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APPROXIMATE CONTROLLABILITY OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH TIME DELAYS IN INPUT AND STATE

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Abstract. This article examines the approximate controllability of a class of second-order ordinary differential equations, where the input and system state, both have non-zero delays. Here, the existence and uniqueness of the mild solution to the problem is derived using a generalized contraction principle and properties of the cosine family of bounded linear operators on a Banach space. Also, the approximate controllability of the system under consideration and the related linear system with delay are achieved under a set of suitable criteria. The methodology employed for establishing the outcomes guarantees that the nonlinear system's approximate controllability remains independent of the approximate controllability of the corresponding linear system with delay. Further, the results are valid whether or not the operators are noncompact.

Keywords. Second order differential equations, control delay, state delay, mild solution, approximate controllability
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1. Introduction

Second order differential equations have drawn a lot of attention because they may be utilized to investigate a wide range of real-world events. It is essential in interpreting a broad variety of phenomena in the domains of biology, physics, and other sciences. Modeling the behavior of an electrical circuit, or the motion of a simple pendulum, are a few examples. The theory of the strongly continuous cosine family of operators is a helpful tool for studying second-order abstract differential equations in infinite dimensional spaces. The study was initiated by Fattorini in 1969 [8, 9]. The theory was then further developed by Travis and Webb, who also carried out more research on second-order evolution equations using the cosine family of operators [23, 24]. Numerous researchers have since, used this theory to investigate various forms of controllability, mild solutions and other solutions for second order differential equations [19, 7].

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Nonlinear systems with time or state delay are studied because of a variety of issues that arise in physical and biological systems. Examples include, how the thyroid gland functions to regulate blood sugar and insulin levels, how the central nervous system behaves during learning, etc. Here, the evolution of the processes is influenced by past information, and such situations are modelled with the help of delay differential equations [10, 12]. The hot shower problem is a well-known example of a finite delay system, that arises because of the time it takes for the water to heat up and flow from the faucet to the shower head. For systems with finite delay in state, one can refer to the works by [22], [17] as well as the references included there.

Many physical systems include time delays, which can occur not only in the system's state, but also in the control, or both. For systems with control delay, the current rate of change of control, are dependent on their preceding values and thus can be best represented with the help of delay differential equations with time delay present in the control function. For example, input delays can result from population dynamics, harmonic oscillator with delayed forcing term, time-consuming data collection, or in biological processes such as gestation delays [3], [4]. When a mobile robot or other teleoperated system, placed in a remote location, is controlled, it gives rise to a transport delay in the communication channel, which becomes an input delay for the device under control [25]. Now, compared to delay in state, there is very little literature on systems with control delay, and most of it is limited to finite-dimensional cases. The study for finite-dimensional case was initiated by Chyung [6], where the author established necessary and sufficient conditions for controllability of a time-invariant linear system with delay in the control function. Sebakhy and Bayoumi [21], developed a simple test to determine the controllability of the same system [6]. Thereafter, results on different types of controllability was established by Olbrot [20] for linear systems with delay in control, and Klamka made a significant contribution to the literature by developing the theory for both linear and nonlinear systems with time variable and distributed delays in control (see, [13]-[16]). For more works related to the controllability of systems with delay in finite dimensional setup, one can refer to [18] and the references therein.

For literature related to controllability of problems with delay in control, in infinite dimension, the study is mostly confined to first order ordinary differential equation (see [5, 2]). While Haq and Sukavanam [11] studied second order retarded systems, Afreen et al [1] focused on second order stochastic systems, and they determined the approximate controllability of the corresponding systems with input delays. In all the mentioned articles, zero initial control is considered and most of the results will fail, if the initial control is taken to be non-zero. However, as mentioned in [5], it is possible to have non-zero initial control. Motivated by the above discussion, this work aims to address the approximate controllability of the following system:

$$\left.\begin{aligned} \omega''(t) &= T\omega(t) + F_0\nu(t) + F_1\nu(t-\alpha) + p\big(t, \omega_{x(t)}, \nu(t)\big),\ t \in (0, \Upsilon], \\ \omega'(0) &= \omega_1, \\ \omega(t) &= \lambda(t),\ t \in [-\beta, 0], \\ \nu(t) &= v(t),\ t \in [-\alpha, 0]. \end{aligned}\right\} \tag{1.1}$$

Here, $0 \leq \alpha \leq \beta < \Upsilon$, $T : D(T) \subset W \to W$ is a closed linear operator generating a strongly continuous cosine family on a Banach space W, the control function $\nu \in L^2([0, \Upsilon], N)$, where N is a Banach space and with $v \in C([-\alpha, 0], N)$ being the initial control. $F_0 : L^2([0, \Upsilon], N) \to L^2([0, \Upsilon], W)$ and $F_1 : N \to W$ are bounded linear operators. $p : [0, \Upsilon] \times C([-\beta, 0], W) \times N \to W$ is a given nonlinear function satisfying some assumptions to be specified later. Also, for $\omega \in C([-\beta, \Upsilon], W)$, the function $\omega_{x(t)} \in C([-\beta, 0], W)$ is defined by $\omega_{x(t)}(t) = \omega(t + x(t))$, $t \in [-\beta, 0]$, where, $x : [0, \Upsilon] \to [0, \Upsilon]$ satisfies $t \geq x(t)$ for all t in the specified interval. $\omega_1 \in \Omega$, which will be defined later and $\lambda \in C([-\beta, 0], W)$ is such that $\lambda(0) \in D(T)$.

