Gulf Journal of Mathematics
Vol 21, Issue 1 (2025) 470-485

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i1.3403

ANALYTICAL SOLUTION OF TWO-DIMENSIONAL FRACTIONAL ADVECTION-DIFFUSION EQUATION WITH INSTANTANEOUS SOURCE AND TIME-VARYING COEFFICIENTS

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ABSTRACT. This article presents an analytical study of the two-dimensional Advection-Diffusion Equation (ADE) incorporating the Caputo-Fabrizio (CF) time-fractional derivative. This work introduces a formulation with time-varying diffusion and advection, featuring an instantaneous point injection at the boundary. Analytical solutions in a half-plane are derived using the Laplace transform with respect to time and the Fourier transform with respect to spatial coordinates. The resulting solution, expressed in terms of special functions and integral representations, encompasses both the Fractional Advection-Diffusion Equation (FADE) and the classical ADE. The analysis shows that, for each fractional order, the concentration profile follows a bel-shaped curve where it begins at a low value, rises to a peak, and then decays with increasing distance from the source. Moreover, temporal variations in diffusion and velocity coefficients significantly affect transport dynamics, underscoring the importance of accounting for time-dependent factors to accurately model real-world fractional transport processes.

Keywords. Advection-Diffusion Equation, Caputo-Fabrizio Fractional Derivative, Temporally Dependent Coefficients, Laplace Transform, Fourier Transform
2020 Mathematics Subject Classification. Primary 35R11; Secondary 26A33

1. INTRODUCTION

The Advection-Diffusion Equation (ADE) serves as a fundamental model to describe the transport process in diverse fields such as environmental engineering, hydrology, and biomedical engineering. To help understand more complicated movement in mixed and porous materials, researchers developed a fractional version of ADE called Fractional Advection-Diffusion Equation (FADE) [27]. The advantage of FADE lies in its use of fractional derivatives, enabling it to model more complex diffusion behaviours, such as different spreading speeds and long-range interactions, which classical ADE cannot effectively capture [25].

Date: Received: Aug 21, 2025; Accepted: Sep 20, 2025.
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This field has seen growing interest in the application of non-singular fractional derivatives to model anomalous diffusion. [1], [4], [15] indicated that the fractional operators with non-singular kernels such as Caputo-Fabrizio (CF) and Atangana-Baleanu effectively capture long-memory and non-local transport dynamics. This, ultimately making them more physically consistent and better suited for complex media.

In addition, several studies underscore the superiority of fractional models over classical ones. For example, the model presented by [21] demonstrated stable transport using finite-memory kernels designed for environmental or engineering applications. In addition, [23] developed the FADE model to analyze pollutant levels and dissolved oxygen under unstable aeration conditions, indicating how memory affects pollutant decay and oxygen recovery in rivers. Similarly, [11] and [18] revealed that FADEs excel in simulating breakthrough curves and contaminant transport in fractured media. Furthermore, [14] highlighted the models' adaptability for media with high spatial and temporal variability.

Several studies have targeted analytical solutions for FADE, particularly in two-dimensional spaces. Researchers such as [2] and [12] employed mathematical transforms to derive solutions; however they primarily assumed a constant coefficient. This limitation restricts applicability to real-world situations where transport parameters vary over time. Meanwhile, [8] excluded advection effects and temporal changes from their analysis.

Instantaneous boundary conditions, which model sudden or pulse-like sources, are essential in simulating real-world events such as contaminant spills or injections [3], [19]. Despite their significance, analytical solutions of two-dimensional FADEs with temporally varying coefficients under instantaneous boundary conditions remain scarce. Correspondingly,this gap limits a comprehensive understanding of transient transport processes in complex media, particularly where time-dependent transport parameters critically affect solute dispersion and advection.

Therefore, this study addresses the research gap by formulating a two-dimensional FADE with a CF time-fractional derivative, time-dependent velocity and diffusion, and an instantaneous source. The equation is mapped to a constant-coefficient model, permitting the straightforward application of integral transforms. Next, the inverse integral transforms yield a compact Bessel representation for the concentration. This framework remains analytically tractable and recovers the classical ADE in the appropriate parameter limits.

2. Preliminaries

The following propositions present some standard properties of limits, which are used throughout this work to handle the behavior of functions involving exponential terms and products.

Proposition 2.1 (Limit of product). *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ *exist and are finite, then* [16]

$$\lim_{x\to a}\big(f(x)g(x)\big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big) = LM. \tag{2.1}$$

Proposition 2.2 (Limit of the exponential function)**.** *Let $f(x)$ be a function such that $\lim_{x\to a} f(x)$ exists and is finite. Then, the limit of the exponential function of $f(x)$ is given by* [16]

$$\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}. \tag{2.2}$$

Recall some basic definitions and the lemma of the time fractional derivative, Caputo-Fabrizio (CF).

