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EXPONENTIAL STABILITY OF TIMOSHENKO TYPE SYSTEM UNDER DIRICHLET-NEUMANN CONDITIONS INVOLVING TIME VARYING DELAYS

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ABSTRACT. In this work, we deal with a linear vibrating system of Timoshenko type in the bounded one dimensional domain under Dirichlet-Neumann boundary conditions driven by variable time delays. Using the approach of the stable family of infinitesimal generators of C_0-semigroups and Kato's norm technique, we show that the system is well posed. Moreover, we prove the exponential stability of the system via the Lyapunov's theorem under the assumption that the propagation speeds are equal.

Keywords. Non-autonomous Timoshenko system, stable family, variable time delays, Lyapunov functional, exponential stability.
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1. Introduction

In this paper, we deal with the following Timoshenko type system

$$\begin{cases} \varrho_1\varphi_{tt} - k[\varphi_x+\psi]_x = 0, & (x,t)\in(0,L)\times(0,+\infty), \\ \\ \varrho_2\psi_{tt} - b\psi_{xx} + k[\varphi_x+\psi] = 0, & (x,t)\in(0,L)\times(0,+\infty) \end{cases} \tag{1.1}$$

under the following initial conditions

$$\varphi(.,0)=\varphi_0(.), \varphi_t(.,0)=\varphi_1(.), \psi(.,0)=\psi_0(.), \psi_t(.,0)=\psi_1(.) \text{ on } (0,L) \tag{1.2}$$

and the Dirichlet-Neumann boundary conditions driven by the time dependent delays of the type

$$\begin{cases} \varphi(0,t)=\psi(0,t)=0, & \forall\, t>0, \\ \varphi_x(0,t)=\psi_x(0,t)=0, & \forall\, t>0, \\ k[\varphi_x+\psi](L,t) = -\alpha_1\sigma_1\varphi_t(L,t) - \alpha_1(1-\sigma_1)\varphi_t(L,t-\tau_1(t)), & \forall\, t>0, \\ b\psi_x(L,t) = -\alpha_2\sigma_2\psi_t(L,t) - \alpha_2(1-\sigma_2)\psi_t(L,t-\tau_2(t)), & \forall\, t>0. \end{cases} \tag{1.3}$$

Here $\varphi:(0,L)\times(0,+\infty)\to\mathbb{R}$ is the transverse displacement of a beam of length L and $\psi:(0,L)\times(0,+\infty)\to\mathbb{R}$ is the rotation angle of a filament while the non-negative

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coefficients ϱ_1, ϱ_2, k, are respectively the mass density, the polar moment of inertia of a cross section, the shear modulus and the positive constant b is the product of Young's modulus of elasticity and the moment of inertia of a cross section. Furthermore, σ_1, σ_2, α_1, α_2 are positive fixed constants due to the restoring force of the movement. By t we denote the time variable, while x is the space variable along the beam of length L. Observe that the functions

$$\tau_i : [0, +\infty) \longrightarrow (0, +\infty),\ i = 1, 2$$

represent the time varying delays functions while $\varphi_0, \varphi_1, \psi_0, \psi_1$ are some given smooth functions. Finally, we consider

$$\varphi_t(x, t - \tau_1(0)) = f_0(x, t - \tau_1(0)),\ t \in (0, \tau_1(0))$$

and

$$\psi_t(x, t - \tau_2(0)) = g_0(x, t - \tau_2(0)),\ t \in (0, \tau_2(0))$$

where the functions

$$f_0 : [0, L] \times [-\tau_1(0), 0] \to \mathbb{R} \quad \text{and} \quad g_0 : [0, L] \times [-\tau_2, 0] \to \mathbb{R}$$

are some given functions which are belonging to a suitable functions space.

Observe that the problem (1.1)-(1.3) takes into account to the case when the spatial variation is linked to the time variation with respect to the boundaries on the one hand, and on the other hand, it ensures the presence of variable delays. The interest of such a study arises when we face the phenomena which occur often in situations of the wave propagation, the diffusion and heat transfer, the deformation of viscoelastic materials. Because the physical property of material for beam or plate, the deformation might not be instantaneous, thus is subject to a delay effect. For illustration, there are many examples in physical, chemical, biological and economic phenomena where the time delay affect the behaviour of a dynamical system. More details for delays effect equations can be seen in [4, 12, 13, 15, 16, 18].

Our purpose is to study the well-posedness and the exponential stability of the solutions of problem (1.1)-(1.3) under the following equal speed wave propagation

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{1.4}$$

Historically, it is worth recalling that the vibrations of a beam or thin plate coinciding with an interval on the x-axis and plates could be described by the following system of partial differential equations:

$$\begin{cases} \rho A \varphi_{tt} = S_x, \\ \rho I \psi_{tt} = M_x - S \end{cases} \tag{1.5}$$

where $\varphi = \varphi(x, t)$ is the transversal displacement of beam, $\psi = \psi(x, t)$ denotes the rotation angle for beam filament, M is the bending moment, S is the shear stress, ρ is the mass density, A is the area and I denotes the inertial moment of the transversal section.

Taking into account the physical properties, the corresponding constitutive laws in the theory of mathematical elasticity are given by

$$\begin{cases} M = EI\psi_x, \\ S = KA(\varphi_x + \psi) \end{cases} \tag{1.6}$$

with E and K are respectively the elastic constant and the shear coefficient. Therefore, the relationship of (1.6) in (1.5) leads to the model describing the transverse vibration of a beam firstly introduced in 1921 by Timoshenko in [19]. This model known as Timoshenko equation is given through the following coupled hyperbolic equations:

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi)_x), & \text{in } (0,L)\times(0,+\infty), \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0,L)\times(0,+\infty) \end{cases} \tag{1.7}$$

where t denotes the time variable, x is the space variable along the beam of length L, φ is the transverse displacement of the beam while coefficients ρ, I_ρ, E, I and K are respectively, the density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus. Although, there are fruitful works on the Timoshenko system (1.7) as reviewed above, many literatures pay attentions to well-posedness and decay results under some few boundary conditions with respect to the physical observed phenomena.