So far it has been observed that, the study of approximate controllability of second order systems with delay in infinite dimensional case, with non-zero initial control, is not well covered in literature. Therefore, in order to close the gap, the study of the second order system (1.1), with delay in both state and control is considered. Unlike the majority of other approaches, here the approximate controllability of the investigated nonlinear system is independent of the approximate controllability of its corresponding linear system with delay. The outcomes hold true regardless of whether the operators are compact or noncompact.

The arrangement of the article is as follows: Section 2 provides an overview of the notations used throughout this work and defines the mild solution for the problem under consideration. The existence and uniqueness results are presented in Section 3. Section 4 includes some results and sufficient conditions which assists in establishing the approximate controllability of (1.1). Lastly, Section 5, provides a summary of the outcomes of this work.

2. Preliminaries

The following notations are used throughout the work:
$(W, \|.\|_W)$ and $(N, \|.\|_N)$ are Banach spaces. $\{T_{\mathcal{C}}(t)\}_{t\in\mathbb{R}}$ and $\{T_{\mathcal{S}}(t)\}_{t\in\mathbb{R}}$ denotes the strongly continuous cosine and sine family of bounded linear operators on W. $\mathcal{C}$ and $\mathcal{S}$ are two positive real numbers such that $\|T_{\mathcal{C}}(t)\|_{B(W)} \leq \mathcal{C}$ and $\|T_{\mathcal{S}}(t)\|_{B(W)} \leq \mathcal{S}$ for all $t \in [0, \Upsilon]$ (by using Uniform boundedness theorem). T is the infinitesimal generator of $\{T_{\mathcal{C}}(t)\}_{t\in\mathbb{R}}$, with domain $D(T)$ defined by

$$D(T) = \{\omega \in W | T_{\mathcal{C}}(t)\omega \text{ is twice continuously differentiable function in } t\},$$

and, $\Omega = \{\omega \in W | T_{\mathcal{C}}(t)\omega \text{ is once continuously differentiable function in } t\}$. For more details on the theory and properties of cosine and sine family of operators, refer to [23, 24].

$(B(N,W), \|.\|_{B_N^W})$ is a Banach space, where $B(N,W)$ denotes the set of all bounded linear maps from N to W. The function spaces

$$C_\alpha = \big(C([-\alpha,0],W), \|.\|_{C_\alpha}\big), \qquad C_\beta = \big(C([-\beta,0],W), \|.\|_{C_\beta}\big)$$
$$\text{and,} \qquad C = \big(C([-\beta,\Upsilon],W), \|.\|_{C}\big),$$

represents Banach spaces with respect to the norms given by

$$\|\xi_1\|_{C_\alpha} = \sup_{t\in[-\alpha,0]} \|\xi_1(t)\|_W, \qquad \|\xi_2\|_{C_\beta} = \sup_{t\in[-\beta,0]} \|\xi_2(t)\|_W$$
$$\text{and,} \qquad \|\xi_3\|_C = \sup_{t\in[-\beta,\Upsilon]} \|\xi_3(t)\|_W,$$

respectively, where $\xi_1 \in C([-\alpha,0],W)$, $\xi_2 \in C([-\beta,0],W)$ and $\xi_3 \in C([-\beta,\Upsilon],W)$. The symbols $\|.\|_{L^2}$ and $\|.\|_{\bar{L}^2}$ denotes the norm on the spaces $L^2([0,\Upsilon],W)$ and $L^2([0,\Upsilon],N)$, respectively.

The mild solution of (1.1) is defined as follows:

Definition.[12] A mild solution of (1.1) is a function $\omega \in C([-\beta,\Upsilon],W)$ satisfying the integral equation

$$\begin{cases} \omega(t) = T_c(t)\lambda(0) + T_s(t)\omega_1 + \displaystyle\int_{\tau=0}^{t} T_s(t-\tau)F_0\nu(\tau)d\tau + \int_{\tau=0}^{t} T_s(t-\tau) \\ \qquad \times F_1\nu(\tau-a)d\tau + \displaystyle\int_{\tau=0}^{t} T_s(t-\tau)p(\tau,\omega_{x(\tau)},\nu(\tau))d\tau, \quad t\in(0,\Upsilon], \\ \omega(t) = \lambda(t), \quad t\in[-\beta,0]. \end{cases}$$

3. Existence result

Assume the following hypothesis:

Assumption 1 the function p is such that:
for each $t \in [0,\Upsilon]$, $p(t,.,.) : C([-\beta,0],W)\times N \to W$ is continuous, and for each $(\omega,\nu) \in C([-\beta,0],W)\times N$, $p(.,\omega,\nu) : [0,\Upsilon] \to W$ is measurable. Also, there exists a positive real number P satisfying

$$\|p(t,\omega_1,\nu_1) - p(t,\omega_2,\nu_2)\|_W \le P\big[\|\omega_1-\omega_2\|_{C_\beta} + \|\nu_1-\nu_2\|_N\big],$$

for all ω_1, $\omega_2 \in C([-\beta,0],W)$ and all ν_1, $\nu_2 \in N$ and each $t \in [0,\Upsilon]$.

