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EXISTENCE AND UNIQUENESS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. In this paper, we study nonlinear fractional differential equations involving the Caputo fractional derivative subject to an integral boundary condition. First, the exact solution of the corresponding linear problem is obtained by applying tools from fractional calculus. To prove the existence of solutions for the nonlinear case, we employ Schauder's and Krasnoselskii's fixed point theorems. Moreover, the uniqueness of the solution is established by using the Banach contraction principle. Finally, several illustrative examples are presented to demonstrate the applicability and effectiveness of the theoretical results.

Keywords. Caputo fractional derivative, Hölder inequality, Schauder's fixed point theorem, completely continuous
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1. Introduction and basic results

Fractional differential equations involving different types of fractional derivatives are models of many phenomena in various domains like engineering and scientific fields. When, we mention these fields, we mean among them: electrodynamics of complex mediums, aerodynamics, signal and image processing, blood flow phenomena, economics, biophysics, control theory, (see [1, 3, 5, 6, 9, 13, 15, 22, 23]).

Boundary value problems for differential equations involving Riemann-Liouville, Caputo, generalized Caputo, Hadamard derivatives, Atangana-Baleanu-Caputo, etc have attracted the interest of many researchers. To identify a few, we refer the reader to [7, 10, 11, 16, 18, 21].

In [8], the authors considered the following boundary value problem:

$$D^{\nu}\kappa(\tau)+f(\tau,\kappa(\tau))=0, \quad \tau\in(0,1),$$

subject to the boundary conditions

$$\kappa^{(i)}(0)=0,\ 0\leq i\leq 2,\ D^{\sigma}\kappa(1)=\lambda I^{\sigma}\kappa(\eta),$$

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where D^{ν}, D^{σ} are the standard Riemann-Liouville fractional derivative of order $\nu \in (3,4]$, $\sigma \in (0,2]$ respectively, I_{0+}^{σ} is the stantard Riemann-Liouville fractional integral of order $\sigma \in (0,2]$ and $0 \leq \frac{\lambda\Gamma(\nu-\sigma)\eta^{\nu+\beta-1}}{\Gamma(\nu+\sigma)} < 1$. To establish the existence results of the considered problem, the authors used the properties of the Green's function and the monotone iteration technique.

In [2], Aljurbua studies the existence of solutions for fractional differential equations with nonlocal boundary conditions (NFDEs):

$$\begin{cases} {}^{c}D^{\nu}\kappa(\tau) = f(\tau, \kappa(\tau)), \quad 1 < \nu \leq 2,\ \tau \in [0,d], \\ a_0\kappa(c) = -b_0\kappa(d),\ a_1\kappa'(c) = -b_1\kappa'(d), \quad 0 < c < d, \end{cases}$$

where $a_i, b_i \in \mathbb{R}^+$ for $i = 1,2$, ${}^{c}D^{\nu}$ represents the Caputo derivative of order ν, and there is a continuous function $f : [0,d] \times \mathbb{R} \to \mathbb{R}$.

In [19], Shivanian studied the following fractional differential equations with nonloal boundary conditions

$$\begin{cases} D_{\tau}^{\nu}\kappa(\tau) + f(\tau, \kappa(\tau), \kappa'(\tau), \kappa''(\tau)) = 0,\ \tau \in (0,1), \\ \kappa(0) = \kappa'(0) = \kappa''(0) = 0,\ (1-\lambda)\kappa(1) = \int_0^1 \kappa(s)\, ds, \end{cases}$$

where $3 < \nu \leq 4$, D_{τ}^{ν} is the left Rieman-Liouville fractional derivative and $f : (0,1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function.

In this paper, we investigate the existence and uniqueness of solutions for a Caputo fractional boundary value problem with a three-point integral condition, where the nonlinearity depends on both the function and its lower order Caputo derivative. As in the following problem:

$$\begin{cases} {}^{c}D^{\nu}\kappa(\tau) = f(\tau, \kappa(\tau), {}^{c}D^{\sigma}\kappa(\tau)), \quad 0 < \sigma \leq 1, 2 < \nu \leq 3, \tau \in [0,T], \\ \kappa(0) = \kappa''(0) = 0, \\ \kappa(T) = \lambda \int_0^{\eta} \kappa(s)ds. \end{cases} \tag{1.1}$$

Here ${}^{c}D^{\nu}$, ${}^{c}D^{\sigma}$ are the Caputo fractional derivatives of order ν and σ, respectively, f a continuous function, η and λ two parameters with $0 < \eta < T$ and $0 < \lambda < \frac{2T}{\eta^2}$.

By using some suitable conditions together with Hölder inequality, we investigate the existence results for the problem (1.1). In Section 2, we present our study that is based on Schauder's fixed point theorem, Krasnoselskii's fixed point theorem and Banach's fixed point theorem. In the last section, we give some illustrative examples.

Now, we recall some definitions and lemmas which can be found in [13].

Definition 1.1. The Riemann-Liouville fractional integral of order $\nu > 0$ for a function $f : (0,+\infty) \to \mathbb{R}$ is defined as

$$I^{\nu}f(\tau) = \frac{1}{\Gamma(\nu)} \int_0^{\tau} (\tau - s)^{\nu-1} f(s)ds,$$

provided the integral exists, and $\Gamma(\nu) = \int_0^{\infty} e^{-\tau}\tau^{\nu-1}d\tau$.

Definition 1.2. Let a function $f : [0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\nu > 0$ is defined as

$$ {}^{c}D^{\nu}f(\tau) = \frac{1}{\Gamma(n-\nu)} \int_0^{\tau} (\tau - s)^{n-\nu-1} f^{(n)}(s)ds, \quad n = [\nu] + 1,$$

where $[\nu]$ denotes the integer part of the real number ν, provided the right side is pointwise defined on $(0, +\infty)$.

Lemma 1.3. *Let $\nu > 0$ and $\kappa \in AC^N[0,T]$. Then the equation*

$$^cD^\nu \kappa(\tau) = 0,$$

has a unique solution

$$\kappa(\tau) = a_0 + a_1\tau + a_2\tau^2 + ... + a_{N-1}\tau^{N-1}, \quad a_i \in \mathbb{R},\ i = 1, 2, ..., N,$$

where N is the smallest integer greater than or equal to ν.

