud et al. [6] solved that the only perfect powers among the Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$. More precisely, the only powers of 2 in the Fibonacci sequence are 1, 2 and 8. For an integer $k \geq 2$, let us consider that the k-generalized Fibonacci sequence or the k-Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ defined by the following linear recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

where the initial conditions

$$F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0 \quad \text{and} \quad F_1^{(k)} = 1.$$

This sequence generalizes the usual Fibonacci sequence which is obtained for $k = 2$. In the same way, let $(P_n)_{n\geq 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and the binary recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for all $n \geq 2$. The n-th terms of the Fibonacci

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and Pell sequences are respectively referred to as Fibonacci and Pell numbers. For more information on these sequences, including a numerous application to these numbers and their relatives, see Koshy book [10]. Another sequence of considerable interest in studying equations that contain the k-Pell sequence, which is as important as the k-Fibonacci sequence. For an integer $k \geq 2$, we also consider that the k-generalized Pell sequence or the k-Pell sequence $(P_n^{(k)})_{n\geq 2-k}$ defined by the following linear recurrence of higher order

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

where the initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0 \quad \text{and} \quad P_1^{(k)} = 1.$$

We refer to $P_n^{(k)}$ as the n-th k-Pell number. This sequence generalizes the usual Pell sequence which corresponds to $k = 2$, i.e., $P_n^{(2)} = P_n$. Below, we shall present the values of these numbers for the first few values of k and $n \geq 1$.

TABLE 1. The first nonzero terms of k-Pell numbers are well-known

k	Name	First nonzero terms
2	Pell	$1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \ldots$
3	3-Pell	$1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, \ldots$
4	4-Pell	$1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, \ldots$
5	5-Pell	$1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, \ldots$
6	6-Pell	$1, 2, 5, 13, 34, 89, 233, 609, 1592, 4162, 10881, 28447, 74371, \ldots$
7	7-Pell	$1, 2, 5, 13, 34, 89, 233, 610, 1596, 4176, 10927, 28592, 74815, \ldots$

Recently in [8], without Bouchikhi, we found all the Fibonacci numbers $2F_n$ that are close to a power of 3, in the sense that we studied the Diophantine inequality

$$|2F_n - 3^m| < 3^{m/2}$$

in positive integers (n, m). This result has been generalized in several ways. In particular, we determined the Diophantine inequality

$$|F_n + F_m - 3^a| < 3^{a/2}$$

in positive integers (n, m, a) with $n \geq m$. As a corollary, we also determined

$$|L_n - 3^a| < 3^{a/2},$$

where L_n is the n-th Lucas number referred to $L_n = F_{n-1} + F_{n+1}$ for all $n \geq 0$ and the positive integers (n, a) with $n \geq a$.

In the present paper, we study all the solutions $(F_n, 3^m)$ to the Diophantine inequality

$$|F_n - 3^m| < 3^{m/2} \tag{1.1}$$

in positive integers (n, m) with $n \geq m$. This motivates us to ask the following profound question.

Question. *What terms of the k-Pell numbers which are close to a power of* 3*?* Following that, we also study all the solutions $(P_n^{(k)}, k, n, m)$ to the Diophantine inequality

$$|P_n^{(k)} - 3^m| < 3^{m/2} \tag{1.2}$$

in the positive integers (k, n, m) with $k \geq 2$.

To answer this question, we will prove our principal results which are formulated in the following theorems and lemmas.

Theorem 1.1. *There are only* 5 *Fibonacci numbers which are close to a power of* 3*. Including, the solutions* $(F_n, 3^m)$ *of the Diophantine inequality* (1.1) *in positive integers* (n, m) *with* $n \geq m$*, are given by*

$$(F_n, 3^m) \in \{(2,3),(3,3),(8,9),(89,81),(233,243)\}$$

and so

$$(n,m) \in \{(3,1),(4,1),(6,2),(11,4),(13,5)\}.$$

Theorem 1.2. *All the solutions* $(P_n^{(k)}, k, n, m)$ *of the Diophantine inequality* (1.2) *in the positive integers* (k, n, m) *with* $k \geq 2$*, are given by*

$$(2,k,2,1),\ 2 \leq k \leq 4, \quad (89,k,6,4),\ k \geq 5, \quad (233,k,7,5),\ k \geq 6,$$
$$(29,2,5,3), \quad (84,3,6,4), \quad (88,4,6,4), \quad (228,4,7,5), \quad \textit{and} \quad (232,5,7,5).$$

Note that the above theorem gives the solutions to the Diophantine equation

$$P_n^{(k)} = 3^m + t \quad \text{with} \quad |t| < 3^{m/2}. \tag{1.3}$$

By assuming that $n \geq k+2$, then we have the following lemma which yields us the bounds on m in terms of n.

Lemma 1.3. *If the integers* (k, n, m) *is a solution of the inequality* (1.2) *with* $n \geq 1$*,* $m \geq 1$*, and* $n \geq k+2$*, then we have the both inequalities*

$$0.44n - 2.76 < m < 0.88n + 0.56 < 1.5n.$$

As with Theorem 1.1 and checking for the small values of n, we may assume that $n \geq 2$. Now, we obtain the following lemma which appears an upper bound on n and m in terms of k.