Theorem 3.1. *Under Assumption 1, our problem* (1.1) *has a unique mild solution, for each* $\nu \in L^2([0,\Upsilon],N)$ *with initial control* $\nu(t) = v(t)$ *in* $[-\alpha,0]$.

Proof. Define Π on $C([-\beta,\Upsilon],W)$ by

$$(\Pi\omega)(t) = \begin{cases} T_c(t)\lambda(0) + T_s(t)\omega_1 + \int_0^t T_s(t-\tau)F_0\nu(\tau)d\tau + \int_0^t T_s(t-\tau) \\ \qquad \times F_1\nu(\tau-a)d\tau + \int_0^t T_s(t-\tau)p(\tau,\omega_{x(\tau)},\nu(\tau))d\tau, & t\in(0,\Upsilon], \\ \lambda(t), & t\in[-\beta,0]. \end{cases}$$

Using Assumption 1 and Hölder's inequality, it can be easily shown that Π is well-defined on $C([-\beta,\Upsilon],W)$. To show that Π has a unique fixed point on $C([-\beta,\Upsilon],W)$, we take the following steps:

Step 1: $\Pi\omega \in C([-\beta,\Upsilon],W)$ for $\omega \in C([-\beta,\Upsilon],W)$.

Let $\omega \in C([-\beta,\Upsilon],W)$. Take $-\beta \leq \delta_1 < \delta_2 \leq 0$, then λ being continuous, we have

$$\|(\Pi\omega)(\delta_2) - (\Pi\omega)(\delta_1)\|_W = \|0_W\|_W \longrightarrow 0 \quad \text{as} \;\; \delta_2 \to \delta_1.$$

Next, for $0 < \delta_1 < \delta_2 \leq \Upsilon$,

$$\|(\Pi\omega)(\delta_2) - (\Pi\omega)(\delta_1)\|_W \leq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6,$$

where,

$$\Delta_1 = \|T_{\mathcal{C}}(\delta_2)\lambda(0) - T_{\mathcal{C}}(\delta_1)\lambda(0)\|_W, \quad \Delta_2 = \left\|T_{\mathcal{S}}(\delta_2)\omega_1 - T_{\mathcal{S}}(\delta_1)\omega_1\right\|_W,$$
$$\Delta_3 = \left\| \int_0^{\delta_2} T_{\mathcal{S}}(\delta_2-\tau)F_0\nu(\tau)d\tau - \int_0^{\delta_1} T_{\mathcal{S}}(\delta_1-\tau)F_0\nu(\tau)d\tau \right\|_W,$$
$$\Delta_4 = \left\| \int_0^{\delta_2} T_{\mathcal{S}}(\delta_2-\tau)F_1\nu(\tau-a)d\tau - \int_0^{\delta_1} T_{\mathcal{S}}(\delta_1-\tau)F_1\nu(\tau-a)d\tau \right\|_W,$$
$$\Delta_5 = \left\| \int_0^{\delta_2} T_{\mathcal{S}}(\delta_2-\tau)p(\tau,\omega_{x(\tau)},\nu(\tau))d\tau - \int_0^{\delta_1} T_{\mathcal{S}}(\delta_1-\tau)p(\tau,\omega_{x(\tau)},\nu(\tau))d\tau \right\|_W.$$

Now, both the quantities

$$\begin{aligned} \Delta_1 &= \|T_{\mathcal{C}}(\delta_2)\lambda(0) - T_{\mathcal{C}}(\delta_1)\lambda(0)\|_W \longrightarrow 0, \\ \text{and,} \quad \Delta_2 &\leq \mathcal{C}\|\omega_1\|_W(\delta_2-\delta_1) \longrightarrow 0, \end{aligned}$$

as $\delta_2 \to \delta_1$, by using the properties of the cosine family of bounded linear operators on W.
For Δ_3:

$$\begin{aligned} \Delta_3 &\leq \left\| \int_{\delta_1}^{\delta_2} T_{\mathcal{S}}(\delta_2-\tau)F_0\nu(\tau)d\tau \right\|_W + \left\| \int_0^{\delta_1} [T_{\mathcal{S}}(\delta_2-\tau) - T_{\mathcal{S}}(\delta_1-\tau)]F_0\nu(\tau)d\tau \right\|_W \\ &\leq \Delta_{31} + \Delta_{32}, \end{aligned}$$

with

$$\begin{aligned} \Delta_{31} &= \int_{\delta_1}^{\delta_2} \|T_{\mathcal{S}}(\delta_2-\tau)F_0\nu(\tau)\|_W d\tau \\ &\leq \mathcal{S}\|F_0\nu\|_{L_2}\sqrt{\delta_2-\delta_1} \longrightarrow 0 \quad \text{as} \;\; \delta_2 \to \delta_1, \end{aligned}$$

and

$$\begin{aligned} \Delta_{32} &= \int_0^{\delta_1} \|[T_{\mathcal{S}}(\delta_2-\tau) - T_{\mathcal{S}}(\delta_1-\tau)]F_0\nu(\tau)\|_W d\tau \\ &\leq \mathcal{C}\int_0^{\delta_1} \|F_0\nu(\tau)\|_W d\tau \\ &\leq \mathcal{C}\sqrt{\Upsilon}\|F_0\nu\|_{L_2}(\delta_2-\delta_1) \longrightarrow 0 \quad \text{as} \;\; \delta_2 \to \delta_1. \end{aligned}$$