Lemma 1.4. *Let $\nu > \sigma > 0$ and $\kappa \in L^p(0,1) \subset L^1(0,1), 0 \leq p \leq +\infty$. Then the next formulas hold.*

(i) $(^cD^\sigma I^\nu \kappa)(\tau) = I^{\nu-\sigma}\kappa(\tau)$,
(ii) $(^cD^\nu I^\nu \kappa)(\tau) = \kappa(\tau)$.

One of our used tools in this study is Hölder inequality.

Lemma 1.5. *(Hölder inequality)[17] Let $f \in L^p([a,b])$ with $0 < a < b$ and $p > 1$, $g \in L^q([a,b])$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1([a,b])$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

Let $f \in L^1([a,b])$, $g \in L^\infty([a,b])$. Then $fg \in L^1([a,b])$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty .$$

In the next Lemma, we characterize the general solution of the linear problem associated with problem (1.1).

Lemma 1.6. *Let $2T \neq \lambda\eta^2$. Then for $h \in C([0,T],[0,\infty))$, the problem*

$$\begin{cases} ^cD^\nu \kappa(\tau) = h(\tau), \quad 2 < \nu \leq 3, \tau \in [0,T], \\ \kappa(0) = \kappa''(0) = 0, \\ \kappa(T) = \lambda \int_0^\eta \kappa(s)ds, \end{cases} \tag{1.2}$$

has a unique solution given by

$$\kappa(\tau) = \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} h(s)ds + \frac{2\tau}{(2T - \lambda\eta^2)\Gamma(\nu)} \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu h(s)ds - \int_0^T (T - s)^{\nu-1} h(s)ds \right).$$

Proof. By Definition 1.1 and Lemma 1.3, the equation (1.2) is reduced to the integral equation,

$$\kappa(\tau) = \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} h(s)ds + a_0 + a_1\tau + a_2\tau^2,$$

where $a_0, a_1, a_2 \in \mathbb{R}$ are arbitrary constants.
We have

$$\kappa''(\tau) = \frac{1}{\Gamma(\nu - 2)} \int_0^\tau (\tau - s)^{\nu-3} h(s)ds + 2a_2.$$

By using $\kappa(0)=0$ and $\kappa''(0)=0$, we get

$$\kappa(\tau)=\frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-s)^{\nu-1}h(s)ds+a_1\tau. \tag{1.3}$$

Integrating the equation (1.3) from 0 to η, we get

$$\begin{aligned}\int_0^{\eta}\kappa(\tau)d\tau &= \frac{1}{\Gamma(\nu)}\int_0^{\eta}\left(\int_0^{\tau}(\tau-s)^{\nu-1}h(s)ds\right)d\tau+a_1\int_0^{\eta}\tau d\tau\\ &=\frac{1}{\nu\Gamma(\nu)}\int_0^{\eta}(\eta-s)^{\nu}h(s)ds+a_1\frac{\eta^2}{2}.\end{aligned}$$

Then, by the condition $\kappa(T)=\lambda\int_0^{\eta}\kappa(s)ds$, we get

$$\frac{1}{\Gamma(\nu)}\int_0^{T}(T-s)^{\nu-1}h(s)ds+a_1T=\lambda\frac{1}{\nu\Gamma(\nu)}\int_0^{\eta}(\eta-s)^{\nu}h(s)ds+a_1\lambda\frac{\eta^2}{2},$$

which implies:

$$a_1=\frac{2}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda}{\nu}\int_0^{\eta}(\eta-s)^{\nu}h(s)ds-\int_0^{T}(T-s)^{\nu-1}h(s)ds\right).$$

Therefore, the solution of the problem (1.2)

$$\begin{aligned}\kappa(\tau)&=\frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-s)^{\nu-1}h(s)ds\\ &\quad+\frac{2\tau}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda}{\nu}\int_0^{\eta}(\eta-s)^{\nu}h(s)ds-\int_0^{T}(T-s)^{\nu-1}h(s)ds\right).\end{aligned}$$

□

2. EXISTENCE RESULTS

This part addresses the proof of existence of solutions for the given problem (1.1).

We define the space

$$\mathfrak{X}=\{\kappa\mid\kappa\in C([0,T],\mathbb{R})\ and\ {}^cD^{\sigma}\kappa\in C([0,T],\mathbb{R})\}$$

equipped with the norm

$$\|\kappa\|_{\mathfrak{X}}=\sup_{\tau\in[0,T]}|\kappa(\tau)|+\sup_{\tau\in[0,T]}|{}^cD^{\sigma}\kappa(\tau)|,$$

where $0<\sigma\leq 1$. $(\mathfrak{X},\|.\|)$ is a Banach space ([20]).

The operator $\mathfrak{G}:\mathfrak{X}\longrightarrow\mathfrak{X}$ is given by

$$\begin{aligned}(\mathfrak{G}\kappa)(\tau)&=\frac{1}{\Gamma(\nu)}\int_0^{\tau}(\tau-s)^{\nu-1}f(s,\kappa(s),{}^cD^{\sigma}\kappa(s))ds\\ &\quad+\frac{2\tau}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda}{\nu}\int_0^{\eta}(\eta-s)^{\nu}f(s,\kappa(s),{}^cD^{\sigma}\kappa(s))ds\right.\\ &\quad\left.-\int_0^{T}(T-s)^{\nu-1}f(s,\kappa(s),{}^cD^{\sigma}\kappa(s))ds\right),\end{aligned}$$

then problem (1.1) has solutions if and only if operator $\mathfrak{G}$ has fixed points. So $\mathfrak{G}'\kappa$ is given by

$$
\begin{aligned}
(\mathfrak{G}'\kappa)(\tau) = & \int_0^\tau \frac{(\tau-s)^{\nu-2}}{\Gamma(\nu-1)} f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds + \frac{2}{(2T-\lambda\eta^2)\Gamma(\nu)} \\
& \times \left(\frac{\lambda}{\nu} \int_0^\eta (\eta-s)^\nu f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds \right. \\
& \left. - \int_0^T (T-s)^{\nu-1} f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds \right),
\end{aligned}
$$