Definition 2.3. The CF time fractional derivative of order $\alpha \in [0, 1)$ is defined as [6]

$${}^{CF}D_t^\alpha f(t) = \frac{1}{1-\alpha}\int_0^t \exp\left(\frac{-\alpha(t-\tau)}{1-\alpha}\right) f'(\tau)d\tau \tag{2.3}$$

The Laplace transform of CF time derivative (2.3) is

$$L\left\{{}^{CF}D_t^\alpha f(t)\right\} = \frac{sL[f(t)] - f(0)}{(1-\alpha)s+\alpha} \tag{2.4}$$

Remark 2.4. CF time-fractional derivatives can be extended to the case $\alpha = 1$. Using (2.4) and taking the limit as $\alpha \to 1$ yields

$$\lim_{\alpha\to 1} L\left\{{}^{CF}D_t^\alpha f(t)\right\} = sLf(t) - f(0) = L\left\{f'(t)\right\} \tag{2.5}$$

The CF derivative employs a smooth exponential kernel and works with standard initial conditions. In contrast, the Riemann–Liouville (RL) derivative employs a power-law kernel. It often requires initial data in the form of fractional integrals or derivatives, which complicates the treatment of sudden changes [10]. Meanwhile, the Laplace transform of the CF operator offers a rational function in s; hence, the calculations and inversions are simpler. On the other hand, the Caputo operator results in terms involving s^α [6]. Another distinguishing feature is that the CF kernel stays bounded at τ, avoiding singularities at $t = 0$. This setup valid in a distributional sense, ensures that the convolution with a Dirac delta function remains finite [5]. Moreover, for two-dimensional advection–diffusion problems, fundamental solutions with CF time derivative typically involve elementary or Bessel functions. Conversely, Caputo or RL formulations often involve Mittag–Leffler functions, which can complicate analysis and computation [20]. Therefore, selecting the CF derivative can improve efficiency in both analysis and solution procedures for these problems.

3. Mathematical Formulation

The ADE describes the transport of a substance, such as pollutants or heat, in a moving fluid due to both advection and diffusion. In particular, this study considers a two-dimensional ADE characterized by the CF time-fractional derivative and temporally dependent velocity and diffusion.

Generally, ADE is presented by

$$\frac{\partial C}{\partial t} + \nabla\cdot(uC) = D\nabla^2 C \tag{3.1}$$

where C denotes the solute concentration $\left[\mathrm{mol}/\mathrm{km}^3\right]$, u indicates the crossflow velocity [km/ year], and D represents the diffusion coefficient [km^2 /year]. In this research, the ADE is considered in the (x, y) plane of an incompressible fluid with a temporal variable t [year]. Therefore, the ADE (3.1) becomes:

$$\frac{\partial C(x,y,t)}{\partial t}=D\left[\frac{\partial^2 C(x,y,t)}{\partial x^2}+\frac{\partial^2 C(x,y,t)}{\partial y^2}\right]-u\left[\frac{\partial C(x,y,t)}{\partial x}+\frac{\partial C(x,y,t)}{\partial y}\right] \tag{3.2}$$

In this study, the initial solute concentration is zero throughout, indicating that no contaminant is present before the process begins. Accordingly, an instantaneous source is introduced at $y=0$, represented by a Dirac delta function to model sudden injection. Moreover, the concentration approaches zero at large distances from the source, ensuring that the solution remains physically meaningful and bounded. Mathematically, these conditions are presented as:

$$C(x,y,0)=0 \tag{3.3}$$

$$C(x,0,t)=C_0\delta(x)\delta(t) \tag{3.4}$$

$$\lim_{x\to\pm\infty} C(x,y,t)=0, \quad \lim_{y\to\infty} C(x,y,t)=0 \tag{3.5}$$

where C_0 is a constant concentration and $\delta(\cdot)$ is the Dirac delta function.

It is assumed that both the solute diffusion and the velocity coefficient are temporally dependent [7]; therefore, consider:

$$D(x,y,t)=D_0 f_1(mt) \tag{3.6}$$

$$u(x,y,t)=u_0 f_2(mt) \tag{3.7}$$

where $D_0\left[\ \mathrm{km}^2/\mathrm{year}\right]$ and $u_0[\ \mathrm{km}/\mathrm{year}]$ are the baseline diffusion and velocity values, respectively, and $f_1(mt), f_2(mt)$ are functions represent temporal variation. Meanwhile, the parameter m has units of [year^{-1}], representing the rate of temporal change. Substituting (3.6)-(3.7) into (3.2) leads to the following equation

$$\frac{\partial C}{\partial t}=D_0 f_1(mt)\left[\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right]-u_0 f_2(mt)\left[\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}\right] \tag{3.8}$$

The presence of time-dependent coefficients implies that physical parameters such as advection and velocity may vary over time. Consequently, the original spatial coordinates may not effectively capture the dynamics of the system. Notably, new space variables allow for a more accurate description of spreading processes, providing better alignment with the underlying physical phenomena. Introducing new space variables, X and Y by transformation

$$X = \int \frac{f_2(mt)}{f_1(mt)} dx \tag{3.9}$$

$$Y = \int \frac{f_2(mt)}{f_1(mt)} dy \tag{3.10}$$

The ADE (3.8) reduces to:

$$\frac{f_2^2(mt)}{f_1(mt)} \frac{\partial C}{\partial t} = D_0 \left[\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} \right] - u_0 \left[\frac{\partial C}{\partial X} + \frac{\partial C}{\partial Y} \right] \tag{3.11}$$