For Δ_4:

$$\begin{aligned}\Delta_4 &\leq \left\| \int_{\delta_1}^{\delta_2} T_{\mathcal{S}}(\delta_2-\tau)F_1\nu(\tau-a)d\tau \right\|_W + \left\| \int_0^{\delta_1} [T_{\mathcal{S}}(\delta_2-\tau)-T_{\mathcal{S}}(\delta_1-\tau)] \right. \\ &\quad \left. \times F_1\nu(\tau-a)d\tau \right\|_W \\ &\leq \Delta_{41}+\Delta_{42}.\end{aligned}$$

Now,

$$\begin{aligned}\Delta_{41} &= \int_{\delta_1}^{\delta_2} \|T_{\mathcal{S}}(\delta_2-\tau)F_1\nu(\tau-\alpha)\|_W d\tau \\ &\leq \mathcal{C}\|F_1\|_{B_N^W}\left[\delta_2\int_{\delta_1}^{\delta_2}\|\nu(\tau-\alpha)\|_N d\tau - \int_{\delta_1}^{\delta_2}\tau\|\nu(\tau-\alpha)\|_N d\tau\right].\end{aligned}$$

Therefore,

$$\Delta_{41} \leq \begin{cases} \mathcal{C}\|F_1\|_{B_N^W}\|v\|_{C\alpha}\left[\delta_2(\delta_2-\delta_1)+\frac{\delta_2^2-\delta_1^2}{2}\right], & 0<\delta_1<\delta_2<\alpha, \\ \mathcal{C}\|F_1\|_{B_N^W}\left[\|v\|_{C\alpha}\left(\delta_2(\alpha-\delta_1)+\frac{\alpha^2-\delta_1^2}{2}\right)\right. & \\ \qquad \left. +\|\nu\|_{\bar{L}^2}\left(\delta_2\sqrt{\delta_2-\alpha}+\sqrt{\frac{\delta_2^3-\alpha^3}{3}}\right)\right], & 0<\delta_1<\alpha<\delta_2, \\ \mathcal{C}\|F_1\|_{B_N^W}\|\nu\|_{\bar{L}^2}\left[\delta_2\sqrt{\delta_2-\delta_1}+\sqrt{\frac{\delta_2^3-\delta_1^3}{3}}\right], & \alpha<\delta_1<\delta_2, \end{cases}$$

and

$$\Delta_{42} \leq \mathcal{C}(\delta_2-\delta_1)\int_0^{\delta_1}\|F_1\nu(\tau-\alpha)\|_W d\tau$$

both of which goes to zero as δ_2 approaches δ_1. Thus $\Delta_4 \to 0$, as $\delta_2 \to \delta_1$.
Finally, $\Delta_5 \leq \Delta_{51}+\Delta_{52}$, where

$$\begin{aligned}\Delta_{51} &= \int_{\delta_1}^{\delta_2} \left\|T_{\mathcal{S}}(\delta_2-\tau)p(\tau,\omega_{x(\tau)},\nu(\tau))\right\|_W d\tau \\ &\leq \mathcal{S}(P\|\omega\|_C+\mathcal{A})(\delta_2-\delta_1)+\mathcal{S}P\|\nu\|_{\bar{L}^2}\sqrt{\delta_2-\delta_1} \longrightarrow 0 \quad \text{as } \delta_2\to\delta_1,\end{aligned}$$

and,

$$\begin{aligned}\Delta_{52} &= \int_0^{\delta_1}\|[T_{\mathcal{S}}(\delta_2-\tau)-T_{\mathcal{S}}(\delta_1-\tau)]p(\tau,\omega_{x(\tau)},\nu(\tau))\|_W d\tau \\ &\leq \mathcal{C}(\delta_2-\delta_1)\int_0^{\delta_1}\|p(\tau,\omega_{x(\tau)},\nu(\tau))\|_W d\tau \\ &\leq \mathcal{C}(P\Upsilon\|\omega\|_C+P\sqrt{\Upsilon}\|\nu\|_{\bar{L}^2}+\mathcal{A}\Upsilon)(\delta_2-\delta_1) \longrightarrow 0 \quad \text{as } \delta_2\to\delta_1,\end{aligned}$$

where, $\mathcal{A}=\sup_{t\in[0,\Upsilon]}\|p(t,0,0)\|_W$.

Thus, $\Pi\omega$ is a continuous function from $[-\beta,\Upsilon]$ to W, for any ω in the continuous space $C([-\beta,\Upsilon],W)$.

Step 2: $\Pi^{(m)}$ is a contraction for some $m\in\mathbb{N}$.