Lemma 1.4. *If the integers* (k, n, m) *satisfy the equation* (1.3) *with* $n \geq k+2$*, then we have the following estimates*

$$n < 1.17 \times 10^{15} \cdot k^5 \cdot \log^3 k \quad \textit{and} \quad m < 1.76 \times 10^{15} \cdot k^5 \cdot \log^3 k.$$

Lemma 1.5. *There is no solution of the inequality* (1.2) *with* $k > 330$ *and* $n \geq k+2$*.*

For proving our principal results, we shall need to use a Baker's theory lower bound on such nonzero linear forms in logarithms of algebraic numbers due to Matveev [11] and Dujella-Pethő [7] to obtain a large upper bound on n and to bound n polynomially in terms of k. To do this, we apply a reduction algorithm originally introduced by a version and a variant of the reduction method of Baker-Davenport [1] to reduce this large bound. We begin by recalling some basic notions from algebraic number theory.

2. Preliminary results

Before proceeding further, we shall recall some facts and properties of the Fibonacci sequence which will be used later. Firstly, it is known that the Binet formula for the general term of this sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds for all $n \geq 0$, where $\alpha := (1+\sqrt{5})/2$ and $\beta := (1-\sqrt{5})/2$ are the roots of the characteristic polynomial $x^2 - x - 1 = 0$ of $(F_n)_{n\geq 0}$. It can be seen that $1 < \alpha < 2$,

$-1 < \beta < 0$ and $\alpha\beta := -1$. The relation between the Fibonacci numbers and α, are given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{2.1}$$

holds for all $n \geq 1$. Additionally, we also shall need the following definitions.

Definition 2.1 (Closeness)**.** An integer x is said to be close to a positive integer y, if it satisfies

$$|x - y| < \sqrt{y}.$$

Definition 2.2 (Mahler measure)**.** Let η be an algebraic number, we define its measure by the following identity

$$M(\eta) = |a_d| \prod_{i=1}^{d} \max\{1, |\eta_i|\},$$

where η_i's are the roots of $f(x) = a_d \prod_{i=1}^{d}(x - \eta_i)$ is the minimal polynomial of η.

Definition 2.3 (Absolute logarithmic height)**.** Let η be an algebraic number of degree d on $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d \prod_{i=1}^{d} (x - \eta_i),$$

where the a_i's are relatively prime integers with $a_d > 0$ and the η_i's are conjugates of η. The usual absolute logarithmic height of η is defined as

$$h(\eta) = \frac{1}{d}\left(\log a_d + \sum_{i=1}^{d} \log\left(\max\{1, |\eta_i|\}\right)\right) = \frac{\log\left(M(\eta)\right)}{d}.$$

In particular, if $\eta = \frac{p}{q} \in \mathbb{Q}$ with $q > 0$ and $\gcd(p, q) = 1$, then $h(\eta) = \log\left(\max\{|p|, q\}\right)$. The following properties [14, Property 3.3] of $h(\eta)$, which will be used in the subsection 4.2 and 4.4. For any algebraic numbers η and κ, then we have the function h has the following basic properties:

$$\begin{aligned} h(\eta \pm \kappa) &\leq h(\eta) + h(\kappa) + \log 2, \\ h(\eta\kappa^{\pm 1}) &\leq h(\eta) + h(\kappa), \\ h(\eta^s) &= |s| h(\eta), \ \ s \in \mathbb{Z}. \end{aligned}$$

Using the notations of the above definition and the next theorem to get an upper bound on the variable n which is very large. Thus, we shall need to reduce that bound a lower bound for linear forms in logarithms due to Matveev [11, Corollary 2.3] or Bugeaud et al. [6, Theorem 9.4] proved the following deep theorem.

Theorem 2.4 (Matveev's theorem)**.** *Assume that $\eta_1, \eta_2, \ldots, \eta_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree d. Let $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Gamma := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 = \prod_{i=1}^{t} \eta_i^{b_i} - 1$$

is nonzero. Then

$$|\Gamma| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d^2(1+\log d)(1+\log B)A_1 \cdots A_t\right),$$

where $A_1, A_2, \ldots, A_t$ *are real numbers,*

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{d \cdot h(\eta_i), |\log \eta_i|, 0.16\} \text{ for all } i = 1, \ldots, t.$$

In order to prove our principal results, we shall also need a variant of the Baker-Davenport [1, Lemma], which is due to Dujella-Pethő [7, Lemma 5(a)]. For this, we use some results from the various properties theory of Fibonacci and k-Pell numbers, continued fractions, Diophantine approximation, and a few applications of a Baker's theory lower bound for nonzero linear forms in logarithms of algebraic numbers. To do this, we state a result to Matveev [11] for the general lower bound on linear forms in logarithms, and we also apply a version of the reduction method of Baker-Davenport [1] to reduce the upper bound which is generally very large. However, we use the one given by Bravo et al. [2, Lemma 1]. For a real number x, we write that $\|x\| := \min\{|x-n|, n \in \mathbb{Z}\}$ the distance of x to the nearest integer. Indeed, we shall also need the following lemma due to Dujella-Pethő [7].

Lemma 2.5 (Dujella-Pethő)**.** *Let* M *be a positive integer and suppose that* p/q *be a convergent of the continued fraction of the irrational number* γ *such that* $q > 6M$*, and let* A*,* B*,* μ *be some real numbers with* $A > 0$ *and* $B > 1$*. Furthermore, let* $\varepsilon := \|\mu q\| - M\|\gamma q\|$*, where* $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\varepsilon > 0$*, then there exists no solution of the inequality*

$$0 < |\vartheta\gamma - \nu + \mu| < AB^{-\omega}$$

in positive integers ϑ*,* ν *and* ω *with*

$$\vartheta \leq M \quad \text{and} \quad \omega \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

We next present the following lemma of Sanchez and Luca [13, Lemma 7].

