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COMPLETE SECOND MOMENT CONVERGENCE RATE PROPERTY ESTIMATOR UNDER A GENERAL CENSORING MODEL

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Abstract. This work uses a new mode of convergence, called complete second moments convergence which represents an innovation in the study of convergences with rates. It implies the rates of other types of convergence such as almost complete convergence and mean square error convergence. The advantage of this mode is that it improves the rates obtained using these two types of convergence. In addition to showing this property on the asymptotic efficiency of the estimator, we propose a new nonparametric estimator of the conditional density function based on the use of two kernels applied to the conditional variables, where the observations are subject to a general censorship including left, right, double and twice censoring.

Keywords. Complete second moments, General censorship, Conditional probability density, Almost complete uniform convergence, Non-parametric estimation.
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1. Introduction

The asymptotic rate of convergence is a crucial concept that quantifies the speed at which a sequence converges to the limit. In this situation and inspired by the work of Madi and Laroussi (2022) [15], we investigate a new mode of convergence, to be exact the properties of complete second order convergence with a rate. The importance of this notion is due to the fact that complete second order convergence (c.s.m.) leads to almost complete convergence (a.c.). The latter is obtains by second-order convergence in different work, like Chow (1988) [4], Liang and Rosalsky (2010) [14] and recently Yu and al (2022) [24]. However, Hsu and Robbins (1947)[6] proposed the notion of almost complete convergence. More recently, this concept has reappeared and is taken up by the community of statisticians to demonstrate the efficiency of nonparametric estimators and Ferraty and Vieu (2006) [5] have summarized this type of convergence. As a first contribution, we study this type of convergence for nonparametric estimation of

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predictive functions. In this field, the kernel method plays a crucial role. Many are the works devoted to this subject such as Roussas (1968) [18], Youndjé (1993) [23], more recently Laksaci and Yousfate (2002) [11]. Hyndman, Bashtannyk and Grunwald (1996) [7], Arora and Taylor (2016) [2] and Wen and Wu (2017) [22] introduced estimation using the kernel method on the pair of random variables (X, Y). And this, to estimate the conditional density function in the context of real data. Second, it is well established that in many situations one cannot observe the variable of interest. This is the case when Y is right-censored, which means that the observation does not correspond to the true realization of Y but that it is inferior to it. Among the works on conditional density and conditional mode in the presence of censored data, we cite in the case of independent data and identically distributed models, Ould Saïd and Cai (2005) [16] who defined a new conditional density estimator and established the uniform almost sure convergence with rate of the proposed estimator in a right-censored model. Salha and El Shekh Ahmed (2009) [19] proposed a modified estimator of the conditional mode, they established the normality and the asymptotic consistency of the proposed estimator whose effectiveness they tested through a simulation study and an application to real data. In the sequel, Khardani, Lemdani and Ould Saïd (2010) [9] defined another conditional mode estimator and established its almost sure convergence and asymptotic normality. Khardani and Semmar (2014) [10] employed a recursive version of the conditional density estimator for which they investigated uniform consistency and asymptotic normality. Kebabi and Messaci (2012) [8] applied almost complete convergence on the regression function for twice censored data. And obtained the almost complete uniform convergence which makes it possible to obtain results on the conditional mode directly. Always, for the same censorship model, Aouicha and Messaci (2019) [1] have calculated the mean square convergence rate of the conditional density estimator. Finally, we propose an estimator for a general censoring model, which groups left, right, double and twice censoring. This type of censoring was proposed by Boukeloua and Messaci (2016) [3] in the estimation of the density function by a kernel applied to X. It was used by Laroussi (2021) [12] for the estimation of the regression function by the least squares method. And also reused by Laroussi (2024) [13] for the B-Spline estimate of the regression function. As a practical Framework, rates of complete second order moment convergence for the conditional density estimator assumes known model for probability of censoring. Our article contains a presentation of the mode of convergence section, a construction section of the estimator with the assumptions used. A section bringing together the results dedicated to the types of convergence with rates. Finally a proofs section.

2. Presentation of the mode of convergence

In all this paper, real valued random variables are defined on a fixed probability space $(\Omega, \mathcal{A}, P)$.

Definition 2.1. A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is considered almost complete second order moment convergent (c.s.m.) to the random variable X if,

$$\sum_{n\in\mathbb{N}} U_n E(X_n - X)^2 < \infty,$$

where $(U_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers.

The following theorem establishes that, if $(X_n)_{n\in\mathbb{N}}$ converges in complete second order moment to X with a rate $\frac{1}{\sqrt{U_n}}$, then it is completely convergent with a rate of $\frac{1}{\sqrt{U_n}}$.

