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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A NON-ISOTHERMAL COUPLED PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS IN A THIN DOMAIN

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Abstract. This manuscript analyzes the asymptotic behavior of a non-isothermal, incompressible Herschel–Bulkley fluid in a thin domain governed by a steady nonlinear system with Tresca friction. We study the limit as the domain thickness tends to zero, prove strong convergence of velocity, temperature, and pressure, and derive the corresponding Reynolds equation.

Keywords. Asymptotic approach, Herschel-Bulkley fluid,Temperature, Reynolds equation, variational inequality.
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1. Introduction

Understanding non-Newtonian fluids under physical and geometric constraints is vital in many scientific and engineering fields. Herschel–Bulkley fluids, known for yield stress and shear-dependent viscosity, are particularly relevant in processes like polymer extrusion, food processing, and geological flows. Their complex rheology leads to nonlinear flow models that differ significantly from Newtonian fluids, see ([8,11,13]).

This work examines the asymptotic behavior of a continuum mechanics problem in a thin layer as its thickness ϵ tends to zero. By reformulating the equations on a fixed domain Ω, independent of ϵ. Previous studies have addressed similar problems with friction boundary conditions,a transitions from Ω^ϵ to a more manageable limit problem.Here, focusing on specific cases. For example, [2,3] study incompressible Newtonian and non-Newtonian fluids in thin 3D domains with nonlinear Tresca conditions, while [17,18] analyze Herschel–Bulkley fluids with Fourier and Tresca boundaries. Non-isothermal coupled problems with friction are treated in [16,21].

This paper is structured as follows: Section 2 introduces the steady-state flow problem for a Herschel–Bulkley fluid in a three-dimensional thin domain, along with essential notations, preliminaries, and functional spaces. Section 3 outlines the asymptotic analysis, considering the small parameter ϵ representing the domain's height, and provides estimates for velocity, pressure, and temperature independent of ϵ, along with convergence results. Section 4 establishes the main findings, including the limit problem and a weak form of the Reynolds equation. Section 5 presents concluding remarks and insights derived from this study. Finally, conclusion is presented in Section 6.

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2. Basic equations

Consider a fixed region ϖ in the plane defined by $s = (s_1, s_2) \in \mathbb{R}^2$. We suppose that ϖ has a Lipschitz-continuous boundary and represents the base of a fluid domain. The upper surface Γ_1^ϵ is described by $s_3 = \epsilon h(s)$, where ϵ is a small positive parameter $(0 < \epsilon < 1)$ approaching zero. The function h is smooth, bounded, and satisfies $0 < h_* < h(s) < h^*$ for all $(s, 0) \in \varpi$. We denote the fluid domain by Ω^ϵ, which is defined as follows:

$$\Omega^\epsilon = \{(s, s_3) \in \mathbb{R}^3 : (s, 0) \in \varpi, 0 < s_3 < \epsilon h(s)\}$$

Let Γ^ϵ be the boundary of Ω^ϵ, where $\Gamma^\epsilon = \bar{\Gamma}_1^\epsilon \cup \bar{\Gamma}_L^\epsilon \cup \bar{\varpi}$, and $\bar{\Gamma}_L^\epsilon$ is the lateral boundary.

We denote by $w^\epsilon = (w_i^\epsilon)_{1\le i\le 3}$ the velocity field, and by $\Sigma^\epsilon = (\Sigma_{ij}^\epsilon)_{1\le i,j\le 3}$ the Cauchy stress tensor, given by the power law:

$$\begin{cases} \Sigma_{ij}^\epsilon = \tilde{\Sigma}_{ij}^\epsilon - \rho^\epsilon \delta_{ij}, \\ \tilde{\Sigma}^\epsilon = \sqrt{2} g^\epsilon (T^\epsilon) \dfrac{D(w^\epsilon)}{|D_{II}(w^\epsilon)|} + \Lambda^\epsilon (T^\epsilon) |D(w^\epsilon)|^{\nu-2} D(w^\epsilon) \text{ if } D(w^\epsilon) \neq 0, \\ \text{ if } D(w^\epsilon) = 0 : |\tilde{\Sigma}^\epsilon| \le g^\epsilon (T^\epsilon) \text{ if } D(w^\epsilon) = 0. \end{cases}$$

Σ^ϵ describes the constitutive behavior of a Herschel–Bulkley fluid, where the consistency parameter Λ^ϵ and the yield stress g^ϵ vary with temperature. The exponent ν, constrained within $1 < \nu < 2$, defines the material's power-law response. Additionally, ρ^ϵ denotes the pressure, while $D(w^\epsilon) = (d_{i,j}(w^\epsilon))_{1\le i,j\le 3}$ represents the strain rate tensor.

$$d_{i,j}(w^\epsilon) = \frac{1}{2}\left(\frac{\partial w_i^\epsilon}{\partial s_j} + \frac{\partial w_j^\epsilon}{\partial s_i}\right)$$

Problem 1. Find a velocity field $w^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$, the pressure ρ^ϵ and a temperature: $\Omega^\epsilon \to \mathbb{R}$ such that

$$w^\epsilon \nabla w^\epsilon - div\,(\Sigma^\epsilon) = f^\epsilon \text{ in } \Omega^\epsilon, \tag{2.1}$$

$$div(w^\epsilon) = 0 \text{ in } \Omega^\epsilon, \tag{2.2}$$

$$w^\epsilon \nabla T^\epsilon - \nabla.\,(K^\epsilon \nabla T^\epsilon) = \Lambda^\epsilon (T^\epsilon) |D(w^\epsilon)|^\nu + \sqrt{2} g^\epsilon (T^\epsilon) |D(w^\epsilon)| - \alpha^\epsilon T^\epsilon \text{ in } \Omega, \tag{2.3}$$

$$w^\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \tag{2.4}$$

$$w^\epsilon \times n = 0 \text{ on } \varpi, \tag{2.5}$$

$$\begin{cases} |\Sigma_\tau^\epsilon| < k^\epsilon \implies w_\tau^\epsilon = 0 \\ |\Sigma_\tau^\epsilon| = k^\epsilon \implies \exists \lambda \ge 0, w_\tau^\epsilon = -\lambda \Sigma_\tau^\epsilon \end{cases} \quad \text{on } \varpi, \tag{2.6}$$

$$T^\epsilon = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \tag{2.7}$$

$$\frac{\partial T^\epsilon}{\partial n} = 0 \text{ on } \varpi. \tag{2.8}$$

The flow dynamics are governed by Equation (2.1) under the assumption of unit density. Equation (2.2) enforces the incompressibility condition, while Equation (2.3) represents energy conservation, where the specific heat is set to one, $K^\epsilon > 0$ denotes the thermal conductivity, and the term $-\alpha^\epsilon T^\epsilon$ represents the influence of an external heat source, where $\alpha^\epsilon > 0$.

The velocity on $\Gamma_1^\epsilon \cup \Gamma_L^\epsilon$ is prescribed in Equation (2.4). Given the absence of flux across ϖ, Equation (2.5) applies. Additionally, Condition (2.6) imposes a Tresca thermal friction law on ϖ, where k^ϵ represents the frictional yield coefficient.

Let $n = (n_1, n_2, n_3)$ be the outward unit normal vector on the boundary Ω^ϵ. The velocity components on Ω^ϵ are given by $w_n^\epsilon = w^\epsilon \cdot n$ (normal) and $w_\tau^\epsilon = w^\epsilon - w_n^\epsilon \cdot n$ (tangential). Similarly, the stress tensor field Σ_n^ϵ has its tangential and normal components on ϖ defined as $\Sigma_\tau^\epsilon = \Sigma^\epsilon \cdot n - \Sigma_n^\epsilon \cdot n$, $\quad \Sigma_n^\epsilon = (\Sigma^\epsilon \cdot n) \cdot n$, respectively.