in Definition 1.2, we get

$$
\begin{aligned}
& {}^cD^\sigma(\mathfrak{G}\kappa)(\tau) \\
& = \frac{1}{\Gamma(1-\sigma)} \int_0^\tau (\tau-s)^{-\sigma} (\mathfrak{G}\kappa)'(\tau)ds \\
& = \frac{1}{\Gamma(\nu-\sigma)} \int_0^\tau (\tau-s)^{\nu-\sigma-1} f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds + \frac{2\tau^{1-\sigma}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)} \\
& \times \left(\frac{\lambda}{\nu} \int_0^\eta (\eta-s)^\nu f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds - \int_0^T (T-s)^{\nu-1} f(s,\kappa(s),{}^cD^\sigma\kappa(s))ds \right).
\end{aligned}
$$

For convenience, we set:

$$
\begin{aligned}
\Theta &= \frac{T^\nu}{\Gamma(\nu+1)} + \frac{2\lambda T^{\nu+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)} + \frac{2T^{\nu+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)} \\
\Theta_1 &= \frac{T^{\nu-\sigma}}{\Gamma(\nu-\sigma+1)} + \frac{2\lambda T^{\nu-\sigma+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)} + \frac{2T^{\nu-\sigma+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)\Gamma(2-\sigma)} \\
\mathfrak{K}_1 &= \frac{T^{\nu-\delta}}{\Gamma(\nu)} \left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta} + \frac{2\lambda T^{\nu-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\nu+1)} \left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta} \\
&\quad + \frac{2T^{\nu-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\nu)} \left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta} \\
\mathfrak{K}_2 &= \frac{T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)} \left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta} + \frac{2\lambda T^{\nu-\sigma-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)} \left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta} \\
&\quad + \frac{2T^{\nu-\sigma-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)} \left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}.
\end{aligned} \tag{2.1}
$$

Before, establishing our first result, we state the well known Schauder's fixed point theorem.

Lemma 2.1. *(**Schauder's fixed point theorem** [12]). Let U be a closed, convex, and nonempty subset of banach space $\mathfrak{X}$. Let $\mathfrak{T}: U \longrightarrow U$ be a continuous mapping such that $\mathfrak{T}(U)$ is a relatively compact subset of $\mathfrak{X}$. Then $\mathfrak{T}$ has at least one fixed point in U.*

Theorem 2.2. *Suppose that function $f \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ fulfills the condition:*

$$(\mathfrak{B}_1)\, |f(\tau,\kappa_1,\kappa_2)| \le w(\tau)+\mu_1|\kappa_1|^{\varrho_1}+\mu_2|\kappa_2|^{\varrho_2}, \forall(\tau,\kappa_1,\kappa_2)\in[0,T]\times\mathbb{R}\times\mathbb{R},$$

and $w\in L^{\frac{1}{\delta}}([0,T],\mathbb{R}^+)$, $\delta\in(0,\nu-1)$, $\mu_i\ge 0$, $0\le\varrho_i<1$, $i=1,2$.
Then, the problem (1.1) has at least one solution on $[0,T]$.

Proof. Denote $\|w\|=\left(\int_0^T|w(s)|^{\frac{1}{\delta}}\,ds\right)^{\delta}$. Let $B_r=\{\kappa\in\mathfrak{X}:\|\kappa\|_{\mathfrak{X}}\le r\}$ with $r>0$ to be defined later. B_r is a closed, and convex subset of the Banach space $\mathfrak{X}$. We will show that there exists $r>0$ such that $\mathfrak{G}$ maps B_r into B_r. For $\kappa\in B_r$, we have

$$\begin{aligned}
|(\mathfrak{G}\kappa)(\tau)| \le{}& \frac{1}{\Gamma(\nu)}\int_0^\tau(\tau-s)^{\nu-1}\left[w(s)+\mu_1|\kappa(s)|^{\varrho_1}+\mu_2|{}^cD^{\sigma}_{0+}\kappa(s)|^{\varrho_2}\right]ds\\
&+\frac{2T}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda}{\nu}\int_0^T(T-s)^{\nu}\left[w(s)+\mu_1|\kappa(s)|^{\varrho_1}+\mu_2|{}^cD^{\sigma}_{0+}\kappa(s)|^{\varrho_2}\right]ds\right.\\
&\left.+\int_0^T(T-s)^{\nu-1}\left[w(s)+\mu_1|\kappa(s)|^{\varrho_1}+\mu_2|{}^cD^{\sigma}_{0+}\kappa(s)|^{\varrho_2}\right]ds\right)\\
\le{}& \frac{\|w\|\,T^{\nu-\delta}}{\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}+\frac{T^{\nu}\left(\mu_1r^{\varrho_1}+\mu_2^{\varrho_2}\right)}{\Gamma(\nu+1)}\\
&+\frac{2\lambda T^{\nu-\delta+2}\,\|w\|}{(2T-\lambda\eta^2)\Gamma(\nu+1)}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta}+\frac{2T^{\nu-\delta+1}\,\|w\|}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}\\
&+\left(\frac{2\lambda T^{\nu+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)}+\frac{2T^{\nu+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)}\right)\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right)\\
\le{}& \left(\frac{T^{\nu-\delta}}{\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}+\frac{2\lambda T^{\nu-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\nu+1)}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta}\right.\\
&\left.+\frac{2T^{\nu-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}\right)\|w\|\\
&+\left(\frac{T^{\nu}}{\Gamma(\nu+1)}+\frac{2\lambda T^{\nu+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)}+\frac{2T^{\nu+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)}\right)\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|{}^cD^\sigma(\mathfrak{G}\kappa)(\tau)\right| \\
&\leq \frac{1}{\Gamma(\nu-\sigma)}\int_0^\tau(\tau-s)^{\nu-\sigma-1}\left[w(s)+\mu_1\left|\kappa(s)\right|^{\varrho_1}+\mu_2\left|{}^cD^\sigma\kappa(s)\right|^{\varrho_2}\right]ds \\
&\quad+\frac{2\tau^{1-\sigma}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)} \\
&\quad\times\left(\frac{\lambda}{\nu}\int_0^\eta(\eta-s)^\nu\left[w(s)+\mu_1\left|\kappa(s)\right|^{\varrho_1}+\mu_2\left|{}^cD^\sigma\kappa(s)\right|^{\varrho_2}\right]ds\right. \\
&\quad\left.+\int_0^T(T-s)^{\nu-1}\left[w(s)+\mu_1\left|\kappa(s)\right|^{\varrho_1}+\mu_2\left|{}^cD^\sigma\kappa(s)\right|^{\varrho_2}\right]ds\right) \\
&\leq \frac{\|w\|\,T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)}\left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta}+\frac{T^{\nu-\sigma}}{\Gamma(\nu-\sigma+1)}\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right) \\
&\quad+\frac{2\lambda T^{\nu-\sigma-\delta+2}\|w\|}{(2T-\lambda\eta^2)\Gamma(\nu+1)\Gamma(2-\sigma)}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta} \\
&\quad+\frac{2\lambda T^{\nu-\sigma+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)}\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right) \\
&\quad+\frac{2T^{\nu-\sigma-\delta+1}\|w\|}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta} \\
&\quad+\frac{2T^{\nu-\sigma+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)\Gamma(2-\sigma)}\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right).
\end{aligned}$$