Although the ADE has been spatially transformed, it still explicitly depends on $f_1(mt)$ and $f_2(mt)$ over time. Correspondingly, a new variable T is introduced by the transformation

$$T = \int_0^t \frac{f_2^2(mt)}{f_1(mt)} \mathrm{d}t \tag{3.12}$$

Therefore, the model now becomes

$$\frac{\partial C}{\partial T} = D_0 \left[\frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} \right] - u_0 \left[\frac{\partial C}{\partial X} + \frac{\partial C}{\partial Y} \right] \tag{3.13}$$

$$C(X, Y, 0) = 0 \tag{3.14}$$

$$C(X, 0, T) = \frac{C_0}{a} \delta(X) \delta(T) \tag{3.15}$$

$$\lim_{X \to \pm\infty} C(X, Y, T) = 0, \quad \lim_{Y \to \infty} C(X, Y, T) = 0 \tag{3.16}$$

where $a = f_2(0)$.
Introducing a new unknown function, $v(X, Y, T)$ such that

$$C(X, Y, T) = \exp \left[\frac{u_0}{2D_0} (X + Y) \right] v(X, Y, T) \tag{3.17}$$

The ADE (3.13) with conditions (3.14)-(3.16) reduces to

$$\frac{\partial v}{\partial T} = D_0 \left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2} \right) - \frac{u_0^2}{2D_0} v \tag{3.18}$$

$$v(X, Y, 0) = 0 \tag{3.19}$$

$$v(X, 0, T) = \frac{C_0}{a} \delta(X) \delta(T) \tag{3.20}$$

$$\lim_{X \to \pm\infty} v(X, Y, T) = 0, \quad \lim_{Y \to \infty} v(X, Y, T) = 0 \tag{3.21}$$

The time derivative in (3.18) is replaced with the CF fractional derivative of order $\alpha \in [0, 1)$. The resulting FADE becomes:

$$ {}^{CF}D_t^{\alpha} v(X,Y,T) = D_0 \frac{\partial^2 v(X,Y,T)}{\partial X^2} + D_0 \frac{\partial^2 v(X,Y,T)}{\partial Y^2} - \frac{{u_0}^2}{2D_0} v(X,Y,T) \quad (3.22) $$

To solve the FADE given in (3.22), the Laplace transform with respect to T, defined by $\bar{v}(X,Y,s) = \int_0^{\infty} v(X,Y,T) \mathrm{e}^{-sT} \,\mathrm{d}s$, is applied first. Then it is followed by the Sine-Fourier transform with respect to Y, given by $\tilde{\bar{v}}(X,\eta,s) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \bar{v}(X,Y,s) \sin(\eta Y) \mathrm{d}Y$, and finally the exponential Fourier transform with respect to X, defined as $\tilde{\bar{v}}^*(\xi,\eta,s) = \int_{-\infty}^{\infty} \tilde{\bar{v}}(X,\eta,s) \mathrm{e}^{-\mathrm{i}\xi X} \,\mathrm{d}X$. Applying these transforms to (3.22) yields an algebraic expression in the transform domain

$$ \tilde{\bar{v}}^*(\xi,\eta,s) = \frac{D_0 \left(\sqrt{\frac{2}{\pi}} \eta \frac{C_0}{a} \right)}{\left[\frac{s\gamma}{s+\alpha\gamma} + D_0 \left(\eta^2\right) + D_0 \left(\xi^2\right) + \frac{u_0^2}{2D_0} \right]} \quad (3.23) $$

where $\gamma = \frac{1}{1-\alpha}$ and $a = f_2(0)$.
By algebraically rearrange (3.23), it can be shown that

$$ \tilde{\bar{v}}^*(\xi,\eta,s) = \tilde{\bar{v}}^*{}_1(\xi,\eta,s) + \tilde{\bar{v}}^*{}_2(\xi,\eta,s) \quad (3.24) $$

where

$$ \tilde{\bar{v}}_1^*(\xi,\eta,s) = \sqrt{\frac{2}{\pi}} \frac{D_0 C_0}{a} \left[\frac{\eta}{\left(D_0\eta^2 + D_0\xi^2 + \frac{u_0^2}{2D_0} + \gamma \right)} \right] \quad (3.25) $$

and

$$ \tilde{\bar{v}}_2^*(\xi,\eta,s) = \sqrt{\frac{2}{\pi}} \frac{D_0 C_0}{a} \left[\frac{\alpha\gamma^2}{\left(D_0\eta^2 + D_0\xi^2 + \frac{u_0^2}{2D_0} + \gamma \right)^2} \cdot \frac{1}{\left[s + \frac{\alpha\gamma \left(D_0\eta^2 + D_0\xi^2 + \frac{u_0^2}{2D_0} \right)}{\left(D_0\eta^2 + D_0\xi^2 + \frac{u_0^2}{2D_0} + \gamma \right)} \right]} \right] \quad (3.26) $$