3. Weak formulation

We consider the following functional framework on Ω^ϵ

$$V^\epsilon = \left\{\varphi \in W^{1,\nu}(\Omega^\epsilon)^3 : \varphi = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon,\ \varphi.n = 0 \text{ on } \varpi\right\},$$
$$V_{div}^\epsilon = \left\{\varphi \in V^\epsilon : div(\varphi) = 0\right\},$$
$$L_0^{\nu'}(\Omega^\epsilon) = \left\{\varphi \in L^{\nu'}(\Omega^\epsilon) : \int_{\Omega^\epsilon} \varphi \; dsds_3 = 0\right\},$$

and

$$W^{1,q}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon) = \left\{\Phi \in W^{1,q}(\Omega^\epsilon)^3 : \Phi = 0 \text{ on } \Gamma_1^\epsilon \cup \Gamma_L^\epsilon\right\}.$$

Problem 2 Find $(w^\epsilon, \rho^\epsilon, T^\epsilon) \in V_{div}^\epsilon \times L_0^{\nu'}(\Omega^\epsilon)(\varpi) \times W^{1,q}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}(\Omega^\epsilon)$, $\left(1 < q < \frac{3}{2}\right)$ such that

$$\left\{\begin{array}{l} B\left(w^\epsilon, w^\epsilon, \varphi - w^\epsilon\right) + a(w^\epsilon, T^\epsilon, \varphi - w^\epsilon) + J(T^\epsilon, \varphi) \\ -J\left(T^\epsilon, w^\epsilon\right) - \left(\rho^\epsilon, div\, \varphi - w^\epsilon\right) \geq \left(f, \varphi - w^\epsilon\right),\ \forall \varphi \in V\left(\Omega^\epsilon\right) \\ C\left(T^\epsilon, \Phi\right) = E\left(w^\epsilon, T^\epsilon, \Phi\right) + F\left(w^\epsilon, T^\epsilon, \Phi\right), \quad \forall \Phi \in W^{1,q'}_{\Gamma_1^\epsilon \cup \Gamma_L^\epsilon}\left(\Omega^\epsilon\right), \end{array}\right. \tag{3.1}$$

where

$$a(w^\epsilon, T^\epsilon, \varphi) = 2\int_{\Omega^\epsilon} \Lambda^\epsilon\left(T^\epsilon\right)\left|D(w^\epsilon)\right|^{\nu-2} D(w^\epsilon) D\left(\varphi\right) dsds_3,$$
$$B(w^\epsilon, w^\epsilon, \varphi) = \int_{\Omega^\epsilon} w^\epsilon \nabla w^\epsilon \varphi dsds_3,$$
$$\left(\rho^\epsilon, div\, \varphi\right) = \int_{\Omega^\epsilon} \rho^\epsilon\, div\, \varphi dsds_3,$$
$$J(T^\epsilon, \varphi) = \int_\omega k^\epsilon \left|\varphi\right| \; ds + \sqrt{2}\int_{\Omega^\epsilon} g^\epsilon\left(T^\epsilon\right)\left|D\left(\varphi\right)\right| dsds_3,$$
$$\left(f^\epsilon, \varphi\right) = \int_{\Omega^\epsilon} f^\epsilon \varphi dsds_3 = \sum_{i=1}^{3}\int_{\Omega^\epsilon} f_i^\epsilon \varphi_i dsds_3,$$
$$E\left(w^\epsilon, T^\epsilon, \Phi\right) = \int_{\Omega^\epsilon} T^\epsilon \nabla\Phi w^\epsilon dsds_3$$
$$C\left(T^\epsilon, \Phi\right) = \int_{\Omega^\epsilon} K^\epsilon \nabla T^\epsilon \nabla \Phi dsds_3,$$
$$F\left(w^\epsilon, T^\epsilon, \Phi\right) = 2\int_{\Omega^\epsilon} \Lambda^\epsilon\left(T^\epsilon\right)\left|D(w^\epsilon)\right|^2 \Phi dsds_3 + 2\int_{\Omega^\epsilon} g^\epsilon\left(T^\epsilon\right)\left|D(w^\epsilon)\right| \Phi dsds_3 + \int_{\Omega^\epsilon} \alpha^\epsilon\left(T^\epsilon\right)\Phi dsds_3.$$

This variational problem is known to have a unique solution, see [11,12,13] for further details.

We suppose the existence of $K_*^\epsilon,\ K_\epsilon^*,\ \nu_*^\epsilon,\ \nu_\epsilon^* \Lambda_*, \Lambda^*,\ g^*$ in $\mathbb{R}$ such that

$$\frac{1}{2} \leq K_*^\epsilon \leq K^\epsilon \leq K_\epsilon^*, \quad 0 \leq \nu_*^\epsilon \leq \nu^\epsilon \leq \nu_\epsilon^*. \tag{3.2}$$

and

$$0 \leq g^\epsilon \leq g^*, 0 \leq \Lambda_* \leq \Lambda^\epsilon \leq \Lambda^*, \quad f^\epsilon \in W^{1,\nu'}\left(\Omega^\epsilon\right)^3. \tag{3.3}$$

Based on previous results that will be valuable in the next sections

$$\left\|\nabla w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)} \leq C\left\|D\left(w^\epsilon\right)\right\|_{L^\nu(\Omega^\epsilon)} \ \ (\textit{Korn inequality}), \tag{3.4}$$

$$\left\|w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)} \leq \epsilon h^* \left\|\frac{\partial w^\epsilon}{\partial \kappa}\right\|_{L^\nu(\Omega^\epsilon)} \ \text{ for } i = 1, 2 \ (\textit{Poincaré inequality}), \tag{3.5}$$

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q},\ \forall\left(x, y\right) \in \mathbb{R}^2,\ (\textit{Young inequality}). \tag{3.6}$$

$$\left(c + d\right)^p \leq \left(2\right)^{p-1}\left(c^p + d^p\right), \quad \forall\left(c, d\right) \in \mathbb{R}_+^*, \quad \forall p > 1. \tag{3.7}$$

$$\left(c + d\right)^p \leq \left(c^p + d^p\right), \quad \forall\left(c, d\right) \in \mathbb{R}_+^*, \quad 0 < p < 1. \tag{3.8}$$

4. Change of the domain and study of convergence

Here, we will apply the technique of scaling in Ω^ϵ on the coordinate s_3 by introducing the change of variables $\kappa = \dfrac{s_3}{\epsilon}$. We get

$$\Omega = \left\{(s,\, \kappa) \in \mathbb{R}^3 :\, (s,0) \in \varpi, \quad 0 < \kappa < h(s) \right\}.$$

The boundary is defined as $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_L \cup \overline{\varpi}$. Additionally, we have

$$w_i^\epsilon(s, s_3) = \hat{w}_i^\epsilon(s,\kappa), i = 1, 2,\ \epsilon^{-1} w_3^\epsilon(s, s_3) = \hat{w}_3^\epsilon(s,\kappa) \text{ and } \epsilon^\nu \rho^\epsilon(s, s_3) = \hat{\rho}^\epsilon(s,\kappa). \tag{4.1}$$

Suppose that

$$\left.\begin{array}{c} \epsilon^{\beta+\nu-2} K^\epsilon(s, s_3) = \hat{K}(s,\kappa),\ \epsilon^{\beta+\nu} \alpha^\epsilon(s, s_3) = \hat{\alpha}(s,\kappa), \\ \Lambda^\epsilon = \hat{\Lambda}, \epsilon^\nu f^\epsilon(s, s_3) = \hat{f}(s,\kappa),\ \epsilon^{\nu-1} g^\epsilon = \hat{g},\ \epsilon^{\nu-1} k^\epsilon = \hat{k}, \end{array}\right\}, \tag{4.2}$$