Theorem 2.2. *Madi and Laroussi* [15]
If a sequence $(X_n)_{n\in\mathbb{N}}$ *of random variables verifies*

$$\sum_{n\in\mathbb{N}} U_n E(X_n - X)^2 < \infty,$$

then

$$X_n - X = O_{a.c.}\left(\frac{1}{\sqrt{U_n}}\right).$$

Where $U_n \to +\infty$ *for* $n \to +\infty$.

One use the fact that $U_n E(X_n - X)^2 \geq P\left(|X_n - X| \geq \frac{1}{\sqrt{U_n}}\right)$, which implies

$$\sum_{n\in\mathbb{N}} P\left(|X_n - X| \geq \frac{1}{\sqrt{U_n}}\right) \leq \sum_{n\in\mathbb{N}} U_n E(X_n - X)^2 < \infty.$$

Remark 2.3. The fact that $U_n \to +\infty$ for $n \to +\infty$ is not a condition for the second moment order convergence like we will see in Theorem 2.4.

Theorem 2.4. *If a sequence* $(X_n)_{n\in\mathbb{N}}$ *of random variables verifies*

$$\sum_{n\in\mathbb{N}} U_n E(X_n - X)^2 < \infty,$$

then

$$X_n - X = O_{a.c.}\left(\frac{1}{\sqrt{R_n}}\right).$$

Where $R_n \underset{n\to+\infty}{\to} +\infty$ *where we write* $R_n = v_n U_n, v_n \underset{n\to+\infty}{\to} +\infty$.

Proof. The proof is done in the same way as the previous theorem, using R_n instead of U_n and the fact that

$$\sum_{n\in\mathbb{N}} \frac{1}{v_n} P\left(|X_n - X| \geq \frac{1}{\sqrt{R_n}}\right) \leq \infty \Rightarrow \left[\frac{1}{v_n} P\left(|X_n - X| \geq \frac{1}{\sqrt{R_n}}\right) \underset{n\to+\infty}{\to} 0\right.,$$

if and only if

$$\left. P\left(|X_n - X| \geq \frac{1}{\sqrt{R_n}}\right) \underset{n\to+\infty}{\to} 0\right],$$

almost surely. □

3. Construction of estimator and hypothesis

In this work, the existence of the joint probability density (with respect to the Lebesgue measure) f of the pair real random variables (X, Y) and f_X (the density of X) and $f(y|x)$ (the probability density function of Y given $X = x$) are presupposed. Let us consider independent identically distributed variables (i.i.d.) $(X_i, Z_i, \delta_i)_{1\leq i\leq n}$ of the same law as (X, Z, δ), $Z = Y$ if and only if the indicator of censoring δ takes the value 0. Inspired by Boukeloua and Messaci (2016) [3] and more recently Laroussi (2021) [13]. We use the general censoring model defined by $\phi(y) = P(\delta = 0|Y = y)$. We write $Z = \max(\min(Y, R), L)$, when Y is right censored by real random variable R and $\min(Y, R)$ is left censored by L. If Y, L and R are positive, independent and $\delta = 1_{L<R<Y} + 2 \times 1_{min(Y,R)\leq L}$, Patilea and Rolin [17] obtained that $\phi(y) = F_L(y)S_R(y)$, it is a twice censored data model. Under the assumption that $L = 0$, we obtain the right censored data model and we get $\phi(y) = S_R(y)$. Turnbull [21] suggests doubly censored data model, where Y is independent of the pair (L, R) and $P(0 \leq L \leq R) = 1$ and $\delta = 1_{Y>R}+2\times 1_{Y<L}$, we get $\phi(y) = S_R(y) - S_L(y)$.

The conditional density function of Y given X is

$$f(y|x) = \frac{f(x, y)}{f_X(x)},$$

where $f(., .)$ is the joint density function of (X, Y) and $f_X(.)$ is the density function of X. Given observations (X_i, Y_i), $i = 1, ..., n$, we introduce a kernel estimator of the conditional density function $f(y|x)$ such that, Y is censored, by

$$\hat{f}_n(y|x) = \sum_{i=1}^{n} W_{n,i}(x)1_{\{\delta_i=0\}}\frac{\frac{1}{b_n}H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)}, \tag{3.1}$$

where

$$W_{n,i}(x) = \frac{H\left(\frac{x-X_i}{a_n}\right)}{\sum_{i=1}^{n} H\left(\frac{x-X_i}{a_n}\right)}$$

is the weight function. H and H_0 are real kernel functions, with $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bandwidths successively which converge to zero. For more simplicity, we decompose the hypotheses in two parts. Let S be a compact subset of $\mathbb{R}$. We can determine the rate of the uniform almost complete convergence and the mean squared error of $\hat{f}_n(y|x)$ on S thanks to the following assumptions.