Lemma 2.6. *If* $\ell \geq 1$*,* $T > (4\ell^2)^\ell$ *and* $T > x/(\log x)^\ell$*, then we have*

$$x < 2^\ell T(\log T)^\ell.$$

A simple fact concerning the exponential function. We list it as a lemma by further reference [5, Lemma 8].

Lemma 2.7. *For any nonzero real number* x*, we have the following conditions:*

i) $0 < x < |e^x - 1|$.
ii) *If* $x < 0$ *and* $|e^x - 1| < 1/2$*, then* $|x| < 2|e^x - 1|$.

Secondly, we recall some facts and properties of k-generalized Pell sequence. However, the characteristic polynomial of this sequence is either

$$\psi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1.$$

In [4, Section 2], it is shown that $\psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\lambda := \lambda(k)$ outside the unit circle. This root is real and positive and satisfies $\lambda(k) > 1$. Too, The other roots are strictly inside the unit circle. Moreover, the same authors [4, Lemma 3.2] showed that

$$\theta^2(1-\theta^{-k}) < \lambda(k) < \theta^2 \quad \text{for} \quad k \geq 2 \quad \text{and} \quad \theta := (1+\sqrt{5})/2. \tag{2.2}$$

To simplify this, on the whole, we shall omit the dependence of $\lambda(k)$ on k and use λ. For $k \geq 2$, let

$$f_k(x) := \frac{x-1}{(k+1)x^2 - 3kx + k - 1} = \frac{x-1}{k(x^2-3x+1)+x^2-1}.$$

According to [3, Lemma 1], it follows that the inequalities

$$0.276 < f_k(\lambda) < 0.5 \quad \text{and} \quad |f_k(\lambda_i)| < 1 \quad \text{for} \quad 2 \leq i \leq k, \tag{2.3}$$

where $\lambda := \lambda_1, \lambda_2, \ldots, \lambda_k$ (the conjugates of λ) are the roots of the characteristic polynomial $\psi_k(x)$. It was deduced in [12, Section 2.4] that $f_k(\lambda)$ is not an algebraic integer. Besides, the same authors in [3, Lemma 1] proved that the logarithmic height of $f_k(\lambda)$ satisfies the inequality

$$h(f_k(\lambda)) < 4k \log\theta + k\log(k+1) \quad \text{for} \quad k \geq 2. \tag{2.4}$$

As with the last notations, Bravo et al. [4, Theorem 3.1] proved that

$$P_n^{(k)} = \sum_{i=1}^{k} f_k(\lambda_i)\lambda_i^n \quad \text{and} \quad |P_n^{(k)} - f_k(\lambda)\lambda^n| < 1/2,$$

which holds for $n \geq 1$ and $k \geq 2$. Therefore, for $n \geq 1$ and $k \geq 2$, it follows that

$$P_n^{(k)} = f_k(\lambda)\lambda^n + e_k(n) \quad \text{with} \quad |e_k(n)| \leq 1/2. \tag{2.5}$$

As well as that, it was shown that the inequality

$$\lambda^{n-2} \leq P_n^{(k)} \leq \lambda^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad k \geq 2. \tag{2.6}$$

Lastly, we finish this section by giving the following estimate [3, Lemma 2]. If $n > 1$ and $k \geq 30$ are integers satisfying $n < \theta^{k/2}$, then we have

$$f_k(\lambda)\lambda^n = \frac{\theta^{2n}}{\theta+2}(1+\xi) \quad \text{with} \quad |\xi| < \frac{4}{\theta^{k/2}}. \tag{2.7}$$

3. Proof of Theorem 1.1

Proof. Assume throughout that equation (1.1) holds. At first, we discuss the two following cases.

- **Case 1.** In the range $1 \leq m \leq 10$, then we have list of the first 10 intervals to $I_m := (3^m - 3^{m/2}, 3^m + 3^{m/2})$ are given by

Table 2. The Fibonacci numbers F_n to the inequality (1.1) with $1 \leq m \leq 10$ are displayed

m	Integers in I_m	The Fibonacci numbers F_n in I_m
1	$2, 3, 4$	$2, 3$
2	$7, \ldots, 11$	8
3	$22, \ldots, 32$	
4	$73, \ldots, 89$	89
5	$228, \ldots, 258$	233
6	$703, \ldots, 755$	
7	$2141, \ldots, 2233$	
8	$6481, \ldots, 6641$	
9	$19543, \ldots, 19823$	
10	$58807, \ldots, 59291$	

In them, we find all the Fibonacci numbers F_n. For following, we get that the only Fibonacci numbers which are close to 3^m with $1 \le m \le 10$ are 2, 3, 8, 89 and 233.

- **Case 2**. For $m > 10$, then by Binet formula of $(F_n)_{n\ge 0}$ reveals

$$\left|F_n - \frac{\alpha^n}{\sqrt{5}}\right| = \frac{|\beta|^n}{\sqrt{5}} = \frac{1}{\sqrt{5}\alpha^n}, \quad \text{because} \quad \beta := -\alpha^{-1}.$$

If we combine it with the inequality (1.1), it comes that

$$\left|3^m - \frac{\alpha^n}{\sqrt{5}}\right| < 3^{m/2} + \frac{1}{\sqrt{5}\alpha^n},$$

If we divide both sides of the last inequality by 3^m, we acquire

$$\left|1 - 3^{-m}\cdot\alpha^n\cdot\sqrt{5}^{-1}\right| < 3^{-m/2} + \frac{1}{3^m\alpha^n\sqrt{5}} < 3^{(1-m)/2}. \tag{3.1}$$