with

$$\beta = \frac{3\,(2-\nu)}{3-\nu}.$$

Let

$$\begin{aligned}
V\,(\Omega) &= \left\{ \hat{\varphi} \in \left(W^{1,\nu}\,(\Omega)\right)^3 :\ \hat{\varphi}.n = 0 \text{ on } \varpi; \hat{\varphi} = 0 \text{ on } \Gamma_1 \cup \Gamma_L \right\}, \\
V_{\mathrm{div}}\,(\Omega) &= \left\{ \hat{\varphi} \in V\,(\Omega) :\ \mathrm{div}\hat{\varphi} = 0 \right\}, \\
L_0^{\nu'}(\Omega) &= \left\{ \varphi \in L^{\nu'}(\Omega) : \int_{\Omega^\epsilon} \varphi \; ds ds_3 = 0 \right\} \\
V_\kappa &= \left\{ \hat{\varphi} \in (L^r\,(\Omega))^2 ; \frac{\partial \hat{\varphi}_i}{\partial \kappa} \in L^\nu\,(\Omega) : \hat{\varphi} = 0 \ \text{ on } \Gamma_1 \cup \Gamma_L \right\}, \\
\tilde{V}_\kappa &= \left\{ \hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \in (L^\nu\,(\Omega))^2\,(\Omega) :\ \hat{\varphi} \text{ satisfy } \ (D^{`}) \right\}, \\
\Pi_\kappa &= \left\{ \hat{\varphi} \in (L^q\,(\Omega))^2 ; \frac{\partial \hat{\phi}_i}{\partial \kappa} \in L^q\,(\Omega) \right\},
\end{aligned}$$

where the condition $D^{`}$ is given by

$$\int_{\varpi} \left(\hat{\varphi}_1 \frac{\partial \phi}{\partial s_1} + \hat{\varphi}_2 \frac{\partial \phi}{\partial s_2} \right) ds d\kappa = 0, \text{ for all } \phi \in C_0^\infty\,(\Omega)$$

By injecting the new data and unknown variables into Equations (3), we demonstrate that $\left(\hat{w}^\epsilon, \hat{\rho}^\epsilon, \hat{T}^\epsilon\right) \in V_{\mathrm{div}}\,(\Omega) \times L_0^{\nu'}(\Omega) \times W^{1,q'}_{\Gamma_1 \cup \Gamma_L}\,(\Omega)$ form a solution to the following problem:

$$\left\{ \begin{array}{l} B_1\,(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) + a_1(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon) + J_1(\hat{T}^\epsilon, \hat{\varphi}) - J_1\left(\hat{T}^\epsilon, \hat{w}^\epsilon\right) \\ -\,(\hat{\rho}^\epsilon\,, div\ (\hat{\varphi})) \geq (\hat{f}, \hat{\varphi} - \hat{w}^\epsilon),\ \forall \hat{\varphi} \in V\,(\Omega) \\ C_1\left(\hat{T}^\epsilon, \hat{\Phi}\right) = E_1\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right) + F_1\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right), \quad \forall \hat{\Phi} \in W^{1,q'}_{\Gamma_1 \cup \Gamma_L}\,(\Omega), \end{array} \right. \tag{4.3}$$

where

$$a_1(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\vartheta} - \hat{w}^\epsilon) = \sum_{i,j=1}^{2} \int_\Omega \left[\epsilon^2 \hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\tilde{D}\left(\hat{w}^\epsilon\right)\right|^{\nu-2} \left(\frac{1}{2}\left(\frac{\partial \hat{w}_i^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial s_i}\right)\right)\right] \frac{\partial(\hat{\vartheta}_i - \hat{w}_i^\epsilon)}{\partial s_j} dsd\kappa$$
$$+ \sum_{i=1}^{2} \int_\Omega \hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\tilde{D}\left(\hat{w}^\epsilon\right)\right|^{r-2} \left(\frac{1}{2}\left(\frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} + \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i}\right)\right) \frac{\partial(\hat{\vartheta}_i - \hat{w}_i^\epsilon)}{\partial \kappa} dsdz$$
$$+ \int_\Omega \left(\hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\tilde{D}\left(\hat{w}^\epsilon\right)\right|^{r-2} \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa}\right) \frac{\partial(\hat{\vartheta}_3 - \hat{w}_3^\epsilon)}{\partial \kappa} dsdz +$$
$$+ \sum_{j=1}^{2} \int_\Omega \epsilon^2 \hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\tilde{D}\left(\hat{w}^\epsilon\right)\right|^{\nu-2} \left(\frac{1}{2}\left(\epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial \kappa}\right)\right) \frac{\partial(\hat{\vartheta}_3 - \hat{w}_3^\epsilon)}{\partial s_j} dsd\kappa,$$

$$B_1\left(\hat{w}^\epsilon, \hat{w}^\epsilon, \hat{\varphi} - \hat{w}^\epsilon\right) = \sum_{i,j=1}^{2} \int_\Omega \epsilon^2 \hat{w}_i^\epsilon \frac{\partial \hat{w}_j^\epsilon}{\partial s_i} \left(\hat{\varphi} - \hat{w}^\epsilon\right) dsd\kappa$$
$$+ \sum_{i=1}^{2} \int_\Omega \epsilon^4 \hat{w}_i^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial s_i} \left(\hat{\varphi} - \hat{w}^\epsilon\right) dsd\kappa + \sum_{i=1}^{2} \int_\Omega \epsilon^2 \hat{w}_3^\epsilon \frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} \left(\hat{\varphi}_i - \hat{w}_i^\epsilon\right) dsd\kappa$$
$$+ \int_\Omega \epsilon^4 \hat{w}_3^\epsilon \frac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \left(\hat{\varphi}_3 - \hat{w}_3^\epsilon\right) dsd\kappa,$$

$$\left(\hat{\rho}^\epsilon, div\left(\hat{\vartheta} - \hat{w}^\epsilon\right)\right) = \int_{\Omega^\epsilon} \hat{\rho}^\epsilon div\left(\hat{\vartheta} - \hat{w}^\epsilon\right) dsd\kappa,$$

$$J_1(\hat{T}^\epsilon, \hat{\vartheta}) = \sqrt{2} \int_\Omega \hat{g}\left(\hat{T}^\epsilon\right) \left|\tilde{D}\left(\hat{\vartheta}\right)\right| dsd\kappa + \int_\varpi \hat{k} |\hat{\vartheta}| ds,$$

$$(\hat{f}^\epsilon, \hat{\vartheta} - \hat{w}^\epsilon) = \sum_{j=1}^{2} \int_\Omega \hat{f}_i(\hat{\vartheta}_i - \hat{w}_i^\epsilon) dsd\kappa + \int_\Omega \epsilon \hat{f}_3(\hat{\vartheta}_3 - \hat{w}_3^\epsilon) dsd\kappa,$$

$$b_1\left(\hat{T}^\epsilon, \hat{\Phi}\right) = \int_\Omega \epsilon^2 \nabla_\epsilon \hat{T}^\epsilon \nabla_\epsilon \hat{\Phi} dsd\kappa = \sum_{i=1}^{2} \int_\Omega \epsilon^2 \hat{K} \frac{\partial \hat{T}^\epsilon}{\partial s_i} \frac{\partial \hat{\Phi}}{\partial s_i} dsd\kappa + \int_\Omega \hat{K} \frac{\partial \hat{T}^\epsilon}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} dsdz,$$

$$E_1\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right) = \int_{\Omega^\epsilon} \epsilon^{\beta+\nu} \hat{T}^\epsilon \nabla \hat{\Phi} \hat{w}^\epsilon dsd\kappa$$

$$C_1\left(\hat{T}^\epsilon, \hat{\Phi}\right) = \int_\Omega \epsilon^2 \nabla_\epsilon \hat{T}^\epsilon \nabla_\epsilon \hat{\Phi} dsd\kappa = \sum_{i=1}^{2} \int_\Omega \epsilon^2 \hat{K} \frac{\partial \hat{T}^\epsilon}{\partial s_i} \frac{\partial \hat{\Phi}}{\partial s_i} dsd\kappa + \int_\Omega \hat{K} \frac{\partial \hat{T}^\epsilon}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} dsdz,$$

$$F_1\left(\hat{w}^\epsilon, \hat{T}^\epsilon, \hat{\Phi}\right) = \int_\Omega \epsilon^\beta \hat{\Lambda}\left(\hat{T}^\epsilon\right) \left|\tilde{D}(\hat{w}^\epsilon)\right|^\nu \hat{\Phi} dsd\kappa + \sqrt{2} \int_\Omega \epsilon^\beta \hat{g}\left(\hat{T}^\epsilon\right) \left|\tilde{D}(\hat{w}^\epsilon)\right| \hat{\Phi} dsd\kappa$$
$$- \int_\Omega \hat{\alpha} \hat{T}^\epsilon \hat{\Phi} dsd\kappa,$$

$$\left|\widetilde{D}\left(\hat{w}^\epsilon\right)\right| = \left(\frac{1}{4} \sum_{i,j=1}^{2} \epsilon^2 \left(\frac{\partial \hat{w}_i^\epsilon}{\partial s_j} + \frac{\partial \hat{w}_j^\epsilon}{\partial s_i}\right)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{w}_i^\epsilon}{\partial \kappa} + \epsilon^2 \frac{\partial \hat{w}_3^\epsilon}{\partial s_i}\right)^2 + \epsilon^2 \left(\frac{\partial \hat{w}_3^\epsilon}{\partial \kappa}\right)^2\right)^{\frac{1}{2}}.$$