In this sense, Zhong-Jie Han and Gen-Qi Xu in [6] considered (1.7) for $x \in (0,1)$ under the following linear Dirichlet-Neumann boundary conditions

$$\begin{cases} \varphi(0,t) = \psi(0,t) = 0, & \forall\ t > 0, \\ k[\varphi_x + \psi](1,t) = \mu_1 v_1(t) + (1-\mu_1)v_1(t-\tau_1), & \forall\ t > 0, \\ EI\psi_x(1,t) = \mu_2 v_2(t) + (1-\mu_2)v_2(t-\tau_2), & \forall\ t > 0 \end{cases}$$

where τ_i and $\mu_i \in (0,1),\ i = 1,2$ are some positive constants and

$$v_i(t-\tau_i) = \widetilde{f}_i(t-\tau_i), \quad t \in (0,\tau_i)$$

and

$$\widetilde{f}_i : \mathbb{R} \to \mathbb{R},\ i = 1,2$$

are the given functions.

In the same way, Kim and Renardy in [10] considered (1.7) subjected to the two boundary controls of the form

$$\begin{cases} K(\varphi_x - \psi)(L,t) = -\alpha\varphi_t(L,t), & t \in [0,+\infty) \\ EI\psi_x(L,t) = -\beta\psi_t(L,t), & t \in [0,+\infty) \end{cases}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy associated to the solution of (1.7).

Deepening these Dirichlet-Neumann boundary conditions, in this work, we introduce the controls which include the time varying delays terms

$$\begin{cases} \varphi(0,t) = \psi(0,t) = 0, \qquad \forall\ t > 0, \\ k(\varphi_x - \psi)(L,t) = -\alpha_1\sigma_1\varphi_t(L,t) - \alpha_1(1-\sigma_1)\varphi_t(L,t-\tau_1(t)) \\ b\psi_x(L,t) = -\alpha_2\sigma_2\psi_t(L,t) - \alpha_2(1-\sigma_2)\psi_t(L,t-\tau_2(t)) \end{cases}$$

which make the problem (1.1)-(1.3) different from those considered in the literature. Observe that the coefficients $\alpha_i, \sigma_i,\ i = 1,2$ are physical constraints which can play an important role in the control of the boundary conditions. Moreover, the time varying delays following the Dirichlet-Neumann boundary conditions lead to obtain an equivalent homogeneous non-autonomous Cauchy problem. Therefore, some theory of non-autonomous Cauchy problem will be applied to obtain the regularity of solution of system (1.1)-(1.3).

In recent years, the partial differential equations with time varying delay effects have become an active area of research. In this list of works, one can observe several authors which have been interested in their studies, some well-posedness results on the one hand

and on the other hand, the stability behaviour results of dynamical systems incorporating delay effects. Among them, let us cite Adimy and Ezzinbi in [1], Ammar-Khodja et *al*. in [2], Datko et *al*. in [4] and also the references therein.
This paper is organised as follows. In section 2, we make some preliminaries, the useful properties and the assumptions used to study the system (1.1)-(1.3). The section 3 is devoted to the well-posedness of classical solution. In the last section, the exponential stability of solution of (1.1)-(1.3) is established via Lyapunov functional.

2. Preliminary

In this section, on the one hand we will claim the important results which appear when we study the non-autonomous dynamical systems and on the other hand, we have to make some sufficient assumptions which play an important role in the rest of this paper. Without loss of generality and for the convenience, we assume that there exists a Banach space Y densely and continuously embedded in X where X denotes a Banach space.
We start giving the following definitions and results which will be used in our work, see [[17], Chapter 4 and Chapter 5] and also [3, 5, 7, 8, 9] for more details.

Definition 2.1. Let $\{S(\lambda)\}_{\lambda\geq 0}$ be a C_0-semigroup with generator $(A, D(A))$ on a Banach space X and let Y be a subspace of X. Y is said A-admissible if it is $S(\lambda)$-invariant for all $\lambda \in \mathbb{R}^+$.

Definition 2.2. Let X be a Banach space. A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0-semigroups on X is said to be stable if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ (called the stability constants) such that

$$(\omega, +\infty) \subset \rho(A(t)) \quad \text{for} \quad t \in [0, T] \tag{2.1}$$

and

$$\Big\| \prod_{j=1}^{k} R(\lambda; A(t_j)) \Big\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda-\omega)^k} \tag{2.2}$$

for $\lambda > \omega$ and any finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and $k = 1, 2, \cdots$
Here, $\rho(A(t))$ denotes the resolvent set of the operator $A(t)$ and $R(\lambda; A(t))$ defines the resolvent operator associated to $A(t)$ at the point λ.

Theorem 2.3. *([17], p.13, theorem 4.3 (Lumer-Phillips)) Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in X.*
If $\mathcal{A}$ is dissipative and there is a $\lambda_0 > 0$ such that the range of $\lambda_0 I - \mathcal{A}$ is X, then $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on X.

The following proposition is an equivalent tool used to bring the stability of the family of operators $\{\mathcal{A}(t) : \ t \in [0, T]\}$.

Proposition 2.4. [9] *For each t, let $||.||_t$ be a new norm in X equivalent to the original one, depending on t smoothly in the sense that*

$$\frac{||x||_t}{||x||_s} \leq e^{c|t-s|}, \qquad x \in X, \qquad s, t \in [0, T]. \tag{2.3}$$

For each t, let $\mathcal{A}(t) \in G(||.||_t, 1, \beta)$. Then $\{\mathcal{A}(t)\}_{t\in[0,T]}$ is a stable family, with $M = e^{2cT}$ with respect to $||.||_t$ for any $t \in [0, T]$.

The space $\mathcal{H}$ below denotes a Hilbert space. The following theorem gives the existence and uniqueness results of the Cauchy problem which we will be concerned.

Theorem 2.5. [9] *Assume that:*
(i) $Y = D(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{H}$*;*
(ii) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$ *for all* $t > 0$*;*
(iii) for all $t \in [0,T]$, $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{H}$ *and the family* $\{\mathcal{A}(t) :\ t \in [0,T]\}$ *is stable with stability constants* C *and* ω *independent of* t*;*
(iv) $\dfrac{d}{dt}\mathcal{A}(t) \in L^{\infty}_{*}\Big([0,T], B(Y,\mathcal{H})\Big)$, *where* $L^{\infty}_{*}\Big([0,T], B(Y,\mathcal{H})\Big)$ *is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(Y,\mathcal{H})$ *of bounded operators from* Y *into* $\mathcal{H}$.
Then, the problem

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}(t)U(t) & \forall\ t > 0, \\ U(0) = U_0 \in D(\mathcal{A}(0)) \end{cases}$$

has a unique solution $U \in C\Big([0,T], D(\mathcal{A}(0))\Big) \cap C^1\Big([0,T], \mathcal{H}\Big)$.