Let ω^1, $\omega^2 \in C([-\beta,\Upsilon],W)$, then for $t \in [-\beta,0]$,

$$\|(\Pi\omega^1)(t)-(\Pi\omega^2)(t)\|_W = 0.$$

Claim For each $m \in \mathbb{N}$ the following inequality holds for all $t > 0$ and ω^1, $\omega^2 \in C([-\beta,\Upsilon],W)$:

$$\|(\Pi^m\omega^1)(t)-(\Pi^m\omega^2)(t)\|_W \leq \frac{(\mathcal{S}Pt)^m}{m!}\sup_{\tau\in[0,t]}\|\omega^1(\tau)-\omega^2(\tau)\|_W.$$

To prove this, proceed by induction on $m \in \mathbb{N}$.

For $t > 0$, we have

$$\begin{aligned}
&\|(\Pi\omega^1)(t)-(\Pi\omega^2)(t)\|_W\\
&\leq \Big\|\int_0^t T_{\mathcal{S}}(t-\tau)\big[p(\tau,\omega^1_{x(\tau)},\nu(\tau))-p(\tau,\omega^2_{x(\tau)},\nu(\tau))\big]d\tau\Big\|_W\\
&\leq \mathcal{S}\int_0^t \|p(\tau,\omega^1_{x(\tau)},\nu(\tau))-p(\tau,\omega^2_{x(\tau)},\nu(\tau))\|_W d\tau\\
&\leq \mathcal{S}P\int_0^t \|\omega^1_{x(\tau)}-\omega^2_{x(\tau)}\|_{C_\beta}d\tau\\
&\leq \mathcal{S}Pt\sup_{\tau\in[0,t]}\|\omega^1(\tau)-\omega^2(\tau)\|_W
\end{aligned}$$

So assume, that the following inequality holds for $m \geq 2$:

$$\begin{aligned}
&\|(\Pi^{m-1}\omega^1)(t)-(\Pi^{m-1}\omega^2)(t)\|_W\\
&\leq \frac{(\mathcal{S}Pt)^{m-1}}{(m-1)!}\sup_{\tau\in[0,t]}\|\omega^1(\tau)-\omega^2(\tau)\|_W,\ \text{for each t} > 0.
\end{aligned}$$

Then, we have for each $t > 0$,

$$\begin{aligned}
&\|(\Pi^m\omega^1)(t)-(\Pi^m\omega^2)(t)\|_W\\
&\leq \Big\|\int_0^t T_{\mathcal{S}}(t-\tau)\big[p\big(\tau,(\Pi^{m-1}\omega^1)_{x(\tau)},\nu(\tau)\big)-p\big(\tau,(\Pi^{m-1}\omega^2)_{x(\tau)},\nu(\tau)\big)\big]d\tau\Big\|_W\\
&\leq \mathcal{S}P\int_0^t \|(\Pi^{m-1}\omega^1)_{x(\tau)}-(\Pi^{m-1}\omega^2)_{x(\tau)}\|_{C_\beta}d\tau
\end{aligned}$$

and thus, using the induction hypotheses, we get

$$\|(\Pi^m\omega^1)(t)-(\Pi^m\omega^2)(t)\|_W \leq \frac{(\mathcal{S}Pt)^m}{m!}\sup_{\tau\in[0,t]}\|\omega^1(\tau)-\omega^2(\tau)\|_W.$$

Let $\Upsilon > 0$ be fixed, then $\frac{(\mathcal{S}P\Upsilon)^m}{m!}$ can be made strictly less than one, for sufficiently large values of m. So, there exist $m_0 \in \mathbb{N}$, with $\frac{(\mathcal{S}P\Upsilon)^{m_0}}{m_0!} < 1$, and therefore we get

$$\|\Pi^{m_0}\omega^1-\Pi^{m_0}\omega^2\|_C \leq \|\omega^1-\omega^2\|_C \ \text{ for all } \omega^1,\omega^2 \in C([-\beta,\Upsilon],W).$$

Thus, Π has a unique fixed point. $\square$

4. Results on approximate controllability

Definition 4.1. (1.1) is said to be approximately controllable on $[0,\Upsilon]$ if given any $\delta > 0$, for every initial state $\lambda \in C([-\beta,0],W)$ and control $v \in C([-\alpha,0],N)$, and any desired final state $\omega_\Upsilon \in W$, there exists a control $\nu \in L^2([0,\Upsilon],N)$, such that the mild solution $\omega(.)$ of (1.1) satisfies

$$\|\omega(\Upsilon) - \omega_\Upsilon\|_W < \delta.$$

The reachable set, $R_\Upsilon(p)$ is defined as

$$R_\Upsilon(p) := \{\omega(\Upsilon) | \nu \in L^2([0,\Upsilon],N)\}.$$

Thus, (1.1) is approximately controllable iff $\overline{R_\Upsilon(p)} = W$.