4.1. **A priori estimates on the velocity and the pressure.**

Theorem 4.1. *For all $1 < \nu < 2$ and under assumptions $(3.2)-(3.3)$ and (4.2), there exists a positive constant C_1 that is independent of ϵ, such that*

$$\sum_{i,j=1}^{2}\left\|\epsilon\frac{\partial \hat{w}_i^\epsilon}{\partial s_j}\right\|_{L^\nu(\Omega)}^{\nu}+\sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{w}_i^\epsilon}{\partial \kappa}\right\|_{L^\nu(\Omega)}^{\nu}+\left\|\epsilon^2\frac{\partial \hat{w}_3^\epsilon}{\partial s_i}\right\|_{L^\nu(\Omega)}^{\nu}\right)+\left\|\epsilon\frac{\partial \hat{w}_3^\epsilon}{\partial \kappa}\right\|_{L^\nu(\Omega)}^{\nu}\leq C. \tag{4.4}$$

$$\left\|\frac{\partial \hat{\rho}^\epsilon}{\partial s_i}\right\|_{W^{-1,\nu'}(\Omega)}\leq C \quad \textit{for } i=1,2 \tag{4.5}$$

$$\left\|\frac{\partial \hat{\rho}^\epsilon}{\partial \kappa}\right\|_{W^{-1,\nu'}(\Omega)}\leq \epsilon C. \tag{4.6}$$

Proof. Choosing $\varphi = 0$ in inequality (3.1), we find

$$B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)+a(T^\epsilon,w^\epsilon,w^\epsilon)+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3$$
$$+\left(\rho,div(w^\epsilon)\right)\leq (f^\epsilon,w^\epsilon),$$

as $B\left(w^\epsilon,w^\epsilon,w^\epsilon\right)=0$ and $div(w^\epsilon)=0$, we obtain

$$a(T^\epsilon,w^\epsilon,w^\epsilon)+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\leq (f^\epsilon,w^\epsilon). \tag{4.7}$$

According to Korn's inequality , there exists a constant $C_k > 0$, independent of ϵ, such that

$$2\Lambda_* C_k \parallel \nabla w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)}^{\nu}\leq a(T^\epsilon,w^\epsilon,w^\epsilon).$$

Using Poincaré's inequality, we have

$$\parallel w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)}\leq \epsilon h^* \parallel \nabla w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)},$$

By inequality (3.5)–(3.6), we obtain

$$\parallel f^\epsilon \parallel_{L^{\nu'}(\Omega^\epsilon)}\parallel w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)}\leq\left(\frac{\epsilon h^*}{\left(\frac{\Lambda_*\nu C_k}{2}\right)^{\frac{1}{\nu}}}\parallel f^\epsilon \parallel_{L^{\nu'}(\Omega^\epsilon)}\right)\left(\left(\frac{\Lambda_*\nu C_k}{2}\right)^{\frac{1}{\nu}}\parallel \nabla w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)}\right)$$
$$\leq\frac{1}{\nu'}\left(\frac{\epsilon h^*\parallel f^\epsilon \parallel_{L^{\nu'}(\Omega^\epsilon)}}{\left(\frac{\Lambda_* r C_k}{2}\right)^{\frac{1}{\nu}}}\right)^{\nu'}+\frac{1}{\nu}\left(\left(\frac{\Lambda_*\nu C_k}{2}\right)^{\frac{1}{\nu}}\parallel \nabla w^\epsilon \parallel_{L^\nu(\Omega^\epsilon)}\right)^{\nu}.$$

Consequently, (4.7) can be rewritten as

$$\begin{aligned}\Lambda_* C_k\left\|\nabla w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)}^{\nu}&+\int_\omega k^\epsilon\left|w^\epsilon\right|\ ds+\sqrt{2}\int_{\Omega^\epsilon}g^\epsilon\left(T^\epsilon\right)\left|D\left(w^\epsilon\right)\right|dsds_3\\&\leq\frac{1}{2}\Lambda_* C_k\left\|\nabla w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)}^{\nu}+\frac{\left(\epsilon h^*\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_*\nu C_k\right)^{\frac{\nu'}{\nu}}}\left\|f^\epsilon\right\|_{L^{\nu'}(\Omega^\epsilon)}^{\nu'}.\end{aligned} \tag{4.8}$$

Multiplying (4.8) by $\epsilon^{\nu-1}$,, and as $\epsilon^{\nu'}\left\|f^\epsilon\right\|_{L^{\nu'}(\Omega^\epsilon)}^{\nu'}=\epsilon^{1-\nu}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}$, we have

$$\begin{aligned}\frac{1}{2}\Lambda_* C_k\epsilon^{\nu-1}\left\|\nabla w^\epsilon\right\|_{L^\nu(\Omega^\epsilon)}^{\nu}&+\sqrt{2}\int_\Omega\hat{g}\left(\hat{T}\right)\left|\tilde{D}\left(\hat{w}^\epsilon\right)\right|dsd\kappa+\int_\varpi\hat{k}\left|\hat{w}^\epsilon\right|ds\\&\leq\frac{\left(h^*\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_*\nu C_k\right)^{\frac{\nu'}{\nu}}}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}\end{aligned} \tag{4.9}$$

So, from (4.9) we deduce (4.4), with $C=(\frac{1}{2}\Lambda_* C_k)^{-1}\dfrac{\left(h^*\right)^{\nu'}}{\nu'\left(\frac{1}{2}\Lambda_*\nu C_k\right)^{\frac{\nu'}{\nu}}}\left\|\hat{f}\right\|_{L^{\nu'}(\Omega)}^{\nu'}$.

For (4.5) and (4.6), we prove as in [19].

□

4.2. **A priori estimates on the temperature.**

Theorem 4.2. *Assuming the conditions of Theorem 4.1 hold, there exists a positive constant C_1 independent of ϵ, such that the following estimate holds:*

$$\left\|\frac{\partial \hat{T}^{\epsilon}}{\partial \kappa}\right\|_{W^{1,q}(\Omega)} \leq C_1, \tag{4.10}$$

and

$$\sum_{i=1}^{2}\left\|\epsilon\frac{\partial \hat{T}^{\epsilon}}{\partial s_i}\right\|_{W^{1,q}(\Omega)} \leq C_1. \tag{4.11}$$

Proof. Choosing $\Phi = \vartheta\left(T^{\epsilon}\right)$ in (4.3) as ϑ is given by

$$\vartheta\left(t\right) = sign\left(t\right)\zeta\int_{0}^{|t|}\frac{d\tau}{\left(1+|t|\right)^{\zeta+1}}dt = sign\left(t\right)\left[1-\frac{1}{\left(1+|t|\right)^{\zeta}}\right],$$

we obtain

$$\zeta K_{*}\int_{\Omega}\frac{\left|\nabla\hat{T}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}} \leq \Lambda^{*}\epsilon^{\beta-2}\int_{\Omega}|\tilde{D}\left(\hat{w}^{\epsilon}\right)|^{\nu}dsd\kappa + \epsilon^{\beta-2}\sqrt{2}g^{*}\int_{\Omega}|\tilde{D}\left(\hat{w}^{\epsilon}\right)|dsd\kappa$$
$$+\epsilon^{\beta+\nu-2}\zeta\int_{\Omega}\frac{\hat{T}^{\epsilon}\left|\nabla\hat{T}^{\epsilon}\right|}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\hat{w}^{\epsilon}dsd\kappa$$