From the obove inequalities, we get

$$\|\mathfrak{G}\kappa\|_{\mathfrak{X}}\leq H_1+H_2\left(\mu_1r^{\varrho_1}+\mu_2r^{\varrho_2}\right),$$

where

$$\begin{aligned}
H_1=&\left[\frac{T^{\nu-\delta}}{\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}+\frac{2\lambda T^{\nu-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\nu+1)}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta}\right. \\
&+\frac{2T^{\nu-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}+\frac{T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)}\left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta} \\
&+\frac{2\lambda T^{\nu-\sigma-\delta+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta} \\
&\left.+\frac{2T^{\nu-\sigma-\delta+1}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}\right]\|w\|
\end{aligned}\tag{2.2}$$

and

$$H_2=\frac{T^{\nu}}{\Gamma(\nu+1)}+\frac{2\lambda T^{\nu+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)}+\frac{2T^{\nu+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)}+\frac{T^{\nu-\sigma}}{\Gamma(\nu-\sigma+1)}+\frac{2\lambda T^{\nu-\sigma+2}}{(2T-\lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)}+\frac{2T^{\nu-\sigma+1}}{(2T-\lambda\eta^2)\Gamma(\nu+1)\Gamma(2-\sigma)}. \tag{2.3}$$

Let $r>0$ with

$$r\geq\max\left\{3H_1,(3H_2\mu_1)^{\frac{1}{1-\varrho_1}},(3H_2\mu_2)^{\frac{1}{1-\varrho_2}}\right\}.$$

Then, for any $\kappa\in B_r$, it follows that

$$\|\mathfrak{G}\kappa\|_{\mathfrak{X}}\leq H_1+H_2\left(\mu_1 r^{\varrho_1}+\mu_2 r^{\varrho_2}\right)\leq\frac{r}{3}+\frac{r}{3}+\frac{r}{3}=r.$$

Since f, we conclude $\mathfrak{G}$ is continuous. The next step is to establish that the families $\mathfrak{G}(\bar{B})$ and ${}^cD^{\sigma}\mathfrak{G}(\bar{B})$ are equicontinuous for every bounded subset $\bar{B}$ of $\mathfrak{X}$, where $\bar{B}$ is any bounded subset of $\mathfrak{X}$. Since f is continuous, we can suppose that $|f(\tau,\kappa(\tau),^c D^{\sigma}\kappa(\tau))|\leq Q$ for any $\kappa\in\bar{B}$ and $\tau\in[0,T]$. Now, for $0\leq\tau_1<\tau_2\leq T$, we have

$$\begin{aligned}
&|(\mathfrak{G}\kappa)(\tau_2)-(\mathfrak{G}\kappa)(\tau_1)|\\
&\leq\left|\frac{1}{\Gamma(\nu)}\int_0^{\tau_1}[(\tau_2-s)^{\nu-1}-(\tau_1-s)^{\nu-1}]f(\tau,\kappa(\tau),^c D^{\sigma}\kappa(\tau))ds\right.\\
&\quad\left.+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\nu-1}f(\tau,\kappa(\tau),^c D^{\sigma}\kappa(\tau))ds\right|\\
&\quad+\frac{2\left|\tau_2-\tau_1\right|}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda}{\nu}\int_0^{\eta}(\eta-s)^{\nu}\left|f(\tau,\kappa(\tau),^c D^{\sigma}\kappa(\tau))\right|ds\right.\\
&\quad\left.+\int_0^T(T-s)^{\nu-1}\left|f(\tau,\kappa(\tau),^c D^{\sigma}\kappa(\tau))\right|ds\right)\\
&\leq\frac{Q}{\nu\Gamma(\nu)}\left|2(\tau_2-\tau_1)^{\nu}+\tau_1^{\nu}-\tau_2^{\nu}\right|+\frac{2Q\left|\tau_2-\tau_1\right|}{(2T-\lambda\eta^2)\Gamma(\nu)}\left(\frac{\lambda\eta^{\nu+1}}{\nu(\nu+1)}+\frac{T^{\nu}}{\nu}\right)\\
&\leq\frac{Q}{\Gamma(\nu+1)}\left|2(\tau_2-\tau_1)^{\nu}+\tau_1^{\nu}-\tau_2^{\nu}\right|+\frac{2QT^{\nu}\left|\tau_2-\tau_1\right|}{(2T-\lambda\eta^2)\Gamma(\nu+2)}(\lambda T+\nu+1)
\end{aligned}$$