The above step turns the Laplace part to become a constant term and a single simple pole, allowing simple inversion. The inverse Laplace and Fourier transforms of $\tilde{\bar{v}}_1^*(\xi,\eta,s)$ yield

$$ v_1(X,Y,T) = \frac{2}{\pi} D_0 C_0 \delta(T) \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\eta \sin(\eta y)}{\left(D_0\xi^2 + D_0\eta^2 + \frac{u_0^2}{2D_0} + \gamma \right)} e^{i\xi X} dn d\xi \quad (3.27) $$

Write $e^{i\xi X} = \cos(\xi X) + i\sin(\xi X)$ and set

$$F(\xi,\eta) = \frac{\eta\,\sin(\eta Y)}{D_0(\xi^2+\eta^2) + \frac{u_0^2}{2D_0} + \gamma}. \tag{3.28}$$

Then $F(-\xi,\eta) = F(\xi,\eta)$ is even in ξ, so

$$\int_{-\infty}^{\infty} F(\xi,\eta)\sin(\xi X)\,d\xi = 0 \tag{3.29}$$

and hence

$$\begin{aligned}\int_{-\infty}^{\infty} F(\xi,\eta)e^{i\xi X}\,d\xi &= \int_{-\infty}^{\infty} F(\xi,\eta)\cos(\xi X)\,d\xi \\ &= 2\int_{0}^{\infty} F(\xi,\eta)\cos(\xi X)\,d\xi.\end{aligned} \tag{3.30}$$

Therefore, the integrand is non-negative up to the cosine factor and is absolutely integrable under the parameters. So (3.27) reduces to

$$v_1(X,Y,T) = \frac{4}{\pi}\,D_0C_0\,\delta(T)\int_0^{\infty}\int_0^{\infty}\frac{\eta\,\sin(\eta Y)\,\cos(\xi X)}{D_0(\xi^2+\eta^2)+\frac{u_0^2}{2D_0}+\gamma}\,d\eta\,d\xi. \tag{3.31}$$

At this stage, although the Laplace and Fourier transforms have been inverted, the solution remains in the form of a double integral over the transformed variables ξ and η. To further simplify the expression and obtain a more tractable representation in physical space, a change of variables from Cartesian coordinates to polar coordinates are applied as

$$\begin{cases}\xi = \rho\cos\theta \\ \eta = \rho\sin\theta\end{cases}$$

where $\rho \in [0,\infty), \theta \in \left[0,\frac{\pi}{2}\right]$. So, (3.31) converts to

$$\begin{aligned}v_1(X,Y,T) = \frac{4}{\pi}\,D_0C_0\delta(T)&\int_0^{\pi/2}\int_0^{\infty}\frac{\rho^2}{D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma} \\ &\times\sin\theta\,\sin(\rho\sin\theta\,Y)\,\cos(\rho\cos\theta\,X)\,d\rho\,d\theta\end{aligned} \tag{3.32}$$

Let $z = \cos\theta$, $\sin\theta = \sqrt{1-z^2}$. The limit of integration z when $\theta = \frac{\pi}{2}$ is 0 and when $\theta = 0$ is 1. **So the integration with respect to θ**

$$\begin{aligned}I_1 &= \int_0^{\pi/2}\sin\theta\sin(Y\rho\sin\theta)\cos(X\rho\cos\theta)d\theta \\ &= \int_0^{1}\sin(Y\rho\sqrt{1-z^2})\cos(X\rho z)dz\end{aligned} \tag{3.33}$$

Based on [17], it is given that

$$\int_0^1 \cos(mz)\sin\left(n\sqrt{1-z^2}\right)dz = \frac{\pi}{2}\left(\frac{n}{\sqrt{m^2+n^2}}\right) J_1\left(\sqrt{m^2+n^2}\right) \tag{3.34}$$

So, assuming $m = X\rho$ and $n = Y\rho$, (3.32) reduces to

$$\begin{aligned} v_1(X,Y,T) = \frac{2YD_0C_0\delta(T)}{a\sqrt{X^2+Y^2}} \int_0^\infty \frac{\rho^2}{\left(D_0\rho^2 + \frac{{u_0}^2}{2D_0} + \gamma\right)} \\ \times J_1\left(\rho\sqrt{X^2+Y^2}\right) \mathrm{d}\rho \end{aligned} \tag{3.35}$$

Using the same technique, $v_2(X,Y,T)$ converts to

$$\begin{aligned} v_2(X,Y,T) = \frac{2\alpha\gamma^2 YC_0D_0}{a\sqrt{X^2+Y^2}} e^{-\alpha\gamma T} \int_0^\infty \frac{\rho^2}{\left(D_0\rho^2 + \frac{u_0^2}{2D_0} + \gamma\right)^2} \\ \times \exp\left(\frac{\alpha\gamma^2 T}{D_0\rho^2 + \frac{u_0^2}{2D_0} + \gamma}\right) J_1\left(\rho\sqrt{X^2+Y^2}\right) \mathrm{d}\rho \end{aligned} \tag{3.36}$$