Using Young's inequality with $q < \frac{3}{2} < p$, we get

$$\epsilon^{\beta+\nu-2}\int_{\Omega}\frac{\hat{T}^{\epsilon}\left|\nabla\hat{T}^{\epsilon}\right|}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\hat{w}^{\epsilon}dsd\kappa \leq \epsilon^{\beta+\nu-2}\frac{1}{q}\int_{\Omega}\left(\frac{\hat{T}^{\epsilon}\left|\nabla\hat{T}^{\epsilon}\right|}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{q}dsd\kappa$$
$$+\epsilon^{\beta+\nu-2}\frac{1}{p}\int_{\Omega}\left(\hat{w}^{\epsilon}\right)^{p}dsd\kappa \tag{4.12}$$

Furthermore

$$\int_{\Omega}|\tilde{D}(\hat{w}^{\epsilon})|^{\nu}dsd\kappa \leq C(\nu)\left[\sum_{i,j=1}^{2}\left\|\epsilon\frac{\partial\hat{w}_i^{\epsilon}}{\partial s_j}\right\|_{L^{\nu}(\Omega)}^{\nu} + \sum_{i=1}^{2}\left\|\frac{\partial\hat{w}_i^{\epsilon}}{\partial\kappa}\right\|_{L^{\nu}(\Omega)}^{\nu}\right.$$
$$\left.+\sum_{i=1}^{2}\left\|\epsilon^{2}\frac{\partial\hat{w}_3^{\epsilon}}{\partial s_i}\right\|_{L^{\nu}(\Omega)}^{\nu} + \left\|\epsilon\frac{\partial\hat{w}_3^{\epsilon}}{\partial\kappa}\right\|_{L^{\nu}(\Omega)}^{\nu}\right]$$

As $C\left(\nu\right) > 0$ depends only on ν, and since $\nu > 1$ and $0 < \epsilon < 1$, it follows that $\epsilon^{\nu-1} \leq 1$. Combining this inequality with (4.4), we deduce that

$$\zeta K_{*}\int_{\Omega}\frac{\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}} \leq \left(\Lambda^{*}C\left(\nu\right)C+\sqrt{2}g^{*}C\right)\epsilon^{\beta-2}$$
$$+\epsilon^{\beta+\nu-2}\zeta\frac{1}{q}\int_{\Omega}\left(\frac{\hat{T}^{\epsilon}\left|\nabla\hat{T}^{\epsilon}\right|}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{q}dsd\kappa \tag{4.13}$$
$$+\epsilon^{\beta+\nu-2}\zeta\frac{1}{p}\int_{\Omega}\left(\hat{w}^{\epsilon}\right)^{p}dsd\kappa$$

Employing Young's inequality for exponents $\frac{2}{q}$ and $\frac{2}{2-q}$, for $q<\frac{3}{2}$, we obtain

$$\begin{aligned}\epsilon^{\beta+\nu-2}\left(\frac{\hat{T}^{\epsilon}\left|\nabla\hat{T}^{\epsilon}\right|}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{q} &\leq \frac{q}{2}\frac{\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}} \\ &+\frac{2-q}{2}\epsilon^{(\beta+\nu-2)\frac{2}{2-q}}\frac{(\hat{T}^{\epsilon})^{\frac{2q}{2-q}}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{(\frac{\zeta+1}{2})\frac{2q}{2-q}}}\end{aligned} \tag{4.14}$$

As $\epsilon^{(\beta+\nu-2)\frac{2q}{2-q}}\leq\epsilon^{(\beta-2)}$ and $\frac{(\hat{T}^{\epsilon})^{\frac{2q}{2-q}}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{(\frac{\zeta+1}{2})\frac{2q}{2-q}}}\leq 1$, we deduce

$$\zeta\left(K_{*}-\frac{1}{2}\right)\int_{\Omega}\frac{\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\leq\left(\Lambda^{*}C\left(\nu\right)C+\sqrt{2}g^{*}C+\frac{2-q}{2q}\zeta\Omega+\frac{1}{p}\zeta C\right)\epsilon^{\beta-2} \tag{4.15}$$

Applying Hölder's inequality with the conjugate powers $\frac{2}{q}$ and $\frac{2}{2-q}$, for $q<\frac{3}{2}$, we find

$$\int_{\Omega}\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{q}dsd\kappa\leq\left(\int_{\Omega}\frac{\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{2}}{\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\zeta+1}}\right)^{\frac{q}{2}}\left(\int_{\Omega}\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{\frac{(\zeta+1)q}{2-q}}\right)^{\frac{2-q}{2}},$$

Using (4.15) yields

$$\int_{\Omega}\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{q}dsd\kappa\leq\left(\frac{1}{\zeta\left(K_{*}-\frac{1}{2}\right)}C^{"}\epsilon^{\beta-2}\right)^{\frac{q}{2}}\left(\int_{\Omega}\left(1+\left|\hat{T}^{\epsilon}\right|\right)^{q*}\right)^{\frac{2-q}{2}},$$

where $C^{"}=\Lambda^{*}C\left(\nu\right)C+\sqrt{2}g^{*}C+\frac{2-q}{2q}\zeta\Omega+\frac{1}{p}\zeta C$ and $q^{*}=\frac{3q}{3-q}\geq\frac{(\zeta+1)q}{2-q}$, from (3.7) and (3.8), we derive

$$\begin{aligned}\int_{\Omega}\left|\nabla_{\epsilon}\hat{T}^{\epsilon}\right|^{q}dsd\kappa &\leq\left(\frac{1}{\zeta\left(K_{*}-\frac{1}{2}\right)}C^{"}\epsilon^{\beta-2}\right)^{\frac{q}{2}} \\ &\times 2^{\left(\frac{2-q}{2}\right)\left(q^{*}-1\right)}\left(\left|\Omega\right|^{\frac{2-q}{2}}+\left(\int_{\Omega}\left|\hat{T}^{\epsilon}\right|^{q*}\right)^{\frac{2-q}{2}}\right).\end{aligned} \tag{4.16}$$

Poincaré–Sobolev inequality implies

$$\begin{aligned}\left(\int_{\Omega}\left|\hat{T}^{\epsilon}\right|^{q*}dsd\kappa\right)^{\frac{1}{q^{*}}} &\leq C'\left\|\nabla_{\epsilon}\hat{T}^{\epsilon}\right\|_{L^{q}(\Omega)} \\ &\leq\left(\frac{1}{\zeta\left(K_{*}-\frac{1}{2}\right)}C^{"}\right)^{\frac{1}{2}} \\ &\times\epsilon^{\frac{\beta}{2}-1}2^{\left(\frac{2-q}{2q}\right)\left(q^{*}-1\right)}C'\left(\left|\Omega\right|^{\frac{2-q}{2q}}+\left(\int_{\Omega}\left|\hat{T}^{\epsilon}\right|^{q*}\right)^{\frac{2-q}{2q}}\right).\end{aligned} \tag{4.17}$$

For all $x>0,\ y>0,\ z>0$ with $0<n<m$, it holds that

$$\text{If } x^{m}\leq y+zx^{n}\text{ then } x\leq\max\left\{1,(y+z)^{\frac{1}{m-n}}\right\}. \tag{4.18}$$