and

$$\begin{aligned}
&|^c D^\sigma(\mathfrak{G}\kappa)(\tau_2) -^c D^\sigma(\mathfrak{G}\kappa)(\tau_1)| \\
&\leq \left| \frac{1}{\Gamma(\nu-\sigma)} \int_0^{\tau_1} [(\tau_2 - s)^{\nu-\sigma-1} - (\tau_1 - s)^{\nu-\sigma-1}] f(\tau, \kappa(\tau), ^c D^\sigma \kappa(\tau)) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\nu-\sigma)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\nu-\sigma-1} f(\tau, \kappa(\tau), ^c D^\sigma \kappa(\tau)) ds \right| \\
&\quad + \frac{2\left|\tau_2^{1-\sigma} - \tau_1^{1-\sigma}\right|}{(2T - \lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)} \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu \left| f(\tau, \kappa(\tau), ^c D^\sigma \kappa(\tau)) \right| ds \right. \\
&\quad \left. + \int_0^T (T-s)^{\nu-1} \left| f(\tau, \kappa(\tau), ^c D^\sigma \kappa(\tau)) \right| ds \right) \\
&\leq \frac{Q}{\Gamma(\nu-\sigma+1)} \left| 2(\tau_2 - \tau_1)^{\nu-\sigma} + \tau_1^{\nu-\sigma} - \tau_2^{\nu-\sigma} \right| \\
&\quad + \frac{2QT^\nu \left|\tau_2^{1-\sigma} - \tau_1^{1-\sigma}\right|}{(2T - \lambda\eta^2)\Gamma(\nu+2)\Gamma(2-\sigma)} (\lambda T + \nu + 1).
\end{aligned}$$

Consequently, we get

$$\sup_{\tau \in \bar{B}} |(\mathfrak{G}\kappa)(\tau_2) - (\mathfrak{G}\kappa)(\tau_1)| + \sup_{\tau \in \bar{B}} |^c D^\sigma(\mathfrak{G}\kappa)(\tau_2) -^c D^\sigma(\mathfrak{G}\kappa)(\tau_1)| \longrightarrow 0 \quad as\ \tau_2 \longrightarrow \tau_1,$$

independent of $\tau \in \bar{B}$. Consequently, the operator $\mathfrak{G} : B_r \longrightarrow B_r$ is equicontinuous and uniformly bounded. By the Arzelá-Ascoli theorem, $\mathfrak{G}(B_r)$ is relatively compact in $\mathfrak{X}$. Applying Lemma 2.1, we conclude that problem (1.1) admits at least one solution on $[0, T]$. □

Now, we use the Krasnoselskii's fixed point theorem of the sum of two operators.

Lemma 2.3. *(**Krasnoselskii's fixed point theorem** [14]) Let M be a closed, bounded, convex, and nonempty subset of a Banach space $\mathfrak{X}$. Let A, B be the operators such that*

(i) *$Az_1 + Bz_2 \in M$ whenever $z_1, z_2 \in M$,*
(ii) *A is compact and continuous,*
(iii) *B is a contraction mapping.*

Then there exists $z_3 \in M$ such that $z_3 = Az_3 + Bz_3$

Theorem 2.4. *Let function $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ fulfill the conditions:*
($\mathfrak{B}_2$) *$|f(\tau, \kappa_1, \kappa_2) - f(\tau, \bar{\kappa_1}, \bar{\kappa_2})| \leq w(\tau)(|\kappa_1 - \bar{\kappa_1}| + |\kappa_2 - \bar{\kappa_2}|)$ for $\tau \in [0, T]$, $\kappa_i, \bar{\kappa_i} \in \mathbb{R}\,(i = 1, 2)$ and $\delta \in (0, \nu - 1)$ and the function $w : [0, T] \to \mathbb{R}^+$ is Lebesgue integrable with power $\frac{1}{\delta}$, that is,*
$w \in L^{\frac{1}{\delta}}([0, T], \mathbb{R}^+)$ with $\|w\| = \left(\int_0^T |w(s)|^{\frac{1}{\delta}} \, ds \right)^\delta$.
($\mathfrak{B}_3$) *$|f(\tau, \kappa_1, \kappa_2)| \leq w(\tau)$, $\forall (\tau, \kappa_1, \kappa_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, and $w \in L^{\frac{1}{\delta}}([0, T]$, $\delta \in (0, \nu - 1)$. Thus, problem (1.1) admits at least one solution on $[0, T]$ whenever*

$$\|w\| \left(\mathfrak{K}_1 - \frac{T^{\nu-\delta}}{\Gamma(\nu)} \left(\frac{1-\delta}{\nu-\delta} \right)^{1-\delta} + \mathfrak{K}_2 - \frac{T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)} \left(\frac{1-\delta}{\nu-\sigma-\delta} \right)^{1-\delta} \right) < 1, \quad (2.4)$$

where $\mathfrak{K}_1, \mathfrak{K}_2$ are defined in (2.1).

Proof. Let $\rho < H_1$, we define the ball $B_\rho = \{\kappa \in \mathfrak{X} : \|\kappa\|_{\mathfrak{X}} \leq \rho\}$ and define the operators A and B on B_ρ are given by:

$$(A\kappa)(\tau) = \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} f(s, \kappa(s), {}^c D^\sigma \kappa(s)) ds$$

and

$$(B\kappa)(\tau) = \frac{2\tau}{(2T - \lambda\eta^2)\Gamma(\nu)} \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu f(s, \kappa(s), {}^c D^\sigma \kappa(s)) ds \right. \\ \left. - \int_0^T (T - s)^{\nu-1} f(s, \kappa(s), {}^c D^\sigma \kappa(s)) ds \right).$$