Applying Matveev's result in Theorem 2.4, we can take the parameters $t := 3$ and

$$(\eta_1, b_1) := (3, -m), \quad (\eta_2, b_2) := (\alpha, n), \quad (\eta_3, b_3) := (\sqrt{5}, -1).$$

Let us start by noticing that η_1, η_2, and η_3 are positive real numbers belonging to $\mathbb{K} := \mathbb{Q}(\sqrt{5})$, so we can take $d := [\mathbb{K} : \mathbb{Q}] = 2$. Indeed, we can choose $h(\eta_1) = \log 3$, $h(\eta_2) = (\log\alpha)/2$, and $h(\eta_3) = \log\sqrt{5}$. For this end, we can take $A_1 := 2.2 > 2h(\eta_1)$, $A_2 := 0.5 > 2h(\eta_2)$, and $A_3 := 1.62 > 2h(\eta_3)$. Thus, according to the inequality (1.1) with (2.1), we get

$$3^m - 3^{m/2} < F_n \le \alpha^{n-1},$$

which is valid for $m < n$. So, we infer that $B := \max\{|-m|, |n|, |-1|\} = n$. Suppose now that $\Gamma := 3^{-m}\cdot\alpha^n\cdot\sqrt{5}^{-1} - 1$. We justify that $\Gamma \ne 0$. Obviously, by the fact that if $\Gamma = 0$, we could have the relation $\alpha^n = 3^m\sqrt{5}$, so that $\alpha^{2n} = 5\cdot 3^{2m} \in \mathbb{Z}$, which is a contradiction since $\alpha^{2n} \notin \mathbb{Z}$. By Matveev's result in Theorem 2.4, we acquire

$$\log|\Gamma| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log n)\cdot 2.2\cdot 0.5\cdot 1.62.$$

By the inequality (3.1), we obtain

$$\log|\Gamma| < (1-m)\log\sqrt{3}.$$

Consequently, we have

$$m - 1 < 3.15\times 10^{12}(1+\log n). \tag{3.2}$$

By using the inequalities (1.1) and (2.1), we acquire

$$\begin{aligned}\alpha^{n-2} \le F_n < 3^m + 3^{m/2} = 3^{m+2}(3^{-2} + 3^{-m/2-2}) &< 3^{m+2}(2^{-2} + 2^{-m/2-2})\\ &< 3^{m+2}\cdot\alpha^{-2},\end{aligned}$$

which leads to

$$n < (m+2)\frac{\log 3}{\log\alpha}\cdot \tag{3.3}$$

If we combine it with the inequality (3.2), and by a calculation with the help of Maple, we acquire

$$m < 1.11\times 10^{14} \quad \text{and} \quad n < 2.5\times 10^{14}.$$

At second, we are going to reduce the upper bounds for m and n. In the case $m > 10$, we have no Fibonacci number equals 3^m. Thus, we separated this case in two parts.

- If $F_n > 3^m$. We begin noticing that
$$3^m \le F_n - 1 < \frac{\alpha^n}{\sqrt{5}},$$
which gives
$$n\log\alpha - m\log 3 - \log\sqrt{5} > 0.$$
Here, we can set $\Lambda := n\log\alpha - m\log 3 - \log\sqrt{5}$. Since $e^\Lambda - 1 > \Lambda$ for all positive Λ, then by using the inequalities (3.1) and (3.3), we get
$$0 < \Lambda < |1-e^\Lambda| < 3^{(1-m)/2} < 3^{\frac{3}{2}-\frac{n\log\alpha}{2\log 3}} < 5.2\cdot(1.26)^{-n}.$$
If we divide both sides of the above inequality by $\log 3$, we acquire
$$0 < n\frac{\log\alpha}{\log 3} - m + \frac{\log(1/\sqrt{5})}{\log 3} < 4.75\cdot(1.26)^{-n}. \tag{3.4}$$
Choosing
$$\gamma := \frac{\log\alpha}{\log 3} \notin \mathbb{Q}, \quad \mu := -\frac{\log\sqrt{5}}{\log 3}, \quad A := 4.75, \quad B := 1.26, \text{ and } \omega := n.$$
Let us consider $q_i > 6M$ be the denominator of the i-th convergent of the continued fraction of γ. Since $n < 2.5\times 10^{14}$, we can take $M := 2.5\times 10^{14}$. For following, we try to apply Lemma 2.5 to the inequality (3.4) becomes
$$q_i := q_{33} = 4509703705422533 > 6M$$
and so
$$\varepsilon := \|\mu q_{33}\| - M\|\gamma q_{33}\| = 0.01198\cdots.$$
We deduce that there is no solution to the inequality (3.4) for
$$n \ge \frac{\log(Aq_{33}/\varepsilon)}{\log B} = 181.84975\cdots$$
and hence in the range
$$\left[\left\lfloor\frac{\log(Aq_{33}/\varepsilon)}{\log B}\right\rfloor + 1, M\right] \supset [182, 2.5\times 10^{14}],$$
which leads to $n < 182$.
- If $F_n < 3^m$. Note that for $\Lambda < 0$, it is seen that
$$0 < -\Lambda < e^{-\Lambda} - 1 = e^{-\Lambda}|e^\Lambda - 1|,$$
it follows that
$$|e^\Lambda - 1| = |1-e^\Lambda| < \sqrt{3}^{1-m} < 5.2\cdot(1.26)^{-n} < \frac{1}{2}.$$
Since $\Lambda < 0$, it implies that $e^\Lambda \in (\frac{1}{2}, 1)$, so that $e^{-\Lambda} < 2$. Therefore, we acquire
$$0 < m\frac{\log 3}{\log\alpha} - n + \frac{\log\sqrt{5}}{\log\alpha} < \frac{2\sqrt{3}}{\log\alpha}\cdot\sqrt{3}^{-m} < 22\cdot(1.26)^{-n}. \tag{3.5}$$
By applying again Lemma 2.5 to the inequality (3.5), we conclude that there is no solution to this inequality for
$$m \ge 188.48247\cdots.$$
Similarly, by through argument, we obtain $m < 189$ and $n < 437$.