In our case, the well-posedness result is made under the following assumptions:
The time-varying delay functions

$$\tau_i : [0,+\infty) \longrightarrow (0,+\infty), \quad i = 1,2$$

satisfies

$$\exists\ d < 1 \quad \forall t > 0, \quad \tau_i'(t) \le d < 1 \tag{2.4}$$

and

$$\exists\ M > 0 \quad \forall t > 0, \quad 0 < \tau_i(0) \le \tau_i(t) \le M. \tag{2.5}$$

Also,

$$\tau_i \in W^{2,\infty}([0,T]), \quad \forall\ T > 0. \tag{2.6}$$

Moreover, we assume that

$$\frac{\alpha_i(1-\sigma_i)}{1-\tau_i'(t)} \le r \le \alpha_i(3\sigma_i - 1), \qquad \frac{1}{3} < \sigma_i < 1,\ i = 1,2 \tag{2.7}$$

where r is some positive constant.

3. Well-posedness of the problem

In this section, we can give the well-posedness of problem (1.1)-(1.3) using the semigroup theory and Kato's norm technique developed in [9, 16, 17]. For this, let us set the following auxiliaries functions

$$z(\xi,t) = \varphi_t(L, t - \xi\tau_1(t)) \text{ for } \xi \in (0,1) \text{ and } t > 0 \tag{3.1}$$

and

$$\omega(\rho,t) = \psi_t(L, t - \rho\tau_2(t)) \text{ for } \rho \in (0,1) \text{ and } t > 0. \tag{3.2}$$

Thus z verifies the following transport equation

$$\begin{cases} \tau_1(t)z_t(\xi,t) + \Big(1 - \xi\tau_1'(t)\Big)z_\xi(\xi,t) = 0, & (\xi,t) \in (0,1)\times(0,+\infty), \\ z(0,t) = \varphi_t(L,t), & t \in (0,+\infty), \\ z(\xi,0) = f_0(L, -\xi\tau_1(0)), & \xi \in (0,1) \end{cases} \tag{3.3}$$

while ω satisfies the following transport equation

$$\begin{cases} \tau_2(t)\omega_t(\rho,t) + \Big(1-\rho\tau_2'(t)\Big)\omega_\rho(\rho,t) = 0, & (\rho,t) \in (0,1)\times(0,+\infty), \\ \omega(0,t) = \psi_t(L,t), & t \in (0,+\infty), \\ \omega(\rho,0) = g_0(L,-\rho\tau_2(0)), & \rho \in (0,1). \end{cases} \tag{3.4}$$

Consequently, the problem (1.1)-(1.3) is equivalent to

$$\begin{cases} \varrho_1\varphi_{tt}(x,t) - k[\varphi_x+\psi]_x(x,t) = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \varrho_2\psi_{tt}(x,t) - b\psi_{xx} + k[\varphi_x+\psi](x,t) = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \tau_1(t)z_t(\xi,t) + \Big(1-\xi\tau_1'(t)\Big)z_\xi(\xi,t) = 0, & \xi \in [0,1],\ t>0, \\ \tau_2(t)\omega_t(\rho,t) + \Big(1-\rho\tau_2'(t)\Big)\omega_\rho(\rho,t) = 0, & \rho \in [0,1],\ t>0, \\ k[\varphi_x+\psi](L,t) = -\alpha_1\sigma_1 z(0,t) - \alpha_1(1-\sigma_1)z(1,t), & \forall\, t>0, \\ b\psi_x(L,t) = -\alpha_2\sigma_2\omega(0,t) - \alpha_2(1-\sigma_2)\omega(1,t), & \forall\, t>0, \\ \varphi(0,t) = \psi(0,t) = 0, & \forall\, t>0, \\ z(0,t) = \varphi_t(L,t) \quad \omega(0,t) = \psi_t(L,t), & \forall\, t>0, \\ z(\xi,0) = f_0(L,-\xi\tau_1(0)), & \xi \in [0,1], \\ \omega(\rho,0) = g_0(L,-\rho\tau_2(0)), & \rho \in [0,1], \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x \in (0,L), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x)), & x \in (0,L). \end{cases} \tag{3.5}$$

Now, we introduce the vector function $U = (\varphi, u, \psi, v, z, \omega)^T$, where

$$u = \varphi_t \quad \text{and} \quad v = \psi_t.$$

Therefore, (3.5) can be written as the following first order non-autonomous equation

$$\begin{cases} \dfrac{d}{dt}U(t) = \mathcal{A}(t)U(t), \qquad \text{for all} \quad t>0, \\ U_0 = U(0) = \Big(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0(L,-\tau_1(0)), g_0(L,-\tau_2(0))\Big)^T \end{cases} \tag{3.6}$$

where the time dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ z \\ \omega \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{k}{\varrho_1}[\varphi_x+\psi]_x \\ v \\ \dfrac{b}{\varrho_2}\psi_{xx} - \dfrac{k}{\varrho_2}[\varphi_x+\psi] \\ \dfrac{\xi\tau_1'(t)-1}{\tau_1(t)}z_\xi \\ \dfrac{\rho\tau_2'(t)-1}{\tau_2(t)}\omega_\rho \end{pmatrix}$$

with domain

$$\begin{aligned} D(\mathcal{A}(t)) \;=\; & \Big\{(\varphi,u,\psi,v,z,\omega) \in \Big(H^2(0,L)\cap V\Big) \times \Big(H^1(0,L)\cap V\Big) \\ & \times \Big(H^2(0,L)\cap V\Big) \times \Big(H^1(0,L)\cap V\Big) \times H^1(0,1) \times H^1(0,1)\Big\} \end{aligned}$$

where V is the Hilbert space defined as follows:

$$V = \{y \in H^1(0, L) : y(0) = y_x(0) = 0\}.$$

Immediately, one observes that the domain of the operator $\mathcal{A}(t)$ is independent of the time t. Thus,

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall\, t > 0. \tag{3.7}$$

Now, we introduce the energy space $\mathcal{H}$ as follows

$$\mathcal{H} = \Big(H^2(0, L) \cap V\Big) \times L^2(0, L) \times \Big(H^2(0, L) \cap V\Big) \times L^2(0, L) \times L^2(0, 1) \times L^2(0, 1).$$