$\mathbf{C_1}$ $f_X > 0$ is twice continuously differentiable around x.
$\mathbf{C_2}$ $\lim_{n\to\infty} na_nb_n = +\infty, \lim_{n\to\infty} \frac{\log n}{na_n} = 0$.
$\mathbf{C_3}$ The joint density $f(., .)$ of the pair (X, Y) is twice continuously differentiable.
$\mathbf{C_4}$ $\int tH(t)dt = 0$ where H is a bounded density with compact support.
$\mathbf{C_5}$ $\int tH_0(t)dt = 0$ where H_0 is a bounded density with compact support.
$\mathbf{C'_5}$ $\int tH_0(t)dt = 0$ where H_0 is a bounded density with compact support, moreover H_0 is Holderian, which is written

$$\exists\beta > 0, \exists C > 0 \text{ such as } \forall x, y \in \mathbb{R},\ |H_0(x) - H_0(y)| \leq C|x - y|^{\beta}.$$

For the censor function, we need the following assumption

$\mathbf{C_6}$ $\exists I > 0$ such that $\forall n \in \mathbb{N}, \forall\ 0 \leq i \leq n : \delta_i = 0,\ I \leq Z\ a.s$ and $\inf_{t\in[x-I,x+I]} \phi(t) > 0$.

4. Results

Our first result is the uniform almost complete convergence with rates of the conditional density function as stated in Theorem (4.1) and the second result deals with the mean square convergence and given in Theorem (4.2). The third one (4.3) give the complete second moment convergence.

Theorem 4.1. *Under the hypotheses* $\mathbf{C_1} - \mathbf{C_4}$, $\mathbf{C_5'}$ *and* $\mathbf{C_6}$, *we have*

$$\sup_{y\in S}\left|\hat{f}_n(y|x) - f(y|x)\right| = O\left({a_n}^2 + {b_n}^2\right) + O_{a.c}\left(\sqrt{\frac{\log n}{na_nb_n}}\right).$$

Theorem 4.2. *Under conditions* $\mathbf{C_1} - \mathbf{C_6}$, *we have*

$$\sup_{y\in S} E(\hat{f}_n(y|x) - f(y|x))^2 = O(a_n^4 + a_n^2b_n^2 + b_n^4) + O\left(\frac{1}{na_nb_n}\right).$$

In the following, we propose the new type of convergence, which produces a better almost complete convergence rate. Using the formula

$$\sum_{n\geq 1} U_nE(\hat{f}_n(y|x) - f(y|x))^2 < \infty \Longrightarrow \sum_{n\geq 1} P\left[\hat{f}_n(y|x) - f(y|x)| > \frac{1}{\sqrt{U}_n}\right] < \infty.$$

where $\lim_{n\to\infty} U_n = +\infty$, we obtain the following theorem

Theorem 4.3. *According to the conditions* $\mathbf{C_1}, \mathbf{C_3} - \mathbf{C_6}$ *and supposing that* $\sum_{n\geq 1}\frac{U_n}{na_nb_n}$ *and* $\sum_{n\geq 1} U_n(a_n^4 + b_n^4)$ *converges, we have*

$$\hat{f}_n(y|x) - f(y|x) = O_{c.s.m}\left(\frac{1}{\sqrt{U}_n}\right). \tag{4.1}$$

Where $(U_n)_{n\in\mathbb{N}}$ *is a non-negative sequence, such that* $\lim_{n\to\infty} U_n = +\infty$.

The corollary above presents two general examples of Theorem 4.3.

Corollary 4.4. *Under conditions* $\mathbf{C_1}, \mathbf{C_3} - \mathbf{C_6}$. *We have*

$$\hat{f}_n(y|x) - f(y|x) = O_{a.c}\left(\sqrt{\frac{\log n}{n^a a_n^2 b_n^2}}\right),$$

if $a > 8$ *and* $a_nb_n = O(\frac{1}{n^\beta}), \beta > 3$. *If* $a \in]0, 1[$ *and* $a_nb_n = O(\frac{1}{n^\beta \log n}), \beta \in]0, 1[$.