For any $z_1, z_2 \in B_\rho$, as in the proof of Theorem 2.2, it can be shown that $\|Az_1 + Bz_2\|_{\mathfrak{X}} \leq H_1 < \rho$. This means that $Az_1 + Bz_2 \in B_\rho$. The operator A is completely continuous as in Theorem 2.2.
For $\kappa_1, \kappa_2 \in \mathfrak{X}$, we have

$$\begin{aligned}
&(B\kappa_1)(\tau) - (B\kappa_2)(\tau) \\
&= \frac{2\tau}{(2T - \lambda\eta^2)\Gamma(\nu)} \\
&\quad \times \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu [f(s, \kappa_1(s), {}^c D^\sigma \kappa_1(s)) - f(s, \kappa_2(s), {}^c D^\sigma \kappa_2(s))] ds \right. \\
&\quad \left. - \int_0^T (T - s)^{\nu-1} [f(s, \kappa_1(s), {}^c D^\sigma \kappa_1(s)) - f(s, \kappa_2(s), {}^c D^\sigma \kappa_2(s))] ds \right).
\end{aligned}$$

From condition $(\mathfrak{B}_2)$ together with Hölder's inequality, we obtain

$$\begin{aligned}
&|(B\kappa_1)(\tau) - (B\kappa_2)(\tau)| \\
&\leq \frac{2T}{(2T - \lambda\eta^2)\Gamma(\nu)} \\
&\quad \times \left(\frac{\lambda}{\nu} \int_0^T (T - s)^\nu w(s) (|\kappa_1(s) - \kappa_2(s)| + |{}^c D^\sigma \kappa_1(s) - {}^c D^\sigma \kappa_2(s)|) \, ds \right. \\
&\quad \left. + \int_0^T (T - s)^{\nu-1} w(s) (|\kappa_1(s) - \kappa_2(s)| + |{}^c D^\sigma \kappa_1(s) - {}^c D^\sigma \kappa_2(s)|) \, ds \right) \\
&\leq \frac{2T \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\nu)} \left(\frac{\lambda}{\nu} \int_0^T (T - s)^\nu w(s) ds + \int_0^T (T - s)^{\nu-1} w(s) \, ds \right) \\
&\leq \frac{2T \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}} \, \|w\|}{(2T - \lambda\eta^2)\Gamma(\nu)} \left[\frac{\lambda}{\nu} \left(\int_0^T (T - s)^{\frac{\nu}{1-\delta}} \, ds \right)^{1-\delta} + \left(\int_0^T (T - s)^{\frac{\nu-1}{1-\delta}} \, ds \right)^{1-\delta} \right] \\
&\leq \frac{2T \, \|w\| \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\nu)} \left[\frac{\lambda}{\nu} \left(\frac{1 - \delta}{\nu - \delta + 1} \right)^{1-\delta} T^{\nu - \delta + 1} + \left(\frac{1 - \delta}{\nu - \delta} \right)^{1-\delta} T^{\nu - \delta} \right].
\end{aligned}$$

We also deduce that

$$\begin{aligned}
&{}^cD^\sigma(B\kappa_1)(\tau) -{}^c D^\sigma(B\kappa_2)(\tau)\\
&= \frac{2\tau^{1-\sigma}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\\
&\quad\times\left(\frac{\lambda}{\nu}\int_0^\eta (\eta-s)^\nu[f(s,\kappa_1(s),{}^cD^\sigma\kappa_1(s)) - f(s,\kappa_2(s),{}^cD^\sigma\kappa_2(s))]ds\right.\\
&\quad\left.-\int_0^T (T-s)^{\nu-1}[f(s,\kappa_1(s),{}^cD^\sigma\kappa_1(s)) - f(s,\kappa_2(s),{}^cD^\sigma\kappa_2(s))]ds\right).
\end{aligned}$$

By $(\mathfrak{B}_2)$ and Hölder's inequality, it follows that

$$\begin{aligned}
&|{}^cD^\sigma(B\kappa_1)(\tau) -{}^c D^\sigma(B\kappa_2)(\tau)|\\
&\le \frac{2T^{1-\sigma}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\\
&\quad\times\left(\frac{\lambda}{\nu}\int_0^T (T-s)^\nu w(s)(|\kappa_1(s)-\kappa_2(s)| + |{}^cD^\sigma\kappa_1(s) -{}^c D^\sigma\kappa_2(s)|)ds\right.\\
&\quad\left.+\int_0^T (T-s)^{\nu-1} w(s)(|\kappa_1(s)-\kappa_2(s)| + |{}^cD^\sigma\kappa_1(s) -{}^c D^\sigma\kappa_2(s)|)ds\right)\\
&\le \frac{2T^{1-\sigma}\|\kappa_1-\kappa_2\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\left(\frac{\lambda}{\nu}\int_0^T (T-s)^\nu w(s)ds + \int_0^T (T-s)^{\nu-1}w(s)ds\right)\\
&\le \frac{2T^{1-\sigma}\|w\|\|\kappa_1-\kappa_2\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\left[\frac{\lambda}{\nu}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta} T^{\nu-\delta+1} + \left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta} T^{\nu-\delta}\right].
\end{aligned}$$

Thus, it follows

$$\begin{aligned}
&\|B\kappa_1 - B\kappa_2\|_{\mathfrak{X}}\\
&= \sup_{\tau\in[0,T]}|(B\kappa_1)(\tau)-(B\kappa_2)(\tau)| + \sup_{\tau\in[0,T]}|{}^cD^\sigma(B\kappa_1)(\tau) -{}^c D^\sigma(B\kappa_2)(\tau)|\\
&\le \|w\|\left(\mathfrak{K}_1 - \frac{T^{\nu-\delta}}{\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta} + \mathfrak{K}_2 - \frac{T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)}\left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta}\right)\\
&\quad\times\|\kappa_1-\kappa_2\|_{\mathfrak{X}},
\end{aligned}$$

using the condition $(\mathfrak{B}_3)$, we conclude that B is a contraction mapping. Thus all the assumptions of Lemma (2.3) are satisfied. Hence the conclusion of Lemma 2.3 implies that problem (1.1) has at least one solution on $[0,T]$. □

Finally, we prove that problem (1.1) admits a unique solution, using **Banach's fixed point theorem** along with Hölder's inequality as the key tools.