For $U_j = \Big(\varphi^j, u^j, \psi^j, v^j, z^j, \omega^j\Big)^T \in \mathcal{H}$, $j = 1, 2$, we endow the space $\mathcal{H}$ with the following inner product

$$\begin{aligned} (U_1, U_2)_{\mathcal{H}} &= \int_0^L \Big[\varrho_1 u^1 u^2 + k(\varphi^1_x + \psi^1)(\varphi^2_x + \psi^2) + \varrho_2 v^1 v^2 + b\psi^1_x \psi^2_x\Big] dx \\ &\quad + r\int_0^1 z^1(\xi) z^2(\xi) d\xi + r\int_0^1 \omega^1(\rho)\omega^2(\rho) d\rho. \end{aligned} \tag{3.8}$$

Also, we define on the same space $\mathcal{H}$ the time dependent inner product

$$\begin{aligned} (U_1, U_2)_{\mathcal{H}_t} &= \int_0^L \Big[\varrho_1 u^1 u^2 + k(\varphi^1_x + \psi^1)(\varphi^2_x + \psi^2) + \varrho_2 v^1 v^2 + b\psi^1_x \psi^2_x\Big] dx \\ &\quad + r\tau_1(t)\int_0^1 z^1(\xi) z^2(\xi) d\xi + r\tau_2(t)\int_0^1 \omega^1(\rho)\omega^2(\rho) d\rho \end{aligned} \tag{3.9}$$

where the positive constant r is defined in (2.7). We can remark that the space $\mathcal{H}$ endowed with the above defined inner products is a Hilbert space.
From now on, we will make our main result of this section which claims that the problem (3.6) admits a unique classical and global solution.

Theorem 3.1. *Assume that (2.4)-(2.7) hold. Then, for any $U_0 \in \mathcal{H}$, there exists a unique solution $U \in C\Big([0, +\infty), \mathcal{H}\Big)$ of system (3.6). Moreover, suppose that $U_0 \in D(\mathcal{A}(0))$. Then,*

$$U \in C\Big([0, +\infty), D(\mathcal{A}(0))\Big) \cap C^1\Big([0, +\infty), \mathcal{H}\Big).$$

Proof. In order to prove Theorem 3.1, since the operator $\mathcal{A}(t)$ depends on time, we will use the variable norm technique developed by Kato in [9].
Therefore, we decompose the proof of Theorem 3.1 through several lemmas.

Lemma 3.2. *$D(\mathcal{A}(0))$ is a dense subset of $\mathcal{H}$.*

Proof. The proof of this lemma will be done following the technique of Lemma 2.1 of [14].
Therefore, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ be orthogonal to all elements of $D(\mathcal{A}(0))$. We have for all $U = (\varphi, u, \psi, v, z, \omega)^T \in D(\mathcal{A}(0))$.

$$\begin{aligned} 0 = \Big(U, F\Big)_{\mathcal{H}} &= \int_0^L \Big[\varrho_1 u f_2 + k(\varphi_x + \psi)(f_{1x} + f_3) + \varrho_2 v f_4 + b\psi_x f_{3x} \\ &\quad + r\int_0^1 z f_5(\xi, t) d\xi + r\int_0^1 \omega f_6(\rho, t) d\rho. \end{aligned} \tag{3.10}$$

Firstly, let us consider $\varphi = u = \psi = z = \omega = 0$ and $v \in \mathcal{D}(0, L)$. Then,

$$U = (0, 0, 0, v, 0, 0)^T \in D(\mathcal{A}(0))$$

and (3.10) leads to

$$\varrho_2 \int_0^L v f_4 dx = 0.$$

Using the density argument of $\mathcal{D}(0, L)$ in $L^2(0, L)$, it follows that $f_4 = 0$.
Also, let us take $\varphi = \psi = v = z = \omega = 0$ and $u \in \mathcal{D}(0, L)$. So,

$$U = (0, u, 0, 0, 0, 0)^T \in D(\mathcal{A}(0))$$

and from (3.10), one can write

$$\int_0^L u f_2 dx = 0.$$

Taking into account that $\mathcal{D}(0, L)$ is dense in $L^2(0, L)$, then one claims that $f_2 = 0$.
Moreover, putting $\varphi = u = \psi = v = z = 0$ and $\omega \in \mathcal{D}(0, 1)$, then

$$U = (0, 0, 0, 0, 0, \omega)^T \in D(\mathcal{A}(0))$$

and we deduce from (3.10) that

$$r \int_0^1 \omega f_6(\rho, t) d\rho = 0.$$

But $\mathcal{D}(0, 1)$ is dense in $L^2(0, 1)$. Hence, $f_6 = 0$.
In the same way, using $\varphi = u = \psi = v = \omega = 0$ and $z \in \mathcal{D}(0, 1)$, one obtains

$$U = (0, 0, 0, 0, z, 0)^T \in D(\mathcal{A}(0))$$

which gives $f_5 = 0$.
Furthermore, let us consider $u = \psi = v = z = \omega = 0$ and $\varphi \in \mathcal{D}(0, L)$. It follows that

$$U = (\varphi, 0, 0, 0, 0, 0)^T \in D(\mathcal{A}(0)).$$

From (3.10)

$$-k \int_0^L \varphi (f_{1x} + f_3)_x dx = 0 \text{ and thus } (f_{1x} + f_3)_x = 0.$$

That is to say, there exists some constant C_1 such that $f_{1x} + f_3 = C_1$. Since $f_3(0) = f_{1x}(0) = 0$ then $C_1 = 0$.
Finally, if we take $\varphi = u = v = z = \omega = 0$ and $\psi \in L^2(0, L)$ we get

$$U = (0, 0, \psi, 0, 0, 0)^T \in D(\mathcal{A}(0)).$$

Consequently, we use (3.10) to obtain

$$\int_0^L \Big[k(f_{1x} + f_3) - b f_{3xx}\Big] \psi dx = 0 \text{ and thus } k(f_{1x} + f_3) - b f_{3xx} = 0.$$

But $f_{1x} + f_3 = 0$. Hence, $f_{3xx} = 0$. Therefore, there exists some constant C_2 such that $f_{3x} = C_2$. Since $f_{3x}(0) = 0$, then $C_2 = 0$ and thus $f_3 = 0$ if we take into account that $f_3(0) = C_3 = 0$.
From $f_3 = 0$, one can see f_1 is a constant function and taking into account that $f_1(0) = 0$, then $f_1 = 0$. This completes the proof of the lemma. $\square$

Lemma 3.3. *Assume that (2.4)-(2.7) hold. Then, the operator*

$$\mathcal{A}(t) - \Lambda(t) I : D(\mathcal{A}(0)) \longrightarrow \mathcal{H}$$

where $\Lambda : [0, +\infty) \to (0, +\infty)$ *is some positive function, is dissipative.*