The following result is helpful to establish the approximate controllability of (1.1):

Theorem 4.2. *Let ω^1, ω^2 be the mild solution of (1.1) corresponding to the control ν^1 and ν^2 respectively, then under Assumption 1, the following inequality is satisfied:*

$$\|\omega^2 - \omega^1\|_C \leq \mathcal{S}\sqrt{\Upsilon}\Big[\|F_0\nu^2 - F_0\nu^1\|_{L^2} + \big(\|F_1\|_{B_N^W} + P\big)\|\nu^2 - \nu^1\|_{\bar{L}^2}\Big] \exp\big(\mathcal{S}P\Upsilon\big).$$

Proof. For $t \in [-\beta,0]$, $\|\omega^2(t) - \omega^1(t)\|_W = 0$. So taking $t \in [0,\Upsilon]$, we have

$$\begin{aligned}
&\|\omega^2(t) - \omega^1(t)\|_W \\
&\leq \Big\| \int_0^t T_{\mathcal{S}}(t-\tau)\big[F_0\nu^2(\tau) - F_0\nu^1(\tau)\big] d\tau \Big\|_W + \Big\| \int_0^t T_{\mathcal{S}}(t-\tau)\big[F_1\nu^2(\tau-\alpha) \\
&- F_1\nu^1(\tau-\alpha)\big] d\tau \Big\|_W + \Big\| \int_0^t T_{\mathcal{S}}(t-\tau)\big[p(\tau,\omega^2_{x(\tau)},\nu^2(\tau)) - p(\tau,\omega^1_{x(\tau)},\nu^1(\tau))\big] d\tau \Big\|_W \\
&\leq \mathcal{S}\sqrt{\Upsilon}\|F_0\nu^2 - F_0\nu^1\|_{L^2} + \mathcal{S}\sqrt{\Upsilon}\|F_1\|_{B_N^W}\|\nu^2 - \nu^1\|_{\bar{L}^2} + \mathcal{S}P\sqrt{\Upsilon}\|\nu^2 - \nu^1\|_{\bar{L}^2} \\
&+ \mathcal{S}P\int_0^t \|\omega^2_{x(\tau)} - \omega^1_{x(\tau)}\|_{C_\beta} d\tau.
\end{aligned}$$

Thus, for each $t \in [0,\Upsilon]$, we have

$$\begin{aligned}
\|\omega^2(t) - \omega^1(t)\|_W \leq{}& \mathcal{S}\sqrt{\Upsilon}\Big[\|F_0\nu^2 - F_0\nu^1\|_{L^2} + \big(\|F_1\|_{B_N^W} + P\big)\|\nu^2 - \nu^1\|_{\bar{L}^2}\Big] \\
&+ \mathcal{S}P\int_0^t \sup_{\theta\in[0,\tau]} \|\omega^2(\theta) - \omega^1(\theta)\|_W d\tau
\end{aligned}$$

Then, applying Gronwall's inequality, we get

$$\|\omega^2 - \omega^1\|_C \leq \mathcal{S}\sqrt{\Upsilon}\Big[\|F_0\nu^2 - F_0\nu^1\|_{L^2} + \big(\|F_1\|_{B_N^W} + P\big)\|\nu^2 - \nu^1\|_{\bar{L}^2}\Big] \exp\big(\mathcal{S}P\Upsilon\big),$$

which is the desired result. □

Define a map Λ on $Ł^2([0,\Upsilon],W)$ by

$$\Lambda g = \int_0^\Upsilon T_{\mathcal{S}}(\Upsilon-\tau)g(\tau)d\tau, \quad \text{for any } g \in Ł^2([0,\Upsilon],W).$$

Then Λ is a bounded linear transformation from $\L^2([0,\Upsilon],W)$ to W. Also, for any fix $\omega \in C([-\beta,\Upsilon],W)$ and $\nu \in \L^2([0,\Upsilon],N)$, the map $\varphi\omega_\nu$ defined by

$$\varphi\omega_\nu(t) = p(t, \omega_{x(t)}, \nu(t)), \qquad \text{for each } t \in [0,\Upsilon],$$

is an element of $\L^2([0,\Upsilon],W)$.
Next, consider the following assumptions:
<u>**Assumption 2**</u> there exists a constant $\mathcal{K} > 0$ such that

$$\|F_0\nu\|_{L^2} \geq \mathcal{K}\|\nu\|_{\bar{L}^2},$$

for any $\nu \in L^2([0,\Upsilon],N)$.
<u>**Assumption 3**</u> for any $\delta > 0$ and $g \in L^2([0,\Upsilon],W)$, there exists a control $\nu \in L^2([0,\Upsilon],N)$ such that

$$\|\Lambda g - \Lambda(F_0\nu)\|_W < \delta,$$

and,

$$\|F_0\nu\|_{L^2} \leq \mathcal{H}\|g\|_{L^2},$$

where $\mathcal{H}$ is a positive real number which does not depend on the function g. Furthermore,

$$\mathcal{H}\left[\frac{\|F_1\|_{B_N^W} + P}{\mathcal{K}} + P\mathcal{S}\Upsilon\left(1 + \frac{\|F_1\|_{B_N^W} + P}{\mathcal{K}}\right)\exp\big(P\mathcal{S}\Upsilon\big)\right] < 1.$$

This property, in addition to the fact that $D(T)$ is dense in W [24], gives the following result on approximate controllability of (1.1):

Theorem 4.3. *Under Assumption 1, Assumption 2, Assumption 3,* (1.1) *is approximately controllable.*