5. The proof of results

In this section C represents a generic constant, and before starting the proof of the previous theorems, note that, under condition $\mathbf{C_3}$ and a Taylor's expansion around (x, y), we obtain

$$\sup_{y\in S} E\left[\frac{1}{a_n b_n} H\left(\frac{x-X}{a_n}\right) H_0\left(\frac{y-Y}{b_n}\right)\right] = \sup_{y\in S} \int\int H(s)H_0(v)f(x-sa_n, y-vb_n)dsdv.$$

This, with conditions $\mathbf{C_4}$ and $\mathbf{C_5}$, shows that the quantity

$$\sup_{y\in S} E\left[\frac{1}{a_n b_n} H\left(\frac{x-X}{a_n}\right) H_0\left(\frac{y-Y}{b_n}\right)\right] \tag{5.1}$$

is bounded.

Proof. of the Theorem 4.1
The proof of this theorem relies on the Definition (3.1) and the following equality

$$\hat{f}_n(y|x) - f(y|x) = \left(\frac{\hat{f}_{n,N}(x,y) - f(x,y)}{f_n(x)}\right) + f(y|x)\left(\frac{f_X(x) - f_n(x)}{f_n(x)}\right),$$

where,

$$\hat{f}_n(y|x) = \frac{\hat{f}_{n,N}(x,y)}{f_n(x)},$$

with

$$\hat{f}_{n,N}(x,y) = \frac{1}{na_n b_n}\sum_{i=1}^{n} 1_{\{\delta_i=0\}} \frac{H\left(\frac{x-X_i}{a_n}\right) H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)},$$

and

$$f_n(x) = \frac{1}{na_n}\sum_{i=1}^{n} H\left(\frac{x-X_i}{a_n}\right)$$

is the nonparametric kernel estimator of $f_X(x)$.

Using the results of Kebabi and Messaci [8] and according to the hypotheses $\mathbf{C_1}$, $\mathbf{C_2}$ and $\mathbf{C_4}$ we obtain $Ef_n(x) - f_X(x) = O(a_n^2)$, $Ef_n(x) - f_n(x) = O_{a.c}\left(\sqrt{\frac{\log n}{na_n}}\right)$ and $\exists \eta > 0, \sum_{n=1}^{\infty} P(f_n(x) < \eta) < \infty$. It is sufficent to prove the following lemmas. □

Lemma 5.1. *According to the assumptions* $\mathbf{C_3} - \mathbf{C_5}$, *we get*

$$\sup_{y\in S}\left|E\hat{f}_{n,N}(x,y) - f(x,y)\right| = O(a_n^2 + b_n^2).$$

Proof. Since (X_i, Z_i, A_i) are i.i.d., we have

$$E\hat{f}_{n,N}(x,y) - f(x,y) = E\left(\frac{1_{\{\delta_1=0\}}}{a_n b_n} \times \frac{H\left(\frac{x-X_1}{a_n}\right) H_0\left(\frac{y-Z_1}{b_n}\right)}{\phi(Z_1)}\right) - f(x,y).$$

On the other hand, as

$$E\left(1_{\{\delta=0\}}\middle|X, Y\right) = \phi(Y), \tag{5.2}$$

and using the conditional expectation, we get

$$\begin{aligned} E\hat{f}_{n,N}(x,y) - f(x,y) &= E\left(\frac{1}{a_n b_n} H\left(\frac{x-X_1}{a_n}\right) H_0\left(\frac{y-Y_1}{b_n}\right)\right) - f(x,y) \\ &= \int\int \frac{1}{a_n b_n} H\left(\frac{x-u}{a_n}\right) H_0\left(\frac{y-v}{b_n}\right) f(u,v)dudv - f(x,y) \\ &= \int\int H(r)H_0(s)f(x-ra_n, y-sb_n)drds - f(x,y). \end{aligned}$$

By using the hypotheses $\mathbf{C_3} - \mathbf{C_5}$ and the Taylor expansion around (x,y), we obtain

$$\sup_{y\in S}\left|E\hat{f}_{n,N}(x,y) - f(x,y)\right| = O(a_n^2 + b_n^2).$$

□

Lemma 5.2. *From the hypotheses* $\mathbf{C_2}$, $\mathbf{C_4}$, $\mathbf{C_5'}$ *and* $\mathbf{C_6}$ *we have*

$$\sup_{y\in S}\left|E\hat{f}_{n,N}(x,y) - \hat{f}_{n,N}(x,y)\right| = O_{a.c}\left(\sqrt{\frac{\log n}{na_n b_n}}\right).$$