Proof. Let $U = (\varphi, u, \psi, v, z, \omega)^T \in D(\mathcal{A}(0))$. Using (3.9),

$$\begin{aligned}(\mathcal{A}(t)U, U)_{\mathcal{H}_t} &= \int_0^L \Big[k(\varphi_x+\psi)_x u + k u_x(\varphi_x+\psi) + bv\psi_{xx} + bv_x\psi_x\Big]dx \\ &+ r\int_0^1 (\xi\tau_1'(t)-1)z_\xi z\, d\xi + r\int_0^1 (\rho\tau_2'(t)-1)\omega_\rho\omega\, d\rho.\end{aligned} \tag{3.11}$$

We take into account the integration by parts with respect respectively to ξ and ρ to obtain

$$\begin{aligned}\int_0^1 (\xi\tau_1'(t)-1)z_\xi(\xi)z(\xi)d\xi &= \frac{1}{2}\int_0^1 (\xi\tau_1'(t)-1)\frac{\partial}{\partial\xi}z^2(\xi)d\xi \\ &= \frac{1}{2}\Big((\tau_1'(t)-1)z^2(1)+z^2(0)\Big) \\ &- \frac{\tau_1'(t)}{2}\int_0^1 z^2(\xi)d\xi\end{aligned}$$

and

$$\begin{aligned}\int_0^1 (\rho\tau_2'(t)-1)\omega_\rho(\rho)\omega(\rho)d\rho &= \frac{1}{2}\Big((\tau_2'(t)-1)\omega^2(1)+\omega^2(0)\Big) \\ &- \frac{\tau_2'(t)}{2}\int_0^1 \omega^2(\rho)d\rho.\end{aligned}$$

Therefore,

$$\begin{aligned}(\mathcal{A}(t)U, U)_{\mathcal{H}_t} &= k(\varphi_x+\psi)(L)u(L) + b\psi_x(L)v(L) \\ &+ \frac{r}{2}\Big((\tau_1'(t)-1)z^2(1)+z^2(0)\Big) - \frac{r\tau_1'(t)}{2}\int_0^1 z^2(\xi)d\xi \\ &+ \frac{r}{2}\Big((\tau_2'(t)-1)\omega^2(1)+\omega^2(0)\Big) - \frac{r\tau_2'(t)}{2}\int_0^1 \omega^2(\rho)d\rho.\end{aligned}$$

Since $u(L) = \varphi_t(L,t) = z(0)$ and $v(L) = \psi_t(L) = \omega(0)$ then,

$$\begin{aligned}(\mathcal{A}(t)U, U)_{\mathcal{H}_t} &= \Big(-\alpha_1\sigma_1 z(0) - \alpha_1(1-\sigma_1)z(1)\Big)z(0) \\ &- \Big(\alpha_2\sigma_2\omega(0) + \alpha_2(1-\sigma_2)\omega(1)\Big)\omega(0) \\ &+ \frac{r}{2}\Big((\tau_1'(t)-1)z^2(1)+z^2(0)\Big) - \frac{r\tau_1'(t)}{2}\int_0^1 z^2(\xi)d\xi \\ &+ \frac{r}{2}\Big((\tau_2'(t)-1)\omega^2(1)+\omega^2(0)\Big) - \frac{r\tau_2'(t)}{2}\int_0^1 \omega^2(\rho)d\rho.\end{aligned}$$

Also, it is known that

$$\begin{aligned}-\alpha_1(1-\sigma_1)z(0)z(1) &\le \frac{\alpha_1(1-\sigma_1)}{2}\Big[z^2(0)+z^2(1)\Big]; \\ -\alpha_2(1-\sigma_2)\omega(0)\omega(1) &\le \frac{\alpha_2(1-\sigma_2)}{2}\Big[\omega^2(0)+\omega^2(1)\Big].\end{aligned}$$

Hence

$$\begin{aligned}&(\mathcal{A}(t)U, U)_{\mathcal{H}_t} \\ \le\ & \frac{1}{2}\Big(r - 3\alpha_1\sigma_1 + \alpha_1\Big)z^2(0) - \frac{1}{2}\Big(r(1-\tau_1'(t)) - \alpha_1(1-\sigma_1)\Big)z^2(1) \\ +\ & \frac{1}{2}\Big(r - 3\alpha_2\sigma_2 + \alpha_2\Big)\omega^2(0) - \frac{1}{2}\Big(r(1-\tau_2'(t)) - \alpha_2(1-\sigma_2)\Big)\omega^2(1) \\ -\ & \frac{r\tau_1'(t)}{2}\int_0^1 z^2(\xi)d\xi - \frac{r\tau_2'(t)}{2}\int_0^1 \omega^2(\rho)d\rho.\end{aligned}$$

Since

$$\begin{aligned}(U,U)_{\mathcal{H}_t} &= \int_0^L \Big[\varrho_1 u^2 + k(\phi_x+\psi)^2 + \varrho_2 v^2 + b\psi_x^2\Big]dx \\ &+ r\tau_1(t)\int_0^1 z^2(\xi)d\xi + r\tau_2(t)\int_0^1 \omega^2(\rho)d\rho\end{aligned}$$

then,

$$(U,U)_{\mathcal{H}_t} \geq r\tau_1(t)\int_0^1 z^2(\xi)d\xi \quad \text{and} \quad (U,U)_{\mathcal{H}_t} \geq r\tau_2(t)\int_0^1 \omega^2(\rho)d\rho.$$

Let us set

$$k_i(t) = \frac{\sqrt{\tau_i'(t)^2+1}}{2\tau_i(t)} > 0,\ \ i=1,2.$$

It follows that

$$-k_1(t)(U,U)_{\mathcal{H}_t} \leq -r\frac{\sqrt{\tau_1'(t)^2+1}}{2}\int_0^1 z^2(\xi)d\xi$$

and

$$-k_2(t)(U,U)_{\mathcal{H}_t} \leq -r\frac{\sqrt{\tau_2'(t)^2+1}}{2}\int_0^1 \omega^2(\rho)d\rho.$$