Theorem 2.5. *Let the function* $f \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *satisfies condition* $(\mathfrak{B}_2)$. *Then there exists a unique solution for problem (1.1) on* $[0,T]$ *if*

$$\|w\|(\mathfrak{K}_1+\mathfrak{K}_2) < 1,$$

where $\mathfrak{K}_1,\mathfrak{K}_2$ *are defined in (2.1).*

Proof. For $\kappa_1, \kappa_2 \in \mathfrak{X}$ and $\tau \in [0, T]$. By applying Hölder's inequality, we get

$$
\begin{aligned}
&|(\mathfrak{G}\kappa_1)(\tau) - (\mathfrak{G}\kappa_2)(\tau)| \\
&\leq \Bigg| \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} [f(s, \kappa_1(s), {}^c D^\sigma \kappa_1(s)) - f(s, \kappa_2(s), {}^c D^\sigma \kappa_2(s))] ds \\
&\quad + \frac{2\tau}{(2T - \lambda\eta^2)\Gamma(\nu)} \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu [f(s, \kappa_1(s), {}^c D^\sigma \kappa_1(s)) - f(s, \kappa_2(s), {}^c D^\sigma \kappa_2(s))] ds \right. \\
&\quad \left. - \int_0^T (T - s)^{\nu-1} [f(s, \kappa_1(s), {}^c D^\sigma \kappa_1(s)) - f(s, \kappa_2(s), {}^c D^\sigma \kappa_2(s))] ds \right) \Bigg| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^\tau (\tau - s)^{\nu-1} w(s) \left(|\kappa_1(s) - \kappa_2(s)| + |{}^c D^\sigma \kappa_1(s) - {}^c D^\sigma \kappa_2(s)| \right) ds \\
&\quad + \frac{2\tau}{(2T - \lambda\eta^2)\Gamma(\nu)} \left(\frac{\lambda}{\nu} \int_0^\eta (\eta - s)^\nu w(s) \left(|\kappa_1(s) - \kappa_2(s)| + |{}^c D^\sigma \kappa_1(s) - {}^c D^\sigma \kappa_2(s)| \right) ds \right. \\
&\quad \left. + \int_0^T (T - s)^{\nu-1} w(s) \left(|\kappa_1(s) - \kappa_2(s)| + |{}^c D^\sigma \kappa_1(s) - {}^c D^\sigma \kappa_2(s)| \right) ds \right) \\
&\leq \frac{\|\kappa_1 - \kappa_2\|_{\mathfrak{X}}}{\Gamma(\nu)} \left(\left(\left(\int_0^T (T - s)^{\nu-1} \, ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^\delta \right. \\
&\quad + \frac{2T \|\kappa_1 - \kappa_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\nu)} \left[\frac{\lambda}{\nu} \left(\left(\left(\int_0^T (T - s)^\nu \, ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^\delta \right. \right. \\
&\quad + \left(\left(\left(\int_0^T (T - s)^{\nu-1} \, ds \right)^{\frac{1}{1-\delta}} \right)^{1-\delta} \left(\int_0^T (w(s))^{\frac{1}{\delta}} ds \right)^\delta \right] \\
&\leq \frac{\|w\| \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}} \, T^{\nu-\delta}}{\Gamma(\nu)} \left(\frac{1-\delta}{\nu-\delta} \right)^{1-\delta} + \frac{2T \, \|w\| \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}}}{(2T - \lambda\eta^2)\Gamma(\nu)} \\
&\quad \times \left[\frac{\lambda}{\nu} \left(\frac{1-\delta}{\nu-\delta+1} \right)^{1-\delta} T^{\nu-\delta+1} + \left(\frac{1-\delta}{\nu-\delta} \right)^{1-\delta} T^{\nu-\delta} \right] \\
&= \mathfrak{K}_1 \, \|w\| \, \|\kappa_1 - \kappa_2\|_{\mathfrak{X}} \, .
\end{aligned}
$$

Analogously, we have

$$\begin{aligned}
&\left|{}^cD^\sigma(\mathfrak{G}\kappa_1)(\tau) -{}^c D^\sigma(\mathfrak{G}\kappa_2)(\tau)\right| \\
&\leq \frac{1}{\Gamma(\nu-\sigma)}\int_0^\tau (\tau-s)^{\nu-\sigma-1}w(s)\left(|\kappa_1(s)-\kappa_2(s)| + |{}^cD^\sigma\kappa_1(s) -{}^c D^\sigma\kappa_2(s)|\right)ds \\
&\quad + \frac{2t^{1-\sigma}}{(2T+\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)}\left(\frac{\lambda}{\nu}\int_0^\eta (\eta-s)^{\nu}w(s)\left(|\kappa_1(s)-\kappa_2(s)| + |{}^cD^\sigma\kappa_1(s) -{}^c D^\sigma\kappa_2(s)|\right)ds\right. \\
&\quad \left. + \int_0^T (T-s)^{\nu-1}w(s)\left(|\kappa_1(s)-\kappa_2(s)| + |{}^cD^\sigma\kappa_1(s) -{}^c D^\sigma\kappa_2(s)|\right)ds\right) \\
&\leq \frac{\|w\|\,\|\kappa_1-\kappa_2\|_{\mathfrak{X}}\,T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)}\left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta} + \frac{2T^{1-\sigma}\,\|w\|\,\|\kappa_1-\kappa_2\|_{\mathfrak{X}}}{(2T-\lambda\eta^2)\Gamma(\nu)\Gamma(2-\sigma)} \\
&\quad \times\left[\frac{\lambda}{\nu}\left(\frac{1-\delta}{\nu-\delta+1}\right)^{1-\delta}T^{\nu-\delta+1} + \left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}T^{\nu-\delta}\right] \\
&= \mathfrak{K}_2\,\|w\|\,\|\kappa_1-\kappa_2\|_{\mathfrak{X}}\,.
\end{aligned}$$

From the above inequalities, we obtain

$$\|\mathfrak{G}\kappa_1 - \mathfrak{G}\kappa_2\|_{\mathfrak{X}} \leq (\mathfrak{K}_1+\mathfrak{K}_2)\,\|w\|\,\|\kappa_1-\kappa_2\|_{\mathfrak{X}}\,.$$