Proof. The compactness of S, we allow to write

$$S \subset \bigcup_{j=1}^{\tau_n}]t_j - l_n, t_j + l_n[,$$

with

$$l_n = C{\tau_n}^{-1} \text{ et } \tau_n = Cn^{1+\frac{1}{\beta}}, \text{ tel que } \beta > 0. \tag{5.3}$$

Let's pose

$$t_y = \arg\min_{t\in\{t_1,\dots,t_{\tau_n}\}} |y-t|.$$

The proof is established using the following inequality

$$\sup_{y\in S}\left|\hat{f}_{n,N}(x,y) - E\hat{f}_{n,N}(x,y)\right| \le A_1 + A_2 + A_3,$$

where

$$A_1 = \sup_{y\in S}\left|\hat{f}_{n,N}(x,y) - \hat{f}_{n,N}(x,t_y)\right|,$$

$$A_2 = \sup_{y\in S}\left|\hat{f}_{n,N}(x,t_y) - E\hat{f}_{n,N}(x,t_y)\right|,$$

and

$$A_3 = \sup_{y\in S}\left|E\hat{f}_{n,N}(x,t_y) - E\hat{f}_{n,N}(x,y)\right|.$$

Treatment of the term A_1 :
Using the hypotheses $\mathbf{C_4}$, $\mathbf{C_5'}$ and $\mathbf{C_6}$ we obtain

$$\begin{aligned} A_1 &= \sup_{y\in S}\frac{1}{na_nb_n}\left|\sum_{i=1}^n H\left(\frac{x-X_i}{a_n}\right)\frac{1_{\{\delta_i=0\}}H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)} - \sum_{i=1}^n H\left(\frac{x-X_i}{a_n}\right)\frac{1_{\{\delta_i=0\}}H_0\left(\frac{t_y-Z_i}{b_n}\right)}{\phi(Z_i)}\right| \\ &\leq \frac{{l_n}^\beta}{na_n{b_n}^{\beta+1}\phi(I)}\sum_{i=1}^n H\left(\frac{x-X_i}{a_n}\right) \\ &\leq \frac{C{l_n}^\beta}{a_n{b_n}^{\beta+1}}. \end{aligned}$$

$A_1 = O\left(\sqrt{\frac{\log n}{na_nb_n}}\right)$ because of the relation (5.3) and the condition $\mathbf{C_2}$.
Treatment of the term A_2 :
For $\epsilon > 0$,

$$\begin{aligned} P(A_2>\epsilon) &\leq P\left(\max_{j=1,\ldots,\tau_n}\left|\hat{f}_{n,N}(x,t_j)-E\hat{f}_{n,N}(x,t_j)\right|>\epsilon\right) \\ &\leq \tau_n\max_{j=1,\ldots,\tau_n}P\left(\left|\hat{f}_{n,N}(x,t_j)-E\hat{f}_{n,N}(x,t_j)\right|>\epsilon\right) \\ &\leq \tau_n\max_{j=1,\ldots,\tau_n}P\left(\frac{1}{n}\left|\sum_{i=1}^n\Gamma_i\right|>\epsilon\right), \end{aligned}$$

where

$$\Gamma_i = V_i - EV_i \text{ avec } V_i = \frac{1}{a_nb_n}1_{\{\delta_i=0\}}\frac{H\left(\frac{x-X_i}{a_n}\right)H_0\left(\frac{t_j-Z_i}{b_n}\right)}{\phi(Z_i)}.$$

Applying Bernstein's inequality (see Ferraty and Vieu [5, Corollary A.9.i]) and using the hypotheses $\mathbf{C_4}$, $\mathbf{C_5}$ and $\mathbf{C_6}$ we get

$$|\Gamma_i| \leq \frac{C}{a_nb_n} := M.$$

To bound $\sigma^2 = Var\Gamma_i$, it suffices to bound $E(V_i)^2$. Since the kernels H and H_0 are positive and bounded, we have

$$E{V_i}^2 \leq CE\left[\frac{1}{{a_n}^2{b_n}^2}H\left(\frac{x-X_i}{a_n}\right)H_0\left(\frac{t_j-Y_i}{b_n}\right)\right].$$

This combined with the relation (5.1) leads to

$$\sigma^2 \leq \frac{C}{a_nb_n}.$$

For $\epsilon < \frac{\sigma^2}{M}$, we obtain

$$P\left[\frac{1}{n}\left|\sum_{i=1}^n\Gamma_i\right|>\epsilon\right] \leq 2\exp\left\{-\frac{n\epsilon^2}{2\sigma^2(1+\epsilon\frac{M}{\sigma^2})}\right\} \leq 2\exp\left\{-\frac{n\epsilon^2a_nb_n}{4C}\right\}.$$