From now on, one obtains

$$\begin{aligned}&(\mathcal{A}(t)U,U)_{\mathcal{H}_t} - (k_1(t)+k_2(t))(U,U)_{\mathcal{H}_t} \\ \leq\ & \frac{1}{2}\Big(r - 3\alpha_1\sigma_1 + \alpha_1\Big)z^2(0) - \frac{1}{2}\Big(r(1-\tau_1'(t)) - \alpha_1(1-\sigma_1)\Big)z^2(1) \\ +\ & \frac{1}{2}\Big(r - 3\alpha_2\sigma_2 + \alpha_2\Big)\omega^2(0) - \frac{1}{2}\Big(r(1-\tau_2'(t)) - \alpha_2(1-\sigma_2)\Big)\omega^2(1) \\ -\ & r\frac{\tau_1'(t)+\sqrt{\tau_1'(t)^2+1}}{2}\int_0^1 z^2(\xi)d\xi - r\frac{\tau_2'(t)+\sqrt{\tau_2'(t)^2+1}}{2}\int_0^1 \omega^2(\rho)d\rho.\end{aligned}$$

Observe that for all $t \geq 0$

$$\tau_i'(t) + \sqrt{\tau_i'(t)^2+1} \geq 0, \quad i=1,2.$$

Consequently, putting $\Lambda(t) = k_1(t)+k_2(t)$, we get

$$\begin{aligned}(\mathcal{A}(t)U,U)_{\mathcal{H}_t} - \Lambda(t)(U,U)_{\mathcal{H}_t} \ \leq\ & \frac{1}{2}\Big(r - 3\alpha_1\sigma_1 + \alpha_1\Big)z^2(0) \\ -\ & \frac{1}{2}\Big(r(1-\tau_1'(t)) - \alpha_1(1-\sigma_1)\Big)z^2(1) \\ +\ & \frac{1}{2}\Big(r - 3\alpha_2\sigma_2 + \alpha_2\Big)\omega^2(0) \\ -\ & \frac{1}{2}\Big(r(1-\tau_2'(t)) - \alpha_2(1-\sigma_2)\Big)\omega^2(1).\end{aligned}$$

From (2.7),

$$-\frac{1}{2}\Big(r(1-\tau_i'(t)) - \alpha_i(1-\sigma_i)\Big) \leq 0 \quad \text{and} \quad \frac{1}{2}\Big(r - 3\alpha_i\sigma_i + \alpha_i\Big) \leq 0,\ \text{with } i=1,2.$$

Then, we can write that

$$(\mathcal{A}(t)U,U)_{\mathcal{H}_t} - \Lambda(t)(U,U)_{\mathcal{H}_t} \leq 0, \quad \forall\, t \geq 0$$

which achieves the proof that the operator $\mathcal{A}(t) - \Lambda(t)I$ is dissipative. □

Now, we have to handle that the operator $\mathcal{A}(t)$ is maximal for all $t \in [0,T]$.

Lemma 3.4. *For all $\lambda > 0$, and $t > 0$,*

$$\lambda I - \mathcal{A}(t) : D(\mathcal{A}(0)) \longrightarrow \mathcal{H} \quad \textit{is maximal.}$$

Proof. For all $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathcal{H}$, we seek

$$U=(\varphi,u,\psi,v,z,\omega)^T\in D(\mathcal{A}(0))$$

a solution of the following equation

$$(\lambda I-\mathcal{A}(t))U=F,\quad t>0,\ \lambda>0. \tag{3.12}$$

The equation (3.12) is equivalent to:

$$\begin{cases}\lambda\varphi-u=f_1,\\ \lambda u-\dfrac{k}{\varrho_1}(\varphi_x+\psi)_x=f_2,\\ \lambda\psi-v=f_3,\\ \lambda v-\dfrac{b}{\varrho_2}\psi_{xx}+\dfrac{k}{\varrho_2}(\varphi_x+\psi)=f_4,\\ \lambda z-\dfrac{\xi\tau_1'(t)-1}{\tau_1(t)}z_\xi=f_5,\\ \lambda\omega-\dfrac{\rho\tau_2'(t)-1}{\tau_2(t)}\omega\rho=f_6.\end{cases} \tag{3.13}$$

Suppose that we have found φ and ψ which appeared in (3.13). Then, from $(3.13)_1$ and $(3.13)_3$ we have the system

$$\begin{cases}u=\lambda\varphi-f_1,\\ v=\lambda\psi-f_3.\end{cases} \tag{3.14}$$

Now, using $(3.13)_2$ and $(3.13)_4$, we have to find $(\varphi,\psi)\in\Big[H^2(0,L)\cap V\Big]^2$ which solves the system

$$\begin{cases}\varrho_1\lambda^2\varphi-k\varphi_{xx}-k\psi_x=\varrho_1(f_2+\lambda f_1)\\ \\ \varrho_2\lambda^2\psi-b\psi_{xx}+k\varphi_x+k\psi=\varrho_2(f_4+\lambda f_3).\end{cases} \tag{3.15}$$

That is equivalent to find $(\varphi,\psi)\in\Big[H^2(0,L)\cap V\Big]^2$ such that

$$\begin{cases}\varrho_1\lambda^2\displaystyle\int_0^L\varphi\tilde\varphi dx+k\int_0^L\varphi_x\tilde\varphi_x dx+k\int_0^L\psi\tilde\varphi_x dx=\int_0^L g_1\tilde\varphi dx,\\ \\ \varrho_2\lambda^2\displaystyle\int_0^L\psi\tilde\psi dx+b\int_0^L\psi_x\tilde\psi_x dx+k\int_0^L\varphi_x\tilde\psi+k\int_0^L\psi\tilde\psi dx=\int_0^L g_2\tilde\psi dx.\end{cases} \tag{3.16}$$

For all $(\tilde\varphi,\tilde\psi)\in W=H_0^1(0,L)\times H_0^1(0,L)$ where $g_1=\varrho_1(f_2+\lambda f_1)$ and $g_2=\varrho_2(f_4+\lambda f_3)$. Consequently, the problem (3.16) is equivalent to the following variational formulation

$$B((\varphi,\psi),(\tilde\varphi,\tilde\psi))=G(\tilde\varphi,\tilde\psi) \tag{3.17}$$

where $B:W\times W\longrightarrow\mathbb{R}$ is the bilinear form and $G:W\longrightarrow\mathbb{R}$ is the linear form, which are defined respectively by:

$$\begin{aligned}B((\varphi,\psi),(\tilde\varphi,\tilde\psi))&=\varrho_1\lambda^2\int_0^L\varphi\tilde\varphi dx+k\int_0^L\varphi_x\tilde\varphi_x dx+k\int_0^L\psi\tilde\varphi_x dx\\&+(\varrho_2\lambda^2+k)\int_0^L\psi\tilde\psi dx+b\int_0^L\psi_x\tilde\psi_x dx+k\int_0^L\varphi_x\tilde\psi dx\end{aligned} \tag{3.18}$$

and

$$G(\tilde{\varphi},\tilde{\psi}) = \int_0^L \Big(g_1\tilde{\varphi} + g_2\tilde{\psi}\Big)dx. \tag{3.19}$$