At last, by a calculation in Maple. Suppose now that $x := \frac{\log F_n}{\log 3}$, where $x > 4$, $n \geq m > 10$ and

$$3^{x+1} - 3^{(x+1)/2} = 3^x(3 - 3^{(-x+1)/2}) > 3^x = F_n.$$

Thus, from the case $F_n > 3^m$, it comes that

$$\frac{\log F_n}{\log 3} < m < \frac{\log F_n}{\log 3} + 1.$$

So, we only need to check whether F_n is in the interval $(3^m, 3^m + 3^{m/2})$ when

$$m = \left\lfloor \frac{\log F_n}{\log 3} \right\rfloor + 1.$$

Through a calculation with the help of Maple, we infer that there is no such n. For the case $F_n < 3^m$. Correspondingly, by through a calculation with helping of Maple, we conclude that no such n exists. This completes the proof. □

4. Proof of Theorem 1.2 and Lemmas

In this section, we give all details about the proof of Theorem 1.2. For this, we discuss two following cases will be considered according to the values on n.

4.1. The case $1 \leq n \leq k+1$.

In [9, Section 2], it is shown that the equality

$$P_n^{(k)} = F_{2n-1} \quad \text{for} \quad 1 \leq n \leq k+1.$$

Therefore, by the inequality (1.2) reveals

$$|F_{2n-1} - 3^m| < 3^{m/2}. \tag{4.1}$$

By Theorem 1.1, we get the following subsequent proposition.

Proposition 4.1. *There are only 3 Fibonacci numbers F_{2n-1} which are close to a power of 3. Namely, the solutions $(F_{2n-1}, 3^m)$ to the Diophantine inequality (4.1) in positive integers (n, m) with $n \geq m$, are as follows*

$$(F_{2n-1}, 3^m) \in \{(2, 3), (89, 81), (233, 243)\}$$

and so

$$(n, m) \in \{(2, 1), (6, 4), (7, 5)\}.$$

Proof. The solutions of the inequality (4.1) were determined by a calculation with a computational number theory system PARI/GP code [15]. □

By the above proposition, we conclude that the solutions $(P_n^{(k)}, k, n, m)$ of the inequality (1.2) with $k \geq 2$, are given by

$$(2, k, 2, 1),\ 2 \leq k \leq 4, \quad (89, k, 6, 4),\ k \geq 5, \quad (233, k, 7, 5),\ k \geq 6.$$

4.2. **The case $n \geq k+2$.** In this subsection, we prove Lemma 1.3. Following that, we get an upper bound on m and n in terms of k about the proof of Lemma 1.4.

Proof of Lemma 1.3. If we combine the inequality (2.6) with the equation (1.2), we acquire

$$3^{m-1} < 3^m - 3^{m/2} < P_n^{(k)} \leq \lambda^{n-1}$$

and

$$\lambda^{n-2} \leq P_n^{(k)} < 3^m + 3^{m/2} < 3^{m+1}.$$

If we take the logarithm of both sides of the two above inequalities, we acquire

$$(n-2)\frac{\log \lambda}{\log 3} - 1 < m \leq (n-1)\frac{\log \lambda}{\log 3} + 1.$$

Since $\theta^2(1-\theta^{-k}) < \lambda < \theta^2$ for $k \geq 2$, we conclude that

$$0.44n - 2.76 < m < 0.88n + 0.56 < 1.5n.$$

This completes the proof. $\square$

Proof of Lemma 1.4. If we substitute the formula (2.5) in the equation (1.3), then we have

$$3^m - f_k(\lambda)\lambda^n = -t + e_k(n).$$

If we take the absolute value of both sides, we get

$$|3^m - f_k(\lambda)\lambda^n| = |-t + e_k(n)| < |t| + |e_k(n)| < 3^{m/2} + \frac{1}{2}\cdot$$

If we divide through by $f_k(\lambda)\lambda^n$, we obtain

$$\begin{aligned} \left|\frac{1}{f_k(\lambda)} \cdot \lambda^{-n} \cdot 3^m - 1\right| &< \frac{3^{m/2}}{f_k(\lambda)\lambda^n} + \frac{1}{2f_k(\lambda)\lambda^n} \\ &< 3.624 \cdot \frac{(3\lambda^{n-1})^{1/2}}{\lambda^n} + \frac{1.812}{\lambda^n} \\ &< \frac{4.925}{\lambda^{n/2}} + \frac{1.812}{\lambda^{n/2}} \\ &< \frac{7}{\lambda^{n/2}}, \end{aligned} \tag{4.2}$$

because the inequalities $3^{m-1} < \lambda^{n-1}$ and $0.276 < f_k(\lambda) < 0.5$ hold for all $k \geq 2$. Let us consider that $\Gamma_1 := \frac{1}{f_k(\lambda)} \cdot \lambda^{-n} \cdot 3^m - 1$. Observe that $\Gamma_1 \neq 0$. Indeed, if this $\Gamma_1 = 0$, then we have that $f_k(\lambda) = 3^m\lambda^{-n}$ and so $f_k(\lambda)$ could be an algebraic integer, which is a contradiction. Thus, $\Gamma_1 \neq 0$. Applying Matveev's result in Theorem 2.4, we can take the parameters $t := 3$ and

$$(\eta_1, b_1) := (1/f_k(\lambda), 1), \quad (\eta_2, b_2) := (\lambda, -n), \quad (\eta_3, b_3) := (3, m).$$

Let us start by noticing that η_1, η_2 and η_3 are positive real numbers belonging to $\mathbb{K}_1 := \mathbb{Q}(\lambda)$ for which $d := [\mathbb{K}_1 : \mathbb{Q}] = k$. Obviously, we can choose $h(\eta_2) = (\log \lambda)/k < (2\log\theta)/k$ and $h(\eta_3) = \log 3$. As a consequence, we acquire $A_2 := 0.97 > k \cdot h(\eta_2)$ and $A_3 := k \log 3 > k \cdot h(\eta_3)$.