Posing $\epsilon = \epsilon_0\sqrt{\frac{\log n}{na_nb_n}}$, we obtain then

$$P\left[\frac{1}{n}\left|\sum_{i=1}^{n}\Gamma_i\right| > \epsilon_0\sqrt{\frac{\log n}{na_nb_n}}\right] \leq 2n^{-\frac{\epsilon_0{}^2}{4C}}.$$

From where by applying the relation (5.3) and with an adequate choice of ϵ_0, we obtain

$$A_2 = O_{a.c}\left(\sqrt{\frac{\log n}{na_nb_n}}\right).$$

Treatment of the term A_3 :
It is enough to notice that $|A_3| \leq EA_1$.

□

Proof. of the Theorem 4.2
Let us remember that

$$\hat{f}_n(y|x) = \frac{\hat{f}_{n,N}(x,y)}{f_n(x)} = \frac{\frac{1}{na_nb_n}\sum_{i=1}^{n} 1_{\{\delta_i=0\}}\frac{H\left(\frac{x-X_i}{a_n}\right)H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)}}{\frac{1}{na_n}\sum_{i=1}^{n} H\left(\frac{x-X_i}{a_n}\right)},$$

where $f_n(x) = \frac{1}{na_n}\sum_{i=1}^{n} H\left(\frac{x-X_i}{a_n}\right)$ and

$$\hat{f}_{n,N}(x,y) = \frac{1}{na_nb_n}\sum_{i=1}^{n} 1_{\{\delta_i=0\}}\frac{H\left(\frac{x-X_i}{a_n}\right)H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)}. \tag{5.4}$$

In the same way as Aouicha and Messaci [1], we have under the conditions $\mathbf{C_1}$, $\mathbf{C_3}$ and $\mathbf{C_5}$

$$\lim_{n\to\infty}\frac{1}{a_n}E\left(H\left(\frac{x-X}{a_n}\right)\right) = \lim_{n\to\infty} E(f_n(x)) = f_X(x), \tag{5.5}$$

and

$$\mathrm{Var}(f_n(x)) = O\left(\frac{1}{na_n}\right). \tag{5.6}$$

Then, we need to deal with the following point. The independence and the equidistribution of $(X_i, Y_i)_{1\leq i\leq n}$ combined with relation (5.1), condition $\mathbf{C_6}$, the boundless of H and H_0 permit to write

$$\begin{aligned}\sup_{y\in S}\mathrm{Var}(\hat{f}_{n,N}(x,y)) &\leq \frac{1}{n^2a_n^2b_n^2}\sup_{y\in S}\sum_{i=1}^{n} E\left[\frac{H\left(\frac{x-X_i}{a_n}\right)H_0\left(\frac{y-Z_i}{b_n}\right)}{\phi(Z_i)}1_{\{\delta_i=0\}}\right]^2 \\ &\leq \frac{C}{na_nb_n\phi^2(I)}\sup_{y\in S} E\left[\frac{1}{a_nb_n}H\left(\frac{x-X}{a_n}\right)H_0\left(\frac{y-Y}{b_n}\right)\right].\end{aligned}$$

We get then the following result

$$\sup_{y\in S}\mathrm{Var}(\hat{f}_{n,N}(x,y)) = O\left(\frac{1}{na_nb_n}\right). \tag{5.7}$$

Now, using $\hat{f}_n(y|x) = \frac{\hat{f}_{n,N}(x,y)}{f_n(x)}$, where $\hat{f}_{n,N}(x,y)$ is defined in (5.4), we use the following decomposition to establish the proof

$$E(\hat{f}_n(y|x) - f(y|x))^2 \leq 2\text{Var}\hat{f}_n(y|x) + 2(E\hat{f}_n(y|x) - f(y|x))^2.$$

Treatment of the term $(E\hat{f}_n(y|x) - f(y|x))^2$.
By applying the usual identity $\frac{1}{U} = \frac{1}{V} - \frac{U-V}{V^2} + \frac{(U-V)^2}{UV^2}(V \neq 0, U \neq 0)$, with $U = f_n(x)$ and $V = Ef_n(x)$, we deduce the following inequality

$$[E\hat{f}_n(y|x) - f(y|x)]^2 \leq 4\left\{\frac{E\hat{f}_{n,N}(x,y) - f(x,y)}{Ef_n(x)}\right\}^2 + 4\left\{\frac{f(y|x)\,[Ef_n(x) - f_X(x)]}{Ef_n(x)}\right\}^2 + 4\Big\{\frac{E\left[\hat{f}_n(y|x)(f_n(x) - Ef_n(x))^2\right]}{[Ef_n(x)]^2}\Big\}^2 + 4\Big\{\frac{\text{Cov}\left[\hat{f}_{n,N}(x,y), f_n(x)\right]}{[Ef_n(x)]^2}\Big\}^2.$$