We claim that the bilinear form $B : W \times W \longrightarrow \mathbb{R}$ is continuous. In fact, for all $X = (\varphi,\psi) \in W$ and $\tilde{X} = (\tilde{\varphi},\tilde{\psi}) \in W$, through Hölder's inequality and Poincaré's inequality respectively, it follows

$$\begin{aligned}
&\Big|B(X,\tilde{X})\Big| \\
\leq\ & \varrho_1\lambda^2\Big(\int_0^L \varphi^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\varphi}^2dx\Big)^{1/2} + k\Big(\int_0^L \varphi_x^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\varphi}_x^2dx\Big)^{1/2} \\
+\ & k\Big(\int_0^L \psi^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\varphi}_x^2dx\Big)^{1/2} + (\varrho_2\lambda^2+k)\Big(\int_0^L \psi^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\psi}^2dx\Big)^{1/2} \\
+\ & b\Big(\int_0^L \psi_x^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\psi}_x^2dx\Big)^{1/2} + k\Big(\int_0^L \varphi_x^2dx\Big)^{1/2}\Big(\int_0^L \tilde{\psi}^2dx\Big)^{1/2}
\end{aligned}$$

and

$$\Big|B(X,\tilde{X})\Big| \leq \nu||X||_W||\tilde{X}||_W.$$

where $\nu > 0$ is some constant. Hence, B is a continuous bilinear form on W.
Moreover,

$$\begin{aligned}
B(X,X) &= \varrho_1\lambda^2\int_0^L \varphi^2dx + k\int_0^L \varphi_x^2dx + 2k\int_0^L \psi\varphi_xdx \\
&+ (\varrho_2\lambda^2+k)\int_0^L \psi^2dx + b\int_0^L \psi_x^2dx.
\end{aligned} \tag{3.20}$$

Taking into account Young's inequality, we have for all $\varepsilon > 0$,

$$\begin{aligned}
-\int_0^L \psi\varphi_xdx &\leq \int_0^L |\psi\varphi_x|dx \\
&\leq \frac{\varepsilon}{2}\int_0^L \psi^2dx + \frac{1}{2\varepsilon}\int_0^L \varphi_x^2dx
\end{aligned}$$

and thus

$$\int_0^L \psi\varphi_xdx \geq -\frac{\varepsilon}{2}\int_0^L \psi^2dx - \frac{1}{2\varepsilon}\int_0^L \varphi_x^2dx.$$

Consequently, we put $1 < \varepsilon < 1 + \dfrac{\varrho_2\lambda^2}{k}$ to obtain

$$\begin{aligned}
B(X,X) &\geq \varrho_1\lambda^2\int_0^L \varphi^2dx + (k-\frac{k}{\varepsilon})\int_0^L \varphi_x^2dx \\
&+ (\varrho_2\lambda^2 + k - k\varepsilon)\int_0^L \psi^2dx + b\int_0^L \psi_x^2dx \\
&\geq \nu_1||X||_W^2.
\end{aligned}$$

where ν_1 is some positive constant. Therefore, the bilinear form $B : W \times W \longrightarrow \mathbb{R}$ is coercive.
Now, let $\tilde{X} = (\tilde{\varphi},\tilde{\psi}) \in W$. Then,

$$\begin{aligned}
|G(\tilde{X})| &= \Big|\int_0^L (g_1\tilde{\varphi} + g_2\tilde{\psi})dx\Big| \\
&\leq \int_0^L |g_1\tilde{\varphi}|dx + \int_0^L |g_2\tilde{\psi}|dx.
\end{aligned}$$

Using Hölder's inequality and Poincaré's inequality respectively, one gets

$$\begin{aligned}|G(\tilde{X})| &\leq L||g_1||_{L^2(0,L)}||\tilde{\varphi}_x||_{L^2(0,L)} + L||g_2||_{L^2(0,L)}||\tilde{\psi}_x||_{L^2(0,L)} \\ &\leq \nu_2\Big(||\tilde{\varphi}||_{H_0^1(0,L)} + ||\tilde{\psi}||_{H_0^1(0,L)}\Big) \\ &\leq \nu_2||\tilde{X}||_W\end{aligned}$$

where $\nu_2 = \max\Big(L||g_1||_{L^2(0,L)}, L||g_2||_{L^2(0,L)}\Big)$. Thus G is continuous on W.

Applying the Lax-Milgram theorem, problem (3.17) admits a unique solution $X = (\varphi, \psi) \in W$. Now, using the classical elliptic regularity, it follows from (3.16) that

$$(\varphi, \psi) \in H^2(0, L) \times H^2(0, L).$$

Thus,

$$(\varphi, \psi) \in H^2(0, L) \cap H_0^1(0, L) \times H^2(0, L) \cap H_0^1(0, L).$$

From $(3.13)_1$, we have:

$$\lambda\varphi_x = u_x + f_{1x} = \varphi_{tx} + f_{1x} \quad \text{and} \quad \lambda\varphi_x(0) = \varphi_{tx}(0) \quad \text{since} \quad f_{1x}(0) = 0.$$

Consequently, it is appropriate to consider

$$\varphi_x(0) = \varphi_{tx}(0) = 0.$$

Similarly, through $(3.13)_3$, one can choose

$$\psi_x(0) = \psi_{tx}(0) = 0.$$

Therefore, we conclude that

$$(\varphi, \psi) \in H^2(0, L) \cap V \times H^2(0, L) \cap V.$$

From $(3.13)_6$, we have the following ordinary differential equation

$$\omega_\rho + \frac{\lambda\tau_2(t)}{1 - \rho\tau_2'(t)}\omega = \frac{\tau_2(t)}{1 - \rho\tau_2'(t)} f_6. \tag{3.21}$$

If $\tau_2'(t) \neq 0$, we obtain respectively the homogeneous solution and the particular solution of (3.21) as follows

$$\omega_H = d_1 \exp\Big(\frac{\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - \rho\tau_2'(t))\Big), \; d_1 \in \mathbb{R}$$

and

$$\omega_P = \exp\Big(\frac{\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - \rho\tau_2'(t))\Big) \int_0^\rho \frac{\tau_2(t)}{1 - s\tau_2'(t)} f_6(x, s) \exp\Big(\frac{-\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - s\tau_2'(t))\Big) ds.$$