Since $v_1(X,Y,T)$ contains the delta function, $\delta(T)$ which suggests that it is only nonzero at $T = 0$

$$\delta(T) = \begin{cases} \infty & T = 0 \\ 0 & T \neq 0 \end{cases}$$

Therefore, for $T > 0$ $v_1, (X,Y,T) = 0$, So

$$\begin{aligned} v(X,Y,T) &= v_2(X,Y,T) \\ &= \frac{2\alpha\gamma^2 YC_0D_0}{a\sqrt{X^2+Y^2}} e^{-\alpha\gamma T} \int_0^\infty \frac{\rho^2}{\left(D_0\rho^2 + \frac{u_0^2}{2D_0} + \gamma\right)^2} \\ &\quad \times \exp\left(\frac{\alpha\gamma^2 T}{D_0\rho^2 + \frac{u_0^2}{2D_0} + \gamma}\right) J_1\left(\rho\sqrt{X^2+Y^2}\right) \mathrm{d}\rho \end{aligned} \tag{3.37}$$

Using the standard Bessel recurrence $J_1(z) = \frac{z}{2}\left[J_0(z) + J_2(z)\right]$, setting $z = \rho\sqrt{X^2+Y^2}$, (3.37) converts to

$$
\begin{aligned}
v(X,Y,T) =&\alpha\gamma^2 Y\frac{C_0}{a}D_0\mathrm{e}^{-\alpha\gamma T}\int_0^\infty \frac{\rho^3}{\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)^2} \\
&\times \exp\left(\frac{\alpha\gamma^2 T}{\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)}\right)\left[J_0\left(\rho\sqrt{X^2+Y^2}\right)\right. \\
&\left.+J_2\left(\rho\sqrt{X^2+Y^2}\right)\right]\mathrm{d}\rho
\end{aligned}
\tag{3.38}
$$

The final solution of FADE is by substituting (3.38) into (3.17)

$$
\begin{aligned}
C(X,Y,T) =&\frac{\alpha\gamma^2 Y C_0 D_0}{a}\exp\left[\frac{u_0}{2D_0}(X+Y)-\alpha\gamma T\right] \\
&\times\int_0^\infty \frac{\rho^3}{\left(D_0\rho^2+\frac{u_0}{2D_0}+\gamma\right)^2}e^{\frac{\alpha\gamma T}{\left(D_0\rho^2+\frac{u_0}{2D_0}+\gamma\right)}} \\
&\times\left[J_0\left(\rho\sqrt{X^2+Y^2}\right)+J_2\left(\rho\sqrt{X^2+Y^2}\right)\right]d\rho
\end{aligned}
\tag{3.39}
$$

To understand the classical case, consider the limit of the CF derivative as $\alpha\to 1$. Thus, using (2.4) and taking the limit $\alpha\to 1$ in (3.39)

$$
\begin{aligned}
&\lim_{\alpha\to 1}\alpha\gamma^2\exp\left(-\alpha\gamma T+\frac{\alpha\gamma^2 T}{\left(D_0\rho^2+\frac{{u_0}^2}{2D_0}+\gamma\right)}\right)\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)^{-2} \\
&=\gamma^2\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)^{-2}\exp\left(\frac{-\gamma T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)}{D_0\rho^2+\frac{{u_0}^2}{2D_0}+\gamma}\right)
\end{aligned}
\tag{3.40}
$$

Since $\gamma=\frac{1}{1-\alpha}$, taking the limit $\alpha\to 1$ is equivalent to taking the limit $\gamma\to\infty$ and using (2.1)

$$
\begin{aligned}
&\lim_{\gamma\to\infty}\gamma^2\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)^{-2}\exp\left(\frac{-\gamma T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)}{D_0\rho^2+\frac{{u_0}^2}{2D_0}+\gamma}\right) \\
&=\lim_{\gamma\to\infty}\gamma^2\left(D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma\right)^{-2}\cdot\lim_{\gamma\to\infty}\exp\left(\frac{-\gamma T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)}{D_0\rho^2+\frac{{u_0}^2}{2D_0}+\gamma}\right)
\end{aligned}
\tag{3.41}
$$

(3.41) is of the indeterminate form (∞/∞) and $e^{-\infty/\infty}$. First, the exponential factor is evaluated by applying L'Hôpital's rule and using (2.2)

$$
\begin{aligned}
\lim_{\gamma\to\infty} \exp\left(\frac{-\gamma T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)}{D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma}\right) &= \exp\left(\lim_{\gamma\to\infty}\frac{-\gamma T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)}{D_0\rho^2+\frac{u_0^2}{2D_0}+\gamma}\right) \\
&= \left(-T\left(D_0\rho^2+\frac{u_0^2}{2D_0}\right)\right)
\end{aligned} \tag{3.42}
$$