Thus, from (4.18) and the fact that $\frac{2-q}{2}<1$, we deduce

$$\int_{\Omega}\left|\hat{T}^{\epsilon}\right|^{q*}dsd\kappa\leq\xi_{\epsilon}, \tag{4.19}$$

where

$$\xi=\max\left[1,\ \gamma\epsilon^{\left(\frac{\beta}{2}-1\right)\left(\frac{1}{q^{*}}-\frac{2-q}{2q}\right)^{-1}\frac{2-q}{2}}\right]=\max\left[1,\ \gamma\epsilon^{3(2-q)\left(\frac{\beta}{2}-1\right)}\right],$$

and

$$\gamma = \left[\left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)} C^{"}\right)^{\frac{1}{2}} 2^{\left(\frac{2-q}{2q}\right)\left(q^*-1\right)} C' \left(|\Omega|^{\frac{2-q}{2q}} + 1\right)\right]^{\left(\frac{1}{q^*} - \frac{2-q}{2q}\right)^{-1}}$$
$$= \left[\left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)} C^{"}\right)^{\frac{1}{2}} 2^{\left(\frac{2-q}{2q}\right)\left(q^*-1\right)} C' \left(|\Omega|^{\frac{2-q}{2q}} + 1\right)\right]^{6}.$$

As $\beta = 3\frac{2-q}{3-q}$, then $3\left(2-q\right)\left(\frac{\beta}{2}-1\right) < 0$. Thus for $\epsilon \leq \beta^{\left[-3\left(\frac{\beta}{2}-1\right)\left(2-q\right)\right]^{-1}}$, we find

$$\xi = \gamma \epsilon^{3\left(\frac{\beta}{2}-1\right)\left(2-q\right)} \geq 1.$$

From (4.16) and (4.20), we obtain

$$\epsilon^q \int_\Omega \left|\nabla_\epsilon \hat{T}^\epsilon\right|^q ds d\kappa \leq \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)} C^{"}\right)^{\frac{q}{2}}$$
$$\times 2^{\left(\frac{2-q}{2}\right)\left(q^*-1\right)} \left(|\Omega|^{\frac{2-q}{2}} \epsilon^{\frac{\beta}{2}q} + \gamma \epsilon^{3\left(\frac{\beta}{2}-1\right)\left(2-q\right) + \frac{\beta}{2}q}\right),$$

as $3\left(\frac{\beta}{2}-1\right)\left(2-q\right) + \frac{\beta}{2}q = 0$ and $\frac{\beta}{2}q > 0$, we get

$$\epsilon^q \int_\Omega \left|\nabla_\epsilon \hat{T}^\epsilon\right|^q ds d\kappa \leq C_1,$$

where

$$C_1 = \left(\frac{1}{\zeta\left(K_* - \frac{1}{2}\right)} C^{"}\right)^{\frac{q}{2}} 2^{\left(\frac{2-q}{2}\right)\left(q^*-1\right)} \left(|\Omega|^{\frac{2-q}{2}} + \gamma\right).$$

Where C_1 is a constant independent of ϵ. Thus, we obtain (4.10) and (4.11)

□

Theorem 4.3. *According to the assumptions of Theorem 4.1 and Theorem 4.2 there exist* $w^\star = (w_1^\star, w_2^\star) \in \tilde{V}_\kappa$, $\rho^\star \in L_0^{\nu'}(\Omega)$ *and* $T^* \in \Pi_\kappa$ *such that:*

$$\begin{cases} \hat{w}_i^\epsilon \rightharpoonup w_i^\star, \textit{weakly in } \tilde{V}_\kappa,\ i = 1,2 \quad , \\[2ex] \epsilon \dfrac{\partial \hat{w}_i^\epsilon}{\partial s_j} \rightharpoonup 0,\ \textit{weakly in}\ \ L^\nu(\Omega),\ i,j = 1,2 \\[2ex] \epsilon \dfrac{\partial \hat{w}_3^\epsilon}{\partial \kappa} \rightharpoonup 0, \quad \textit{weakly in}\ \ L^\nu(\Omega) \\[2ex] \epsilon^2 \dfrac{\partial \hat{w}_3^\epsilon}{\partial s_i} \rightharpoonup 0,\ i = 1,2,\ \textit{weakly in}\ \ L^\nu(\Omega), \end{cases} \tag{4.20}$$

$$\epsilon \hat{w}_3^\epsilon \rightharpoonup 0, \quad \textit{weakly in}\ \ L^\nu(\Omega), \tag{4.21}$$

$$\hat{\rho}^\epsilon \rightharpoonup \rho^\star,\ \rho^\star \textit{ depend only of } s, \textit{weakly in}\ \ L^{\nu'}(\Omega), \tag{4.22}$$

$$\begin{cases} \hat{T}^\epsilon \rightharpoonup T^* \textit{ weakly in } \Pi_\kappa, \\[2ex] \frac{\partial \hat{T}^\epsilon}{\partial s_i} \rightharpoonup 0,\ \textit{weakly in } L^q\left(\Omega\right),\ i = 1,2 \end{cases} \tag{4.23}$$

Proof. The convergence of (4.21) and (4.22) follows directly from inequality 4.4. Furthermore, applying a suitable test function to (4.5) and (4.6) yields (4.2). By inequality (4.11), we have

$$\left\|\hat{T}^\epsilon\right\|_{L^\nu(\Omega)} \leqslant \bar{h} \left\|\frac{\partial \hat{T}^\epsilon}{\partial \kappa}\right\|_{L^\nu(\Omega)} \leqslant \bar{h} C$$

Since $\hat{T}^\epsilon$ is bounded in Π_κ, there exists $\hat{T}^\epsilon$ in Π_κ that converges weakly to to $\hat{T}^*$ in V_κ.
Moreover, inequality (4.10) shows that $\epsilon \left\|\frac{\partial \hat{T}^\epsilon}{\partial s_i}\right\|_{L^\nu(\Omega)} \leqslant C$, therefore $\epsilon \frac{\partial \hat{T}^\epsilon}{\partial s_i}$ converge to $\frac{\partial \hat{T}^*}{\partial s_i}$, and since $\hat{T}^\epsilon$ converge to $\hat{T}^*$ in Π_κ, we have that $\epsilon \frac{\partial \hat{T}^*}{\partial s_i}$ converges to zero in $L^\nu(\Omega)$.

□

5. Study of the limit problem

This section introduces the equations that are satisfied by w^* and ρ^* within the domain Ω, along with the inequalities governing the trace of the velocity $\frac{\partial w^*}{\partial \kappa}(s,0)$ and the stress $w^*(s,0)$ on the boundary ϖ.

Theorem 5.1. *Under the same hypotheses of Theorem 4.3,the solution (w^*, ρ^*, T^*) satisfies the following inequalities.*

$$\sum_{i=1}^{2}\int_{\Omega}\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}(T^*)\left(\sum_{i=1}^{2}\left(\frac{\partial w_i^*}{\partial\kappa}\right)^2\right)^{\frac{\nu-2}{2}}\frac{\partial(w_i^*)}{\partial\kappa}\frac{\partial(\hat{\varphi}_i-w_i^*)}{\partial\kappa}dsd\kappa$$
$$-\int_{\Omega}\rho^*(s)\left(\frac{\partial\hat{\varphi}_1}{\partial s_1}+\frac{\partial\hat{\varphi}_2}{\partial s_2}\right)dsd\kappa+\int_{\Omega}\hat{g}(T^*)\left(\left|\frac{\partial\hat{\varphi}}{\partial\kappa}\right|-\left|\frac{\partial w^*}{\partial\kappa}\right|\right)dsd\kappa$$
$$+\int_{\varpi}\hat{k}\left(|\hat{\varphi}|-|w^*|\right)ds\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i(\hat{\varphi}_i-w_i^*)dsd\kappa,\ \forall\hat{\varphi}\in W_{\Gamma_1\cup\Gamma_L},\tag{5.1}$$

$$\int_{\Omega}\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}(T^*)\left|\frac{\partial w^*}{\partial\kappa}\right|^{\nu}dsd\kappa+\int_{\Omega}\hat{g}(T^*)\left|\frac{\partial w^*}{\partial\kappa}\right|dsd\kappa$$
$$+\int_{\varpi}|w^*|\,\hat{k}ds=\int_{\Omega}w^*\hat{f}dsd\kappa\ ,\tag{5.2}$$

$$\int_{\Omega}\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}(T^*)\left|\frac{\partial w^*}{\partial\kappa}\right|^{\nu-2}\frac{\partial\hat{\Phi}}{\partial\kappa}\frac{\partial w^*}{\partial\kappa}dsd\kappa+\int_{\Omega}\hat{g}(T^*)\left|\frac{\partial\hat{\Phi}}{\partial\kappa}\right|dsd\kappa$$
$$+\int_{\varpi}\left|\hat{\Phi}\right|\hat{k}ds\geq\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa,\ \forall\hat{\Phi}\in\Sigma(K),\tag{5.3}$$

and

$$-\frac{\partial}{\partial\kappa}\left(K\frac{\partial T^*}{\partial\kappa}\right)+\hat{\alpha}T^*=0\quad in\ L^q(\Omega),\tag{5.4}$$

where

$$W_{\Gamma_1\cup\Gamma_L}=\left\{\hat{\varpi}=(\hat{\varpi}_1,\hat{\varpi}_2)\in W^{1,\nu}(\Omega)^2,\hat{\varpi}=0\ on\ \Gamma_1\cup\Gamma_L\right\},$$

and

$$\tilde{\Sigma}(K)=\left\{\hat{\Phi}=\left(\hat{\Phi}_1,\hat{\Phi}_2\right)\in W^{1,\nu}(\Omega)^2:\hat{\Phi}\ satisfy\ \ D^{`}\right\}.$$

Remark 5.2. The demonstration of this theorem is based on an application of Minty's Lemma.