For further details, refer to the work of Aouicha and Messaci [1]. Using condition $\mathbf{C_2}$ and formula (5.5), we get

$$\sup_{y\in S}(E\hat{f}_n(y|x) - f(y|x))^2 \leq C(T_{1,n} + T_{2,n} + T_{3,n} + T_{4,n}),$$

where $T_{j,n}$ $(1 \leq j \leq 4)$ are defined and treated below.

As (X_i, Z_i, A_i) are i.i.d. and applying formula (5.4), we obtain $E\hat{f}_{n,N}(x,y) - f(x,y) = E\left(\frac{1}{a_nb_n}1_{\{\delta=0\}}\frac{H(\frac{x-X}{a_n})H_0(\frac{y-Y}{b_n})}{\phi(Y)}\right) - f(x,y)$.
Moreover, relation (5.2) yield to

$$E\hat{f}_{n,N}(x,y) - f(x,y) = E\left[\frac{1}{a_nb_n}H(\frac{x-X}{a_n})H_0(\frac{y-Y}{b_n})\right] - f(x,y).$$

Proceeding as to prove (5.1), we get
$\sup_{y\in S}\left|E\hat{f}_{n,N}(x,y) - f(x,y)\right| = O(a_n^2 + b_n^2)$, we deduce that
$T_{1,n} := \sup_{y\in S}\left(E\hat{f}_{n,N}(x,y) - f(x,y)\right)^2 = O({a_n}^4 + {b_n}^4 + a_n^2b_n^2)$.

Under the hypotheses $\mathbf{C_2}$ and $\mathbf{C_4}$ and by using Aouicha and Messaci [1], we obtain $T_{2,n} := \sup_{y\in S}\left[f(y|x)\,(Ef_n(x) - f_X(x))\right]^2 = O({a_n}^4)$.

Since $\hat{f}_n(y|x) = \sum_{i=1}^n W_{n,i}(x)1_{\{\delta_i=0\}}\frac{\frac{1}{b_n}H_0(\frac{y-Z_i}{b_n})}{\phi(Z_i)}$ and using $\sum_{i=1}^n |W_{n,i}(x)| = 1$, we have under the hypotheses $\mathbf{C_4}$, $\mathbf{C_5}$ and $\mathbf{C_6}$ in addition to the formula (5.6), $T_{3,n} := \left\{E\left[\hat{f}_n(y|x)(f_n(x) - Ef_n(x))^2\right]\right\}^2 \leq \left[\frac{C}{g_n}\text{Var}\left(f_n(x)\right)\right]^2 = O\left(\frac{1}{na_nb_n}\right)$.

applying Cauchy-Shwartz inequality along with the use of (5.7) and (5.6), we obtain

$$\begin{aligned} T_{4,n} &:= \sup_{y\in S}\left[\operatorname{Cov}\left(\hat{f}_{n,N}(x,y), f_n(x)\right)\right]^2 \\ &\leq \sup_{y\in S}\left[\operatorname{Var}\left(\hat{f}_{n,N}(x,y)\right)\operatorname{Var}\left(f_n(x)\right)\right] \\ &= O\left(\frac{1}{n^2 a_n^2 b_n}\right), \end{aligned} \tag{5.8}$$

which, under the hypothesis $\mathbf{C_3}$, gives $T_{4,n} = O\left(\frac{1}{na_nb_n}\right)$.

Then, we can deduce that

$$\sup_{y\in S}\left(E\hat{f}_n(y|x) - f(y|x)\right)^2 = O\left(\frac{1}{na_nb_n} + a_n^4 + b_n^4 + a_n^2 b_n^2\right). \tag{5.9}$$

Treatment of the term $\sup_{y\in S}\operatorname{Var}(\hat{f}_n(y|x))$.