Solve the rational term using L'Hôpital

$$
\lim_{\gamma\to\infty}\frac{\gamma^2}{\left(D_0p^2+\frac{u_0^2}{2D_0}+\gamma\right)^2} = \lim_{\gamma\to\infty}\frac{2\gamma}{2\left(D_0p^2+\frac{u_0^2}{2D_0}+\gamma\right)} = 1 \tag{3.43}
$$

Substituting (3.42)-(3.43) into (3.39) and using the relation of Bessel $J_0(z)+J_2(z)=\frac{2}{z}J_1(z)$, the fundamental solution for classical ADE ($\alpha=1$)

$$
\begin{aligned}
C(X,Y,T) =& \frac{YC_0D_0}{a}\exp\left[\frac{u_0}{2D_0}(X+Y)-\frac{u_0^2}{2D_0}T\right] \\
&\times\int_0^\infty \rho^3 e^{-D_0\rho^2 T}\left[\frac{2J_1\left(\rho\sqrt{x^2+Y^2}\right)}{\rho\sqrt{x^2+Y^2}}\right]d\rho
\end{aligned} \tag{3.44}
$$

Using the formula $\int_0^\infty x^{v+1}e^{-ax^2}J_v(bx)dx = \frac{b^v}{(2a)^{v+1}}e^{-\frac{b^2}{4a}}, a>0, b>0, \mathrm{Re}(v)>-1$ [17], the classical ADE solution can be written as

$$
\begin{aligned}
C(X,Y,T) =& \frac{2YC_0D_0\exp\left[\frac{u_0}{2D_0}(X+Y)-\frac{u_0^2}{2D_0}T\right]}{a\sqrt{X^2+Y^2}} \\
&\times\left[\frac{\sqrt{X^2+Y^2}}{(2D_0T)^2}\exp\left(-\frac{\left(\sqrt{X^2+Y^2}\right)^2}{4D_0T}\right)\right]
\end{aligned} \tag{3.45}
$$

By simplying (3.45), the final classical ADE solution is

$$
C(X,Y,T) = \frac{YC_0}{2D_0T^2a}\exp\left[\frac{u_0}{2D_0}(X+Y)-\frac{u_0^2}{2D_0}T-\frac{X^2+Y^2}{4D_0T}\right] \tag{3.46}
$$

where $a=f_2(0)$.

To validate the analytical method and highlight its generality, the fundamental solution (3.39) derived in this work was compared with the findings of [13]. Their study investigated the two-dimensional FADE with Caputo-Fabrizio fractional derivatives under the assumption of constant diffusion and advection coefficients. By setting $D(x,y,t)=D_0$ and $u(x,y,t)=u_0$ in our model, the derived solution simplifies and exactly recovers their results.

This agreement confirms the correctness of the general formulation and demonstrates that the proposed method naturally extends existing fractional ADE models to accommodate temporally varying transport parameters. It also reinforces the analytical consistency of the Caputo-Fabrizio framework in modeling non-classical diffusion in both static and dynamic environments.

4. Results

The analytical solution is plotted using the input parameters $C_0 = 1.0$ mol/km^3, $D_0 = 1.71$ km^2/ year,$u_0 = 1.60$ km/ year, at times $t = 0.75, 2.5, 5.0$ years. These parameter values are obtained from [9].

Figure 1 illustrates how fractional order α influences the spread of concentration over time. The temporal variations used are $f_1(mt) = e^{mt}, f_2(mt) = e^{-mt}$ with $m = 1.0$ year^{-1}. In general,the concentration profile forms a bell-shaped curve. It begins small, rise to a peak, and then decaying as it moves farther from the source. Also, the concentration decreases with time for all α. At $t = 0.75$ years, as α increases, the peak concentration increases with the peak concentration occuring when $0.5 < x < 1$,. By $t = 2$ years, as α increases, concentration continues to increase but only until $\alpha = 0.7$. The profile $\alpha = 0.8$ flattens with a lower peak concentration than $\alpha = 0.7$, but spreads wider, reflecting an increase in the rate of diffusion as α approaches 1. The classical case shows a lower peak concentration compared to most fractional orders, indicating faster solute dispersion. At $t = 5$ year, the behaviour remains similar, while concentration increases with increasing α but only up to $\alpha = 0.5$ in the region $x < 3$. Starting from $\alpha = 0.7$, as α increases, the concentration decreases with the classical order showing the lowest concentration for $x < 3$.

Table 1 summarizes different combinations of temporally dependent and uniform solute diffusion and flow velocity functions, denoted $f_1(mt)$ and $f_2(mt)$ respectively. Each case represents a distinct scenario of how the diffusion coefficient $D = D_0 f_1(mt)$ and velocity $u = u_0 f_2(mt)$ vary over time.

Table 1. Various combinations of temporal variation in diffusion and velocity.

Case	$f_1(mt)$	$f_2(mt)$	Description of diffusion, $D = D_0 f_1(mt)$ and velocity, $u = u_0 f_2(mt)$
1	$\exp(mt)$	$\exp(-mt)$	Diffusion grows exponentially as velocity decays exponentially.
2	$\exp(-mt)$	$\exp(mt)$	Diffusion decays exponentially as velocity grows exponentially.
3	1	1	Uniform diffusion in uniform flow.