Proof. For (5.1) considering that $div(\hat{w}^{\epsilon})=0$ in $L^q(\Omega)$ and using Minty's Lemma [17] , then (4.3) is equivalent to:

$$B_1(\ \hat{\varphi},\ \hat{\varphi},\ \hat{\varphi}-\ \hat{w}^{\epsilon})+a_1(\ \hat{T}^{\epsilon},\ \hat{\varphi},\ \hat{\varphi}-\ \hat{w}^{\epsilon})+J_1(\ \hat{T}^{\epsilon},\ \hat{\varphi})-J_1\left(\ \hat{T}^{\epsilon},\ \hat{w}^{\epsilon}\right)$$
$$-(\rho^{\epsilon}\,,div\ \hat{\varphi})\geq(\ \hat{f}^{\epsilon},\ \hat{\varphi}-\ \hat{w}^{\epsilon}),\ \forall\ \hat{\varphi}\in V(\Omega),\tag{5.5}$$

Applying Theorem 4.3 and the lower semicontinuity of the convex functional J_1, i.e. $\left(\liminf J_1\left(\hat{T}^{\epsilon},\hat{w}^{\epsilon}\right)\geq J_1(T^*,w^*)\right)$, we conclude that

$$\sum_{i=1}^{2}\int_{\Omega}\frac{1}{2}\hat{\Lambda}(T^*)\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{\varphi}_i}{\partial\kappa}\right)^2\right)^{\frac{\nu-2}{2}}\frac{\partial(\hat{\varphi}_i)}{\partial\kappa}\frac{\partial(\hat{\varphi}_i-w_i^*)}{\partial\kappa}dsd\kappa-$$
$$\int_{\Omega}\rho^*\left(\frac{\partial\hat{\varphi}_1}{\partial s_1}+\frac{\partial\hat{\varphi}_2}{\partial s_2}\right)dsd\kappa-\int_{\Omega}\rho^*\frac{\partial\hat{\varphi}_3}{\partial\kappa}dsd\kappa+J_1(T^*,\hat{\varphi})-J_1(T^*,w^*)$$
$$\geq\sum_{j=1}^{2}\int_{\Omega}\hat{f}_i(\hat{\varphi}_i-w_i^*)dsd\kappa,$$

and since ρ^* independent of κ then $\int_\Omega \rho^* \frac{\partial \hat{\varphi}_3}{\partial \kappa} ds d\kappa = 0$,so

$$\sum_{i=1}^{2} \int_\Omega \frac{1}{2} \hat{\Lambda}(T^*) \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\varphi}_i}{\partial \kappa} \right)^2 \right)^{\frac{\nu-2}{2}} \frac{\partial (\hat{\varphi}_i)}{\partial \kappa} \frac{\partial (\hat{\varphi}_i - w_i^*)}{\partial \kappa} ds d\kappa$$
$$- \int_\Omega \rho^* \left(\frac{\partial \hat{\varphi}_1}{\partial s_1} + \frac{\partial \hat{\varphi}_2}{\partial s_2} \right) ds d\kappa + J_1(T^*, \hat{\varphi}) \tag{5.6}$$
$$- J_1(T^*, w^*) \geq \sum_{j=1}^{2} \int_\Omega (\hat{\varphi}_i - w_i^*) \hat{f}_i ds d\kappa \ . \tag{5.7}$$

By Minty's Lemma, the expresion (5.2) is equivalent to (5.3). The same for (5.6), when ϵ leads to 0, then the equation (3.4) leads as follow:

$$\int_\Omega \widehat{K} \frac{\partial T^*}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa + \int_\Omega \hat{\alpha}(T^*) ds d\kappa = 0, \quad \forall \hat{\Phi} \in W^{1,q'}_{\Gamma_1 \cup \Gamma_L}(\Omega). \tag{5.8}$$

Using Green's formula, we obtain

$$-\frac{\partial}{\partial \kappa} \left(\widehat{K} \frac{\partial T^*}{\partial \kappa} \right) + \hat{\alpha}(T^*) = 0, \text{ in } L^q(\Omega) \tag{5.9}$$

□

Theorem 5.3. *The variational inequality (4.4) is equivalent to the following system*

$$\int_\Omega \hat{\Lambda}(T^*) \left(\frac{1}{2} \right)^{\frac{\nu}{2}} \left| \frac{\partial w^*}{\partial \kappa} \right|^\nu ds d\kappa + \int_\Omega \hat{g}(T^*) \left| \frac{\partial w^*}{\partial \kappa} \right| ds d\kappa + \int_\varpi \hat{k} |w^*| ds = \int_\Omega \hat{f} w^* ds d\kappa \tag{5.10}$$

and

$$\int_\Omega \hat{\Lambda}(T^*) \left(\frac{1}{2} \right)^{\frac{\nu}{2}} \left| \frac{\partial w^*}{\partial \kappa} \right|^{\nu-2} \frac{\partial w^*}{\partial \kappa} \frac{\partial \hat{\Phi}}{\partial \kappa} ds d\kappa + \int_\Omega \hat{g}(T^*) \left| \frac{\partial \hat{\Phi}}{\partial \kappa} \right| ds d\kappa$$
$$+ \int_\varpi \hat{k} \left| \hat{\Phi} \right| ds \geq \int_\Omega \hat{f} \hat{\Phi} ds d\kappa, \ \forall \hat{\Phi} \in \Sigma(V), \tag{5.11}$$

Proof. taking $\hat{\varphi} = 2w^*$ and $\hat{\varphi} = 0$ in (5.2), respectively. we find (5.10).
For (5.11), we take $\hat{\varphi} = \hat{\Phi} + w^*$ for all $\hat{\Phi} \in \Sigma(V)$. □