Proof. The result is proved by showing that $D(T)$ is contained in $\overline{R_\Upsilon(p)}$. Let $c \in D(T)$. Then, the aim is to show that, for any given $\delta > 0$, there exists a mild solution $\omega^\delta(\cdot)$ of our problem (1.1) corresponding to a control $\nu^\delta \in L^2([0,\Upsilon],W)$ such that $\|c - \omega^\delta(\Upsilon)\|_W < \delta$, that is,

$$\|c - T_c(\Upsilon)\lambda(0) - T_{\mathcal{S}}(\Upsilon)\omega_1 - \Lambda F_0\nu^\delta - \Lambda F_1\nu_\alpha^\delta - \Lambda\varphi\omega_\nu^\delta\|_W < \delta,$$

where, for any $\delta > 0$, $\varphi\omega_\nu^\delta \in \L^2([0,\Upsilon],W)$ is the function defined by

$$\varphi\omega_\nu^\delta(t) := p(t, \omega_{x(t)}^\delta, \nu^\delta(t)), \qquad \text{for each } t \in [0,\Upsilon],$$

and, $F_1\nu(\cdot - \alpha) = F_1\nu_\alpha$.

Let ν^1 be an any arbitrarily fixed control function in the space $L^2([0,\Upsilon],N)$, then according to Theorem 3.1, there exists a unique mild solution ω^1 of (1.1) corresponding to ν^1. Let h be the C^1 function from $[0,\Upsilon]$ to W such that

$$\Lambda h = c - T_c(\Upsilon)\lambda(0) - T_{\mathcal{S}}(\Upsilon)\omega_1.$$

Then, for $h - F_1\nu_\alpha^1 - \varphi\omega_\nu^1 \in L^2([0,\Upsilon],W)$, there exists a control $\nu^2 \in L^2([0,\Upsilon],N)$, such that

$$\|\Lambda h - \Lambda F_0\nu_2 - \Lambda F_1\nu_\alpha^1 - \Lambda\varphi\omega_\nu^1\|_W < \frac{\delta}{r^2},$$

where r is any fixed real number such that $r \geq 2$.
Let ω^2 be the unique mild solution of (1.1) corresponding to ν^2, then as

$$F_1(\nu_\alpha^2 - \nu_\alpha^1) + \varphi\omega_\nu^2 - \varphi\omega_\nu^1 \in L^2([0,\Upsilon], W),$$

there exists a control $v^2 \in L^2([0,\Upsilon], N)$, such that

$$\|\Lambda F_1\nu_\alpha^2 - \Lambda F_1\nu_\alpha^1 + \Lambda\varphi\omega_\nu^2 - \Lambda\varphi\omega_\nu^1 - \Lambda F_0 v^2\|_W < \frac{\delta}{r^3},$$

and

$$\|F_0 v^2\|_{L^2} \leq \mathcal{H}\Big[\|F_1\nu_\alpha^2 - F_1\nu_\alpha^1\|_{L^2} + \|\varphi\omega_\nu^2 - \varphi\omega_\nu^1\|_{L^2}\Big].$$

Therefore, we need to obtain the estimates for $\|F_1\nu_\alpha^2 - F_1\nu_\alpha^1\|_{L^2}$ and $\|\varphi\omega_\nu^2 - \varphi\omega_\nu^1\|_{L^2}$.
Using the initial control function v, we get

$$\|F_1\nu_\alpha^2 - F_1\nu_\alpha^1\|_{L^2} \leq \|F_1\|_{B_N^W}\|\nu^2 - \nu^1\|_{\bar{L}^2}.$$

Next, we have

$$\|\varphi\omega_\nu^2 - \varphi\omega_\nu^1\|_{L^2} \leq P\sqrt{\Upsilon}\|\omega^2 - \omega^1\|_C + P\|\nu^2 - \nu^1\|_{\bar{L}^2}.$$

Using these estimates and Theorem 4.2, we have

$$\begin{aligned}
\|F_0 v^2\|_{L^2} &\leq \mathcal{H}\Big[\Big(\|F_1\|_{B_N^W} + P\Big)\|\nu^2 - \nu^1\|_{\bar{L}^2} + P\sqrt{\Upsilon}\|\omega^2 - \omega^1\|_C\Big] \\
&\leq \mathcal{H}\Big[\Big(\|F_1\|_{B_N^W} + P\Big)\|\nu^2 - \nu^1\|_{\bar{L}^2} + P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\|F_0\nu^2 - F_0\nu^1\|_{L^2} \\
&+ P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\Big(\|F_1\|_{B_N^W} + P\Big)\|\nu^2 - \nu^1\|_{\bar{L}^2}\Big] \\
&\leq \mathcal{H}\Big[\Big(\|F_1\|_{B_N^W} + P\Big)\Big(1 + P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\Big)\|\nu^2 - \nu^1\|_{\bar{L}^2} \\
&+ P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\|F_0\nu^2 - F_0\nu^1\|_{L^2}\Big] \\
&\leq \mathcal{H}\left[\frac{\|F_1\|_{B_N^W} + P}{\mathcal{K}}\Big(1 + P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\Big) + P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\right] \\
&\times \|F_0\nu^2 - F_0\nu^1\|_{L^2} \\
&\leq \mathcal{H}\left[\frac{\|F_1\|_{B_N^W} + P}{\mathcal{K}} + P\mathcal{S}\Upsilon\exp\big(P\mathcal{S}\Upsilon\big)\left(\frac{\|F_1\|_{B_N^W} + P}{\mathcal{K}} + 1\right)\right] \\
&\times \|F_0\nu^2 - F_0\nu^1\|_{L^2}.
\end{aligned}$$