Figure 2 demonstrates how different temporal variations in the diffusion coefficient D and the velocity u influence the spread of the substance at two fractional orders, representing lower α ($\alpha = 0.3$) and higher α ($\alpha = 0.7$). For both fractional orders α in all three cases of temporal variation, the magnitude of the concentration decreases with increasing time. For lower α, at $t = 0.75$ years, Case 2 shows the highest concentration due to slower diffusion and faster velocity. However, as time increases, Case 1 becomes the dominant case, showing higher concentrations because it spreads more slowly. By $t = 5$ years, Case 2 shows a marked decrease in concentration, approaching zero, because the concentration profile becomes very spread out due to high velocity while simultaneously being diluted due to low diffusion. For higher α, Case 1 always shows the highest concentration peak. This is due to faster diffusion and slower velocity, which causes the concentration to remain more concentrated near the source. The profiles in all cases are relatively similar at $t = 0.75$ years, with Case 2 and Case 3 showing slightly lower peak concentrations. As time increases, Case 2 shows a rapid decline in concentration over time due to its faster dispersion, especially evident at $t = 5$ years. As α increases, the concentration spreads faster and becomes more diluted over time, particularly in Case 2, where increasing velocity and decreasing diffusion cause rapid dispersion. In contrast, at lower α values, concentration remains more localized and decays more slowly, especially in Case 1, where slower diffusion and velocity changes prevent rapid spreading.

5. Conclusion

This study presents an analytical solution to the two-dimensional advection-diffusion equation with time-dependent dispersion and velocity, incorporating the Caputo-Fabrizio time-fractional derivative. The solution is derived using Laplace and Fourier transforms and expressed in a form suitable for evaluating concentration profiles under instantaneous boundary conditions. In general, concentration profile forms a bell-shaped curve and decreases with time for all fractional orders α. Temporal variations in diffusion and velocity further affect concentration behavior. Case 1 sustains higher concentrated peaks over time, while Case 2 accelerates solute dispersion, leading to rapid concentration decline. These findings emphasize the importance of incorporating fractional calculus and time-dependent parameters to more accurately model complex transport processes in real-world systems.

6. Acknowledgements

This work is part of a research project with the Q.J130000.2754.03K99 cost center supported by Universiti Teknologi Malaysia.