Theorem 5.4. *Let us set*

$$\Sigma^* = \tilde{\Sigma}^* - \nabla \rho^* \text{ and } \tilde{\Sigma}^* = \left(\frac{1}{2} \right)^{\frac{\nu}{2}} \hat{\Lambda}(T^*) \left| \frac{\partial w^*}{\partial \kappa} \right|^{\nu-2} \frac{\partial w^*}{\partial \kappa} + \hat{g}(T^*) \pi, \tag{5.12}$$

then

$$-\frac{\partial}{\partial \kappa} \left[\frac{1}{2} \hat{\Lambda}(T^*) \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial w_i^*}{\partial \kappa} \right)^2 \right)^{\frac{\nu-2}{2}} \frac{\partial w^*}{\partial \kappa} + \hat{g}(T^*) \frac{\frac{\partial w^*}{\partial \kappa}}{\left| \frac{\partial w^*}{\partial \kappa} \right|} \right] + \nabla \rho^* = \hat{f} \ , \text{ in } W^{-1,\nu'}(\Omega)^2 . \tag{5.13}$$

with $\|\pi\|_{L^\infty(\Omega)^2} \leq 1$. *and* $\pi \in L^\infty(\Omega)^2$

Proof. If $\frac{\partial w^*}{\partial \kappa} = 0$, and according to we deduce tha $\left| \tilde{\Sigma}^* \right| < \hat{g}(T^*)$. For all $\hat{\Phi} \in \Sigma(V)$, taking $\hat{\Phi} = -\hat{\Phi}$ then $\hat{\Phi} = \hat{\Phi}$, in (5.10), we find

$$\left| F\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa} \right) \right| \leq \int_\Omega \hat{g}(T^*) \left| \frac{\partial \hat{\Phi}}{\partial \kappa} \right| ds d\kappa + \int_\varpi \hat{k} \left| \hat{\Phi} \right| ds$$

where

$$F\left(\hat{k}\hat{\Phi}, \frac{\partial \hat{\Phi}}{\partial \kappa} \right) = \int_\Omega \left(\frac{1}{2} \right)^{\frac{\nu}{2}} \hat{\Lambda}(T^*) \left| \frac{\partial w^*}{\partial \kappa} \right|^{\nu-2} \frac{\partial \hat{\Phi}}{\partial \kappa} \frac{\partial w^*}{\partial \kappa} ds d\kappa - \int_\Omega \hat{f} \hat{\Phi} ds d\kappa \tag{5.14}$$

According to Hahn-Banach's theorem, then, $\exists(\pi,\chi)\in L^{\infty}(\Omega)^2\times L^{\infty}(\varpi)^2$, where $\|\chi\|_{L^{\infty}(\varpi)^2}\leq 1$ $\|\pi\|_{L^{\infty}(\Omega)^2}\leq 1$,with

$$F\left(\hat{k}\hat{\Phi},\frac{\partial\hat{\Phi}}{\partial\kappa}\right)=-\int_{\varpi}\chi\hat{k}\hat{\Phi}ds-\int_{\Omega}\pi\hat{g}\left(T^{*}\right)\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa. \tag{5.15}$$

Furthermore, according to equations (5.9), (5.14), we have:

$$\int_{\Omega}\hat{g}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|dsd\kappa+\int_{\varpi}\hat{k}\left|w^{*}\right|ds=\int_{\varpi}\chi\hat{k}w^{*}ds+\int_{\Omega}\pi\hat{g}\left(T^{*}\right)\frac{\partial w^{*}}{\partial\kappa}dsd\kappa. \tag{5.16}$$

Using the equalities (5.13) and (5.14), we get:

$$\begin{aligned}&\int_{\Omega}\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds+\\&\int_{\Omega}\pi\hat{g}\left(T^{*}\right)\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa-\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa=0.\end{aligned} \tag{5.17}$$

Now, from the equality (5.15), it follows:

$$\int_{\varpi}\hat{k}\left(\left|w^{*}\right|-\chi w^{*}\right)ds+\int_{\frac{\partial w^{*}}{\partial\kappa}\neq 0}\hat{g}\left(T^{*}\right)\left(\left|\frac{\partial w^{*}}{\partial\kappa}\right|-\pi\frac{\partial w^{*}}{\partial\kappa}\right)dsd\kappa=0.$$

As $\|\pi\|_{\Omega,\infty}\leq 1,\|\chi\|_{\varpi,\infty}\leq 1$ We conclude

$$\left|\frac{\partial w^{*}}{\partial\kappa}\right|=\pi\frac{\partial w^{*}}{\partial\kappa}\text{ and }\left|w^{*}\right|-\chi w^{*}.$$

Moreover, if $\left|\frac{\partial w^{*}}{\partial\kappa}\right|\neq 0$, by (5.11), we have

$$\tilde{\Sigma}^{*}=\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}+\hat{g}\left(T^{*}\right)\frac{\partial w^{*}/\partial\kappa}{\left|\partial w^{*}/\partial\kappa\right|}.$$

then, $\left|\tilde{\Sigma}^{*}\right|=\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-1}+\hat{g}\left(T^{*}\right)>\hat{g}\left(T^{*}\right),$

Thus:

$$\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\hat{\Lambda}\left(T^{*}\right)\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}=\begin{cases}\tilde{\Sigma}^{*}-\hat{g}\left(T^{*}\right)\dfrac{\partial w^{*}/\partial\kappa}{\left|\partial w^{*}/\partial\kappa\right|},\text{ if }\left|\tilde{\Sigma}^{*}\right|>\hat{\alpha},\\0,\quad\text{if }\left|\tilde{\Sigma}^{*}\right|\leq\hat{\alpha}\end{cases} \tag{5.18}$$

From equation (5.16), there exists $p^{*}\in L^{r'}(\Omega)^2$ such that

$$\begin{aligned}&\int_{\Omega}\hat{\Lambda}\left(T^{*}\right)\left(\frac{1}{2}\right)^{\frac{\nu}{2}}\left|\frac{\partial w^{*}}{\partial\kappa}\right|^{\nu-2}\frac{\partial w^{*}}{\partial\kappa}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds+\\&\hat{\alpha}\int_{\Omega}\pi\hat{g}\left(T^{*}\right)\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa-\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa=-\int_{\Omega}\nabla\rho^{*}\hat{\Phi}dsd\kappa,\ \forall\hat{\Phi}\in\Sigma(V)\end{aligned} \tag{5.19}$$

Using (5.17)and (5.18) becomes

$$\int_{\Omega}\tilde{\Sigma}^{*}\frac{\partial\hat{\Phi}}{\partial\kappa}dsd\kappa+\int_{\varpi}\chi\hat{k}\hat{\Phi}ds=\int_{\Omega}\hat{f}\hat{\Phi}dsd\kappa-\int_{\Omega}\nabla\rho^{*}\hat{\Phi}dsd\kappa, \tag{5.20}$$

From (5.19), we deduce (5.12) with $\hat{\Phi}\in W_0^{1,\nu}(\Omega)^2$

□

In the next theorem, we prove that our problem converges to the Reynolds equation:

Corollary 5.5. *Under the same assumptions as in the above theorems, the following holds*

$$\begin{aligned}&\int_{\varpi}\left[\frac{h^{3}}{12}\nabla\rho^{*}+\tilde{F}+\int_{0}^{h}\int_{0}^{y}A^{*}(s,\zeta)\hat{\Lambda}\left(T^{*}(s,\zeta)\right)\frac{\partial w^{*}(s,\xi)}{\partial\xi}d\xi dy\right.\\&\left.-\frac{h}{2}\int_{0}^{h}\hat{g}\left(T^{*}(s,\zeta)\right)\left|\frac{\partial u^{*}}{\partial\kappa}\right|(s,\xi)\,d\xi\right].\nabla\vartheta(s)\,ds=0\ \forall\vartheta\in W^{1,\nu}(\varpi)\end{aligned} \tag{5.21}$$

where

$$\tilde{F}(s) = \int_0^h F(s,y)\,dy - \frac{h}{2}F(s,h),\; F(s,y) = \int_0^h \int_0^\xi \hat{f}(s,t)\,dt d\xi$$

$$A^*(s,\xi) = \frac{1}{2}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial w^*}{\partial \kappa}(s,\xi)\right)^2\right)^{\frac{\nu-2}{2}}.$$

Proof. To prove (5.20), we perform a double integration of equation (5.12) over the interval $[0,\kappa]$. Setting $\kappa = h$, we get the desired result. □

Remark 5.6. The equality (5.20) admits a unique solution and is $(w^*, \rho^*, T^*) \in V_\kappa \times L_0^{\nu'}(\varpi) \times W^{-1,q}(\Omega)$

6. Conclusion

In this work, we investigated the asymptotic behavior of a non-isothermal, incompressible Herschel–Bulkley fluid in a thin domain governed by a nonlinear stationary model with Tresca friction. We first analyzed the limit as the domain thickness tends to zero, then established strong convergence of the velocity, temperature, and pressure fields. Finally, we derived the corresponding limit problem and obtained the associated Reynolds-type equation. Future research may explore several extensions of the present study. One direction is the analysis of time-dependent (unsteady) flows of Herschel–Bulkley fluids in thin domains, which would better capture transient behaviors.

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