By using properties of logarithmic height and the inequality (2.4), it shows that

$$h(\eta_1) = h(f_k(\lambda)) < 4k\log\theta + k\log(k+1) < 4.5k\log k \quad \text{for} \quad k \geq 2.$$

Therefore, we can set $A_1 := 4.5k^2 \log k > k \cdot h(\eta_1)$. Additionally, by Lemma 1.3, we infer that $B := \max\{|b_1|, |b_2|, |b_3|\} = \max\{1, n, m\}$, so we can choose $B = 2n$. Thus, by Matveev's result in Theorem 2.4 tells us that

$$\begin{aligned}\log|\Gamma_1| &> -6.9 \times 10^{11} \cdot k^5 \cdot \log k \cdot (1 + \log k)(1 + \log(2n)) \\ &> -6.04 \times 10^{12} \cdot k^5 \cdot \log^2 k \cdot \log n,\end{aligned}$$

where we used the facts that $1 + \log k < 2.5 \log k$ and $1 + \log(2n) < 3.5 \log n$, which is valid for $k \geq 2$ and $n \geq 2$. If we combine the above inequality with the inequality (4.2), we acquire

$$\frac{n}{2} \log \lambda - \log 7 < 6.04 \times 10^{12} \cdot k^5 \cdot \log^2 k \cdot \log n$$

and so

$$n < 1.21 \times 10^{13} \cdot k^5 \cdot \log^2 k \cdot \log n,$$

which implies that

$$\frac{n}{\log n} < 1.21 \times 10^{13} \cdot k^5 \cdot \log^2 k.$$

By applying Lemma 2.6 with $T = 1.21 \times 10^{13} \cdot k^5 \cdot \log^2 k$, $x = n$, and $\ell = 1$, it comes that

$$\begin{aligned}n &< 2(1.21 \times 10^{13} \cdot k^5 \cdot \log^2 k) \log(1.21 \times 10^{13} \cdot k^5 \cdot \log^2 k) \\ &< (2.42 \times 10^{13} \cdot k^5 \cdot \log^2 k)(30.2 + 5 \log k + 2 \log(\log k)) \qquad (4.3) \\ &< 1.17 \times 10^{15} \cdot k^5 \cdot \log^3 k,\end{aligned}$$

where we have used the fact that $30.2 + 5 \log k + 2 \log(\log k) < 48 \log k$ is valid for all $k \geq 2$. Finally, by using Lemma 1.3 with the inequality (4.3), we acquire

$$m < 1.76 \times 10^{15} \cdot k^5 \cdot \log^3 k.$$

This finishes the proof of Lemma 1.4. □

Subsequently, we will distinguish the cases to the value of k is small or large.

4.3. **In the case $2 \leq k \leq 330$.** To reduce the last bound on n according to the inequality (4.3), we set

$$\Lambda_1 := n \log \lambda - m \log 3 + \log(f_k(\lambda)).$$

By note that $\Gamma_1 = e^{-\Lambda_1} - 1 \neq 0$. Then $\Lambda_1 \neq 0$. If $\Lambda_1 < 0$, then by Lemma 2.7 with the inequality (4.2) tells us that

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = |\Gamma_1| < \frac{7}{\lambda^{n/2}}.$$

If $\Lambda_1 > 0$, then we get $1 - e^{-\Lambda_1} = |e^{-\Lambda_1} - 1| < 1/2$. Therefore, we obtain

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = e^{\Lambda_1}|e^{-\Lambda_1} - 1| < \frac{14}{\lambda^{n/2}}.$$

Thus, in both cases we obtain

$$0 < |\Lambda_1| < \frac{14}{\lambda^{n/2}}.$$

If we divide through by $\log 3$, we acquire

$$\left| n \frac{\log \lambda}{\log 3} - m + \frac{\log(f_k(\lambda))}{\log 3} \right| < \frac{12.8}{\lambda^{n/2}}.$$

Applying Lemma 2.5 to the above inequality, we can take the parameters

$$\gamma := \frac{\log \lambda}{\log 3} \notin \mathbb{Q}, \quad \mu := \frac{\log (f_k(\lambda))}{\log 3}, \quad A := 12.8, \quad B := \lambda, \text{ and } \omega := n/2.$$

In view of the range $2 \leq k \leq 330$, we determine a suitable approximation of the expansion γ and a convergent p_i/q_i of the continued fraction of expansion γ. These convergents fulfill q_i exceeds $6M_k$ and $\varepsilon := \varepsilon(k) = \|\mu q_{53}\| - M_k\|\gamma q_{53}\| > 0$ with $M = M_k := \lfloor 1.17 \times 10^{15} \cdot k^5 \cdot \log^3 k \rfloor$. This value serves as an upper bound on n according to Lemma 1.4. A quick computation with the help of Maple reveals that

$$q_i := q_{53} = 63598223074656670671604561 98488 > 6M_k$$

and

$$\min_{2 \leq k \leq 330} \varepsilon(k) > 0.17879.$$

Additionally, the maximum value of ω over all $k \in [2, 330]$ is

$$\frac{n}{2} \geq \frac{\log (Aq_{53}/\varepsilon)}{\log B} = 78.38481 \cdots .$$

Finally, we write by a brute force calculation with the help of Maple to determine all the solutions of the equation (1.3) for the ranges $2 \leq k \leq 330$, $0 \leq n \leq 156$ and $0 \leq m \leq 137$ (because $m < 0.88n + 0.56$) with $n \geq k + 2$. Also, we find the remaining solutions are listed in Theorem 1.2.