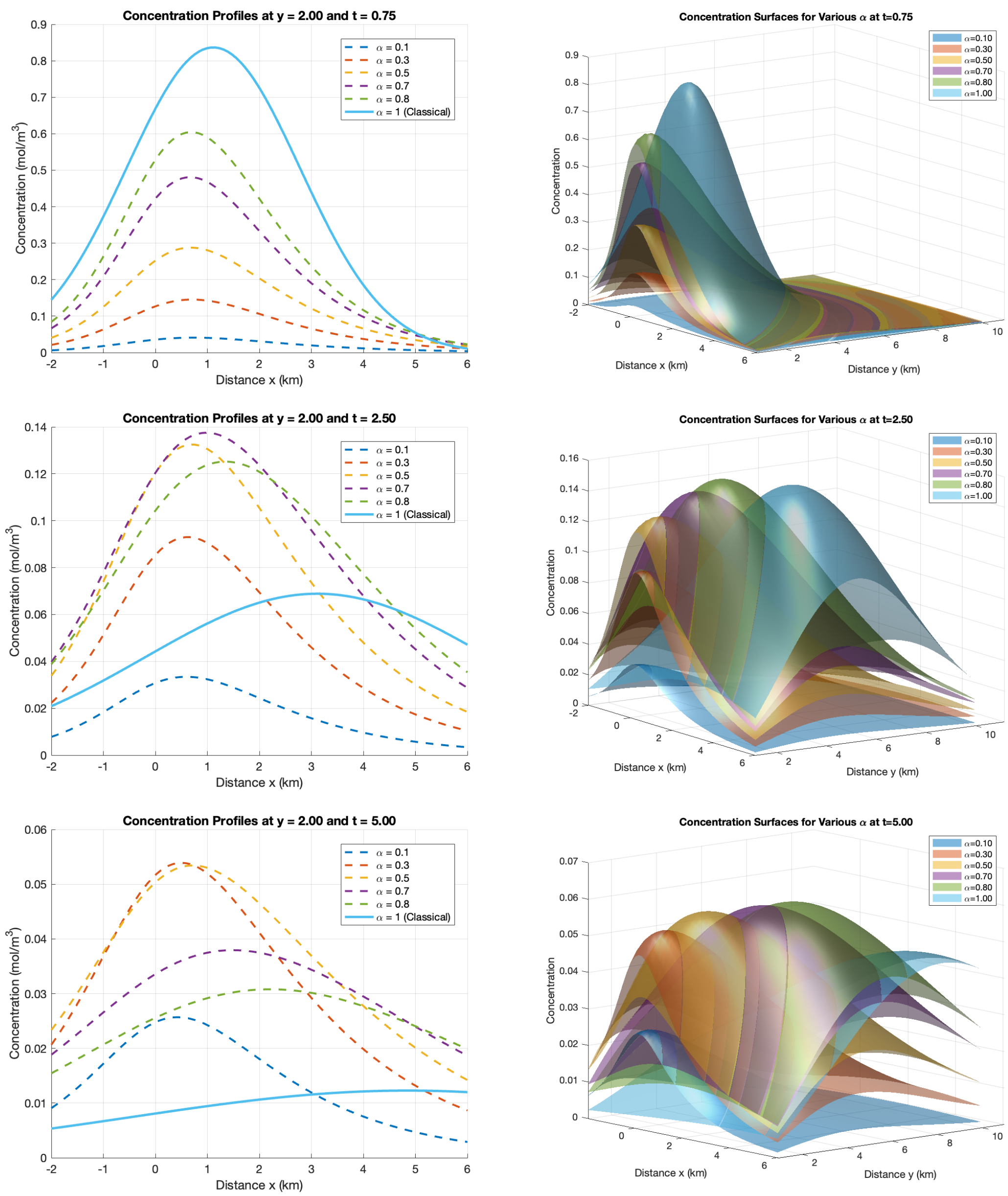

FIGURE 1. Comparison of concentration profiles for different fractional orders α at various time steps.

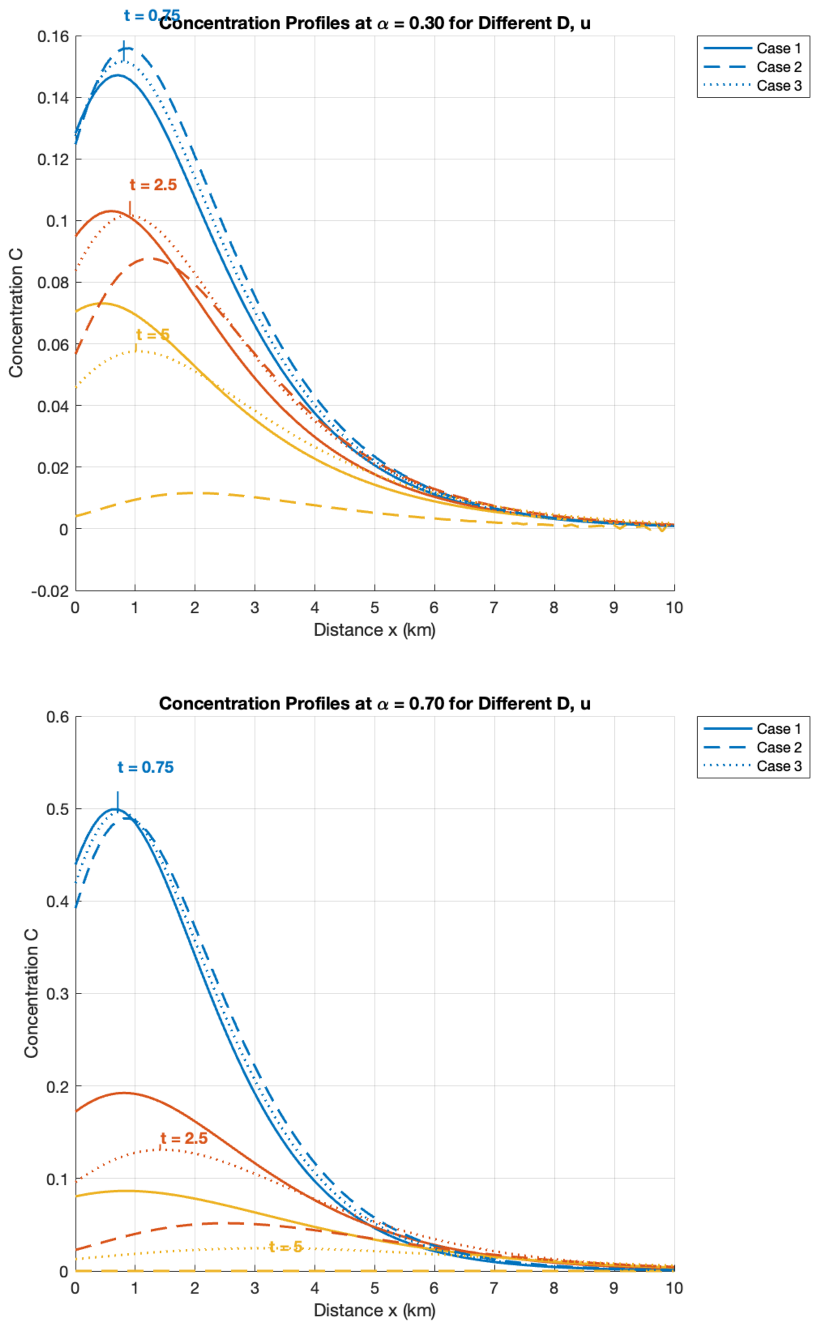

FIGURE 2. Concentration profile of concentration for different combination of temporal variations.

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