Given the condition $\|w\|\,(\mathfrak{K}_1+\mathfrak{K}_2) < 1$, the operator $\mathfrak{G}$ constitutes a contraction. By Banach's fixed point theorem, $\mathfrak{G}$ admits a unique fixed point, ensuring a unique solution to problem (1.1). □

Corollary 2.6. *Let the function $f \in C\left([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$ satisfies the folowing condition:*
$(\mathfrak{B}_2)'$ $|f(\tau,\kappa_1,\kappa_2) - f(\tau,\bar{\kappa_1},\bar{\kappa_2})| \leq H_1\left(|\kappa_1-\bar{\kappa_1}| + |\kappa_2-\bar{\kappa_2}|\right)$ *for all* $\tau\in[0,T]$, $\kappa_i,\bar{\kappa_i}\in\mathbb{R}\,(i=1,2)$, *where $H_1>0$ is a constant. If the inequality*

$$H_1(\Theta+\Theta_1) < 1,$$

holds, with Θ,Θ_1 defined in (2.1) and H_1 as in (2.2). Consequently, problem (1.1) has a unique solution on $[0,T]$.

3. Examples

Consider problem (1.1) with the given data:

$$T=1.5,\qquad \eta=1.25,\qquad \lambda=1,\qquad \nu=2.5,\qquad \sigma=0.5,\qquad \delta=0.5. \tag{3.1}$$

Using the given values of the parameters in (2.1) , we find that

$$\begin{aligned}
&\mathfrak{K}_1 \simeq 3.477680593, \qquad \mathfrak{K}_2 \simeq 3.485012168. \\
&\Theta \simeq 3.301293165, \qquad \Theta_1 \simeq 3.4025962053.
\end{aligned} \tag{3.2}$$

As an example for Theorem 2.4, let us consider

$$\begin{aligned}
&f\left(\tau,\kappa(\tau),{}^cD^{1/2}\kappa(\tau)\right) = \\
&\frac{1}{10(\tau+2)^{1/3}}\left(\frac{1}{5} + \frac{2\,|\kappa(\tau)|}{2+(\kappa(\tau))^2} + \arctan\left({}^cD^{1/2}\kappa(\tau)+1\right)\right)
\end{aligned} \tag{3.3}$$

in (1.1) observe that

$$\begin{aligned}&\left|f\left(\tau,\kappa_1(\tau),{}^c D^{1/2}\kappa_1(\tau)\right)-f\left(\tau,\kappa_2(\tau),{}^c D^{1/2}\kappa_2(\tau)\right)\right|\\&\le\frac{1}{10(\tau+2)^{1/3}}\left(|\kappa_1-\kappa_2|+\left|{}^cD^{1/2}\kappa_1-{}^c D^{1/2}\kappa_2\right|\right).\end{aligned}$$

Here $w(\tau)=\frac{1}{10(\tau+2)^{1/3}}$ with $\|w\|\simeq 0.088040917$. Using the values of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ given by (3.2), we find that

$$\|w\|\left(\mathfrak{K}_1-\frac{T^{\nu-\delta}}{\Gamma(\nu)}\left(\frac{1-\delta}{\nu-\delta}\right)^{1-\delta}+\mathfrak{K}_2-\frac{T^{\nu-\sigma-\delta}}{\Gamma(\nu-\sigma)}\left(\frac{1-\delta}{\nu-\sigma-\delta}\right)^{1-\delta}\right)\simeq 0.445112708<1.$$

Since the conditions of Theorem 2.4 are satisfied, therefore there exists a one solution of problem (1.1) with the data (3.1) and (3.3) on $[0,T]$.

For the illustration of Theorem 2.5, we take

$$\begin{aligned}&f\left(\tau,\kappa(\tau),{}^c D^{1/2}\kappa(\tau)\right)=\\&\frac{1}{50(\tau+2)^{1/3}}\left(\frac{3}{7}+\ln(|\kappa(\tau)|+1)+\frac{2\left|{}^cD^{1/2}\kappa(\tau)\right|}{2+|{}^cD^{1/2}\kappa(\tau)|}\right)\end{aligned}\tag{3.4}$$

in (1.1) and note that

$$\begin{aligned}&\left|f\left(\tau,\kappa_1(\tau),{}^c D^{1/2}\kappa_1(\tau)\right)-f\left(\tau,\kappa_2(\tau),{}^c D^{1/2}\kappa_2(\tau)\right)\right|\\&\le\frac{1}{50(\tau+2)^{1/3}}\left(|\kappa_1-\kappa_2|+\left|{}^cD^{1/2}\kappa_1-{}^c D^{1/2}\kappa_2\right|\right).\end{aligned}$$

Here $w(\tau)=\frac{1}{50(\tau+2)^{1/3}}$ with $\|w\|\simeq 0.017608183$. Using the values of $\mathfrak{K}_1$ and $\mathfrak{K}_2$ given by (3.2), we find that

$$\|w\|\left(\mathfrak{K}_1+\mathfrak{K}_2\right)\simeq 0.122600371<1.$$

Given that the assumptions of Theorem 2.5 are satisfied, a unique solution exists for problem (1.1) subject to (3.1) and (3.4) on $[0,T]$.

We now proceed to demonstrate Corollary 2.6 through the following example

$$f(\tau,\kappa(\tau),{}^c D^{1/2}\kappa(\tau))=H_1\left(\frac{3}{11}+\ln|\kappa(\tau)|+\left|{}^cD^{1/2}\kappa(\tau)\right|\right),\tag{3.5}$$

which is a Lipschitz function with Lipschitz constant H_1. Notice that condition $H_1(\Theta+\Theta_1)<1$ is verified for $0<H_1<0.149167139$ (Θ and Θ_1 are given by (3.2)). Hence, by Corollary 2.6, there exists a unique solution for problem (1.1) under (3.1) and (3.5) on $[0,T]$.

4. Conclusion

The results obtained in this work demonstrate that combining fractional calculus tools with fixed point theorems provides an efficient framework for analyzing nonlinear fractional differential equations with integral boundary conditions. The established existence and uniqueness theorems, together with illustrative examples, confirm both the theoretical relevance and the potential applicability of the proposed approach.

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