4.4. **In the case** $k > 330$**.** In this case, we shall need to proof of Lemma 1.5, which will complete the proof of Theorem 1.2.

Proof of Lemma 1.5. By referring to Lemma 1.4, then could we have

$$n < 1.17 \times 10^{15} \cdot k^5 \cdot \log^3 k < \theta^{k/2} \quad \text{for all} \quad k > 330.$$

Therefore, by using the inequalities (2.5), (2.7), and (1.3), we obtain

$$\left| 3^m - \frac{\theta^{2n}}{\theta + 2} \right| = \left| \frac{\theta^{2n}}{\theta + 2}\xi + e_k(n) - t \right| < \frac{4\theta^{2n}}{(\theta + 2)\theta^{k/2}} + \frac{1}{2} + 3^{m/2}.$$

Multiplying through by $(\theta + 2)\theta^{-2n}$ and using the following facts that $3 < \theta^3$, $m < n$, we acquire

$$\begin{aligned} \left|(\theta + 2) \cdot \theta^{-2n} \cdot 3^m - 1\right| &< \frac{4}{\theta^{k/2}} + \frac{\theta + 2}{2\theta^{2n}} + \frac{3^{m/2}(\theta + 2)}{\theta^{2n}} \\ &< \frac{4}{\theta^{k/2}} + \frac{1.81}{\theta^{2n}} + \frac{3.62\theta^{3m/2}}{\theta^{2n}} \\ &< \frac{4}{\theta^{k/2}} + \frac{5.43}{\theta^{n/2}} \\ &< \frac{9.5}{\theta^{k/2}}. \end{aligned} \tag{4.4}$$

Let us now that

$$\Gamma_2 := (\theta + 2)\theta^{-2n}3^m - 1.$$

Observe that $\Gamma_2 \neq 0$. Obviously, then we have $3^m = \theta^{2n}/(\theta + 2)$, which is a contradiction. Thus, we can apply Matveev's result in Theorem 2.4 to the inequality (4.4) appears that the parameters $t := 3$ and

$$(\eta_1, b_1) := (\theta + 2, 1), \quad (\eta_2, b_2) := (\theta, -2n), \quad (\eta_3, b_3) := (3, m).$$

We begin by noticing that η_1, η_2 and η_3 are positive real numbers belonging to $\mathbb{K}_2 := \mathbb{Q}(\theta)$ for which $d := [\mathbb{K}_2 : \mathbb{Q}] = 2$. Following that, then we have $h(\eta_2) = (\log\theta)/2$ and $h(\eta_3) = \log 3$. In view of the properties of the logarithmic height of η_1, we obtain

$$h(\eta_1) = h(\theta+2) \leq h(\theta) + h(2) + \log 2 \leq \frac{\log\theta}{2} + \log 4.$$

Therefore, we can choose $A_1 := 3.26 > 2h(\eta_1)$, $A_2 := 0.5 > 2h(\eta_2)$, and $A_3 := 2.2 > 2h(\eta_3)$. Again, we deduce that $B := \max\{|b_1|, |b_2|, |b_3|\} = \max\{1, 2n, m\}$, so we can take $B = 3n$. For following, by using Theorem 2.4 tells us that

$$\log|\Gamma_2| > -1.05 \times 10^{13} \log n, \tag{4.5}$$

where we used the fact that $1 + \log(3n) < 3\log n$ holds for $n \geq 3$. Now, we set the inequalities (4.4) and (4.5) together to get

$$k < 4.4 \times 10^{13} \log n. \tag{4.6}$$

By using Lemma 1.4, we get

$$\log n < \log(1.17 \cdot 10^{15} \cdot k^5 \cdot \log^3 k) < 12\log k \quad \text{for} \quad k \geq 330. \tag{4.7}$$

By using the inequalities (4.6) and (4.7), we obtain

$$\frac{k}{\log k} < 5.3 \times 10^{14}.$$

Applying again Lemma 2.6, we acquire

$$k < 2(5.3 \times 10^{14}) \log(5.3 \times 10^{14}) < 3.6 \times 10^{16}.$$

By using Lemma 1.4, it turns out that

$$n < 3.92 \times 10^{102}. \tag{4.8}$$

To reduce the last bound on n, we set

$$\Lambda_2 := 2n\log\theta - m\log 3 + \log(1/(\theta+2)).$$

Let us consider that $\Gamma_2 = e^{-\Lambda_2} - 1 \neq 0$. Thus, $\Lambda_2 \neq 0$. So, we obtain

$$0 < |\Lambda_2| < \frac{19}{\theta^{k/2}}.$$

If we divide through by $\log 3$, we acquire

$$\left|2n\frac{\log\theta}{\log 3} - m + \frac{\log(1/(\theta+2))}{\log 3}\right| < \frac{17.3}{\theta^{k/2}}. \tag{4.9}$$