The variance of $\hat{f}_n(y|x)$ is treated using the following identity (see Scott [20, formula 8.12]).

$$\begin{aligned} \operatorname{Var}(\hat{f}_n(y|x)) = {} & \frac{\operatorname{Var}\left(\hat{f}_{n,N}(x,y)\right)}{(Ef_n(x))^2} - 2E\hat{f}_{n,N}(x,y)\frac{\operatorname{Cov}\left(\hat{f}_{n,N}(x,y), f_n(x)\right)}{(Ef_n(x))^3} \\ & + \operatorname{Var}(f_n(x))\frac{\left(E\hat{f}_{n,N}(x,y)\right)^2}{(Ef_n(x))^4}. \end{aligned}$$

Since Ef_n converges and according to the condition $\mathbf{C_2}$, we obtain

$$\sup_{y\in S}\operatorname{Var}(\hat{f}_n(y|x)) \leq C(T'_{1,n} + T'_{2,n} + T'_{3,n}).$$

It suffices to examine each term $T'_{1,n}$, $T'_{2,n}$ and $T'_{3,n}$.

Relation (5.7) leads to $T'_{1,n} := \sup_{y\in S}\operatorname{Var}\left(\hat{f}_{n,N}(x,y)\right) = O\left(\frac{1}{na_nb_n}\right)$.

Under relations (5.4) and (5.1), we see that $\sup_{y\in S} E\hat{f}_{n,N}(x,y)$ is bounded. So proceeding as to prove (5.8) with the use of the condition $\mathbf{C_3}$, we get that

$$T'_{2,n} := \sup_{y\in S}\left[E\hat{f}_{n,N}(x,y)\operatorname{Cov}\left(\hat{f}_{n,N}(x,y), f_n(x)\right)\right] = O\left(\frac{1}{na_nb_n}\right).$$

Using relation (5.6), hypothesis $\mathbf{C_3}$ and the fact that $\sup_{y\in S} E\hat{f}_{n,N}(x,y)$ is bounded, we obtain

$$T'_{3,n} := \operatorname{Var}\left(f_n(x)\right)\sup_{y\in S}\left(E\hat{f}_{n,N}(x,y)\right)^2 = O\left(\frac{1}{na_nb_n}\right).$$

So, we can concluding that

$$\sup_{y\in S}\operatorname{Var}(\hat{f}_n(y|x)) = O\left(\frac{1}{na_nb_n}\right). \tag{5.10}$$

□

Proof. of the Theorem 4.3
Using the following decomposition

$$E(\hat{f}_n(y|x) - f(y|x))^2 \leq 2\mathrm{Var}\hat{f}_n(y|x) + 2(E\hat{f}_n(y|x) - f(y|x))^2,$$

it suffices to show that

$$\sum_{n\geq 1} U_n Var\hat{f}_n(y|x) < +\infty$$

and

$$\sum_{n\geq 1} U_n (E\hat{f}_n(y|x) - f(y|x))^2 < +\infty.$$

From the relation (5.9), we obtain

$$\sum_{n\geq 1} U_n (E\hat{f}_n(y|x) - f(y|x))^2 \leq C \sum_{n\geq 1} U_n (a_n^4 + b_n^4) < +\infty.$$

From the relation (5.10), we obtain

$$\sum_{n\geq 1} U_n Var\hat{f}_n(y|x) \leq \sum_{n\geq 1} \frac{CU_n}{na_nb_n} < +\infty.$$

The proof is then ended. □

Proof. of the Corollary 4.4
We use the second moment complete convergence, for $U_n = \frac{n^\alpha a_n^2 b_n^2}{\log n}$, $\alpha > 1$ and $a_nb_n = O\left(\frac{1}{n^\beta}\right)$, $\beta > 3$. We have, $\sum_{n\geq 1} U_n Var\hat{f}_n(y|x) < +\infty$ and $\sum_{n\geq 1} U_n(E\hat{f}_n(y|x) - f(y|x))^2 < +\infty$. Then we use the Theorem 2.4 for $v_n = n^\gamma, \gamma > 7$ then $a = \gamma + \alpha \in]0, 1[$.
For $U_n = \frac{n^\alpha a_n^2 b_n^2}{\log n}$, $\alpha \in]0, 1[$ and $a_nb_n = O\left(\frac{1}{n^\beta \log n}\right)$, $\beta \in]0, 1[$. We have, the result then we use the Theorem 2.4 for $v_n = n^\gamma, \gamma > 2$ then $a = \gamma + \alpha > 2$.

□

6. CONCLUSION

In this paper, we introduced and analyzed the complete second-moment convergence for a conditional density estimator under a general censoring model. This new type of convergence provides a sharper control of the estimation error and leads to faster convergence rates compared to those obtained with almost complete convergence or mean squared error. Beyond its theoretical novelty, this result clarifies the asymptotic behavior of nonparametric estimators in censored settings and strengthens the mathematical foundations of their efficiency. Future work may explore extensions to other classes of estimators, as well as potential applications in survival analysis and reliability theory, where censoring naturally arises.

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