Take $\nu^3 := \nu^2 - v^2$, then $\nu^3 \in L^2([0,\Upsilon], N)$, and

$$\|\Lambda h - \Lambda F_0\nu^3 - \Lambda F_1\nu_\alpha^2 - \Lambda\varphi\omega_\nu^2\|_W < \frac{\delta}{r^2} + \frac{\delta}{r^3}.$$

Proceeding in this way, we get a sequence $\nu^m \in L^2([0,\Upsilon], N)$, such that

$$\|\Lambda h - \Lambda F_0\nu^{m+1} - \Lambda F_1\nu_\alpha^m - \Lambda\varphi\omega_\nu^m\|_W < \sum_{i=2}^{m+1}\frac{\delta}{r^i},$$

with

$$\begin{aligned}&\|F_0\nu^{m+1}-F_0\nu^{m}\|_{L^2}\\ &\leq \mathcal{H}\left[\frac{\|F_1\|_{B_N^W}+P}{\mathcal{K}}+P\mathcal{S}\Upsilon\exp\left(P\mathcal{S}\Upsilon\right)\left(\frac{\|F_1\|_{B_N^W}+P}{\mathcal{K}}+1\right)\right]\|F_0\nu^{m}-F_0\nu^{m-1}\|_{L^2}.\end{aligned}$$

Therefore, we get a Cauchy sequence $\{F_0\nu^m\}_{m\in\mathbb{N}}$ in $L^2([0,\Upsilon],W)$, and hence there exists a $\mathcal{B}$ in $L^2([0,\Upsilon],W)$ such that

$$F_0\nu^m \longrightarrow \mathcal{B} \quad \text{in } L^2([0,\Upsilon],W).$$

Also, it can be shown that there exists an $m_0\in\mathbb{N}$, such that

$$\|\Lambda F_0\nu^m-\Lambda\mathcal{B}\|_W<\frac{\delta}{2r^{m_0+2}} \quad \text{for all} \quad m\geq m_0.$$

Therefore,

$$\begin{aligned}&\|\Lambda h-\Lambda F_0\nu^{m_0}-\Lambda F_1\nu_\alpha^{m_0}-\Lambda\varphi\omega_\nu^{m_0}\|_W\\ \leq&\|\Lambda h-\Lambda F_0\nu^{m_0+1}-\Lambda F_1\nu_\alpha^{m_0}-\Lambda\varphi\omega_\nu^{m_0}\|_W+\|\Lambda F_0\nu^{m_0+1}-\Lambda F_0\nu^{m_0}\|_W\\ <&\delta,\end{aligned}$$

that is, for any given $\delta>0$ and $c\in D(T)$, there exists a control $\nu_{m_0}\in L^2([0,\Upsilon],N)$, such that

$$\|c-T_c(\Upsilon)\lambda(0)-T_s(\Upsilon)\omega_1-\Lambda F_0\nu^{m_0}-\Lambda F_1\nu_\alpha^{m_0}-\Lambda\varphi\omega_\nu^{m_0}\|_W<\delta.$$

Thus, $\overline{R_\Upsilon(p)}\supseteq W$, that is, (1.1) is approximately controllable on $[0,\Upsilon]$. □

Due to the presence of delay in input, we get the following result as a corollary of the above result, Theorem 4.3:

Corollary 4.4. *Assume Assumption 2 and Assumption 3 hold, then the linear system with delay corresponding to* (1.1)*, that is,*

$$\left.\begin{aligned}z''(t)&=Tz(t)+F_0\nu(t)+F_1\nu(t-\alpha),\ t\in(0,\Upsilon],\\ z'(0)&=\omega_1,\\ z(t)&=\lambda(t),\ t\in[-\beta,0],\\ \nu(t)&=v(t),\ t\in[-\alpha,0],\end{aligned}\right\}$$

is approximately controllable on $[0,\Upsilon]$*.*

5. Conclusion and future directions

In this article, the existence and approximate controllability results of a second order ordinary differential equation with state and control delays are studied, via cosine family of operators. First, by applying mathematical induction and properties of the cosine family, a fixed point theorem gives the existence and uniqueness of the mild solution of the considered differential equation. Next, for showing the approximate controllability of the system, the denseness of the domain of the cosine family's generator has been used tactically to our advantage. Under the assumptions, an inequality involving the difference of the mild solutions is established, using which a Cauchy sequence is constructed in a complete metric space, which then employs the definition of reachable sets and the operator Λ, to establish the approximate controllability of the nonlinear system.

As a future direction, it is possible to extend this work to investigate the approximate controllability of instantaneous or non-instantaneous impulsive differential equations with delay in state and control.

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