Next, we try to apply a Lemma 2.5 to the above inequality reveals that the parameters

$$\gamma := \frac{\log\theta}{\log 3} \notin \mathbb{Q}, \quad \mu := -\frac{\log(\theta+2)}{\log 3}, \quad A := 17.3, \quad B := \theta, \text{ and } \omega := k/2.$$

Taking $M := 7.84 \times 10^{102}$, which is upper bound on $2n$ by the inequality (4.8). By utilizing Maple, we find that $q_i := q_{191}$ is the denominator of the i-th convergent of the continued fraction of expansion γ. According to hypotheses of Lemma 2.5, we obtain

$$\varepsilon = 4 \times 10^{60} \quad \text{and} \quad \frac{k}{2} \geq \frac{\log(Aq_{191}/\varepsilon)}{\log B} = 212.02398\cdots.$$

Hence, for $k \leq 424$, by using Lemma 1.4, we get

$$n < 3.55 \times 10^{30}.$$

To apply again Lemma 2.6 to the inequality (4.9) with $M := 7.1 \times 10^{30}$ which is upper bound on $2n$ by the above inequality, we acquire

$$q_i := q_{63} = 10114219148903598839961343100020 9 > 6M \text{ and } \varepsilon > 0.30035.$$

So, we get

$$\frac{k}{2} \geq \frac{\log\left(Aq_{63}/\varepsilon\right)}{\log B} = 161.56622\cdots.$$

As a consequence, we deduce that $k \leq 323$. This contradicts our working assumption that $k > 330$. Finally, we conclude that inequality (1.2) doesn't admit any solution for all $k > 330$. This finishes the analysis of these cases and therefore the proof of Lemma 1.5 and that of Theorem 1.2. □

4.5. **The Code.** In this subsection, for the sake of completeness, we present the PARI/GP code [15] used to confirm Proposition 4.1.

```
{
for(n = 1, 330, for(m = 1, 330, if(n >= m,
a = fibonacci(2*n - 1) - 3^m;
if(abs(a) < 3^(m/2),
print([fibonacci(2*n - 1), 3^m, n, m])))))
}
```

This completes the proofs of our principal results.

5. Conclusions

In summary, we found all Fibonacci numbers that are close to a power of 3 and the solutions $(F_n, 3^m)$ of the inequality (1.1) in positive integers (n, m) with $n \geq m$. Subsequently, we also found all k-Pell numbers that are close to a power of 3 and all the solutions $(P_n^{(k)}, k, n, m)$ of the inequality (1.2) in the positive integers (k, n, m) with $k \geq 2$.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Baker, A., & Davenport, H. (1969). The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. Quart. J. Math. Oxford Ser. (2) 20(1), 129-137. https://doi.org/10.1093/qmath/20.1.129
2. Bravo, J. J., Gómez, C. A., & Luca, F. (2016). Power of two as sums of two k-Fibonacci numbers. Miskolc Math. Notes 17(1), 85-100. https://doi.org/10.18514/MMN.2016.1505
3. Bravo, J. J., & Herrera, J. L. (2020). Repdigits in generalized Pell sequences. Arch. Math. (Brno) 56(4), 249-262. http://doi.org/10.5817/AM2020-4-249
4. Bravo, J. J., Herrera, J. L., & Luca, F. (2021). On a generalization of the Pell sequence. Math. Bohema. 146(2), 199-213. https://doi.org/10.21136/MB.2020.0098-19
5. Bravo, J. J., Herrera, J. L., & and Luca, F. (2021). Common values of generalized Fibonacci and Pell sequences. J. Number Theory 226, 51-71. https://doi.org/10.1016/j.jnt.2021.03.001

6. Bugeaud, Y., Mignotte, M., & Siksek, S. (2006). Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. Ann. of Math. 163(3), 969-1018. https://doi.org/10.4007/annals.2006.163.969
7. Dujella, A., & Pethő, A. (1998). A generalization of a theorem of Baker and Davenport. Q. J. Math. Oxford 49(3), 291-306. https://doi.org/10.1093/qmathj/49.3.291
8. Gaha, A., & Mezroui, S. (2025). On the Fibonacci numbers and their sums which are close to a power of 3. Gulf J. Math. 19(2), 47-60. https://doi.org/10.56947/gjom.v19i2.2758
9. Kiliç, E. (2013). On the usual Fibonacci and generalized order-k Pell numbers. Ars Combin. 109, 391-403.
10. Koshy, T. (2014). Pell and Pell-Lucas numbers with applications. Springer New York, 431. https://link.springer.com/book/10.1007/978-1-4614-8489-9
11. Matveev, E. M. (2000). An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II. Izv. Math. 64(6), 1217-1269. https://doi.org/10.1070/IM2000v064n06ABEH000314
12. Normenyo, B. V., Rihane, S. E., & Togbé, A. (2023). Common terms of k-Pell numbers and Padovan or Perrin numbers. Arab. J. Math. 12, 219-232. https://doi.org/10.1007/s40065-022-00407-8
13. Sanchez, S. G., & Luca, F. (2014). Linear combinations of factorials and S-units in a binary recurrence sequence. Ann. Math. Qué. 38, 169-188. https://doi.org/10.1007/s40316-014-0025-z
14. Waldshmidt, M. (2000). Diophantine approximation on linear algebraic groups: Transcendence properties of the exponential function in several variables. Springer-Verag Berlin Heidelberg. https://link.springer.com/book/10.1007/978-3-662-11569-5
15. The PARI Group, PARI/GP, Version 2.15.4, Bordeaux. (2023). Available from https://pari.math.u-bordeaux.fr/

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