Consequently, the solution of (3.21) is

$$\begin{aligned}\omega(\rho) &= \psi(L, t) \exp\Big(\frac{\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - \rho\tau_2'(t))\Big) \\ &+ \exp\Big(\frac{\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - \rho\tau_2'(t))\Big) \int_0^\rho \frac{\tau_2(t)}{1 - s\tau_2'(t)} f_6(x, s) \exp\Big(\frac{-\lambda\tau_2(t)}{\tau_2'(t)} \ln(1 - s\tau_2'(t))\Big) ds.\end{aligned}$$

If $\tau_2'(t) = 0$, then

$$\begin{aligned}\omega(\rho) &= \psi(L, t) \exp\Big(- \lambda\tau_2(t)\rho\Big) \\ &+ \exp\Big(- \lambda\tau_2(t)\rho\Big) \int_0^\rho \tau_2(t) f_6(x, s) \exp\Big(\lambda\tau_2(t)s\Big) ds.\end{aligned} \tag{3.22}$$

Similarly, from the equation $(3.13)_5$, we have the following ordinary differential equation

$$z_\xi + \frac{\lambda\tau_1(t)}{1-\xi\tau_1'(t)}z = \frac{\tau_1(t)}{1-\xi\tau_1'(t)}f_5 \tag{3.23}$$

which is solved if $\tau_1'(t) \neq 0$ by

$$z(\xi) = \varphi(L,t)\exp\Big(\frac{\lambda\tau_1(t)}{\tau_1'(t)}\ln(1-\xi\tau_1'(t))\Big)$$
$$+\exp\Big(\frac{\lambda\tau_1(t)}{\tau_1'(t)}\ln(1-\xi\tau_1'(t))\Big)\int_0^\xi \frac{\tau_1(t)}{1-s\tau_1'(t)}f_5(x,s)\exp\Big(\frac{-\lambda\tau_1(t)}{\tau_1'(t)}\ln(1-s\tau_1'(t))\Big)ds$$

and if $\tau_1'(t) = 0$ by

$$\begin{aligned} z(\xi) &= \varphi(L,t)\exp\Big(-\lambda\tau_1(t)\xi\Big) \\ &\quad+\exp\Big(-\lambda\tau_1(t)\xi\Big)\int_0^\xi \tau_1(t)f_5(x,s)\exp\Big(\lambda\tau_1(t)s\Big)ds. \end{aligned} \tag{3.24}$$

It follows that

$$(z,\omega) \in \Big[H^1(0,1)\Big]^2.$$

Therefore, the operator $\lambda I - \mathcal{A}(t)$ is surjective for any fixed $t \in (0,+\infty)$ and $\lambda > 0$. $\square$

The following lemma shows the stability statement of the family of operators $\{\mathcal{A}(t)\}_{t\geq 0}$.

Lemma 3.5. *$\{\mathcal{A}(t)\}_{t\in[0,T]}$ is a stable family with respect to the norm $||.||_t$.*

Proof. For any $\Theta = (\varphi,u,\psi,v,z,\omega)^T \in \mathcal{H}$, we have to prove that

$$\frac{||\Theta||_t}{||\Theta||_s} \leq e^{\frac{c}{2\tau(0)}|t-s|}, \quad (t,s)\in[0,T]^2$$

where c is some positive constant and we conclude through Proposition 2.4. Taking into account the definitions of the norms $||.||_t$ and $||.||_s$, one can write for all $t,s \in [0,T]$,

$$\begin{aligned} &||\Theta||_t^2 - ||\Theta||_s^2 e^{\frac{c}{\tau(0)}|t-s|} \\ &= \Big(1-e^{\frac{c}{\tau(0)}|t-s|}\Big)\int_0^L \Big[\varrho_1 u^2 + k(\varphi_x+\psi)^2 + \varrho_2 v^2 + b\psi_x^2\Big]dx \\ &+r\Big(\tau_1(t)-\tau_1(s)e^{\frac{c}{\tau(0)}|t-s|}\Big)\int_0^1 z^2(\xi)d\xi \\ &+r\Big(\tau_2(t)-\tau_2(s)e^{\frac{c}{\tau(0)}|t-s|}\Big)\int_0^1 \omega^2(\rho)d\rho. \end{aligned}$$

Immediately, $1-e^{\frac{c}{\tau(0)}|t-s|} \leq 0$ since $c>0$. So

$$\Big(1-e^{\frac{c}{\tau(0)}|t-s|}\Big)\int_0^L \Big[\varrho_1\varphi^2 + k(\phi_x+\psi)^2 + \varrho_2 u^2 + b\psi_x^2\Big]dx \leq 0.$$

Now, it remains to show that $\tau_i(t) - \tau_i(s)e^{\frac{c}{\tau(0)}|t-s|} \leq 0, \quad i=1,2.$
Indeed, through mean value theorem, there exists $\alpha \in]s,t[$ such that

$$\tau_i(t) = \tau_i(s) + \tau_i'(\alpha)(t-s)$$

which leads to

$$\frac{\tau_i(t)}{\tau_i(s)} \leq 1 + \frac{|\tau_i'(\alpha)|}{\tau_i(s)}|t-s|.$$

Coming back to (2.4)-(2.5), we obtain

$$\frac{\tau_i(t)}{\tau_i(s)} \leq 1 + \frac{c}{\tau(0)}|t-s| \leq e^{\dfrac{c}{\tau(0)}|t-s|}$$

which yields

$$r\Big(\tau_i(t) - \tau_i(s)e^{\dfrac{c}{\tau(0)}|t-s|}\Big) \leq 0.$$

Hence,

$$||\Theta||_t^2 - ||\Theta||_s^2 e^{\dfrac{c}{\tau(0)}|t-s|} \leq 0, \quad \forall\ \Theta \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}.$$

□

Recalling that the operator $\lambda I - \mathcal{A}(t)$ is surjective for any fixed $t \in (0,+\infty)$ and $\lambda > 0$. Consequently, the operator

$$\lambda I - \widetilde{\mathcal{A}}(t) = (\lambda + \Lambda(t))I - \mathcal{A}(t)$$

is surjective for any fixed $t \in (0,+\infty)$ and $\lambda > 0$.
Now, we will finish with the last lemma that ends the conditions of Theorem 2.5.

Lemma 3.6. *Assume that* (2.4)-(2.7) *hold. Then,*

$$\frac{d}{dt}\widetilde{\mathcal{A}}(t) \in L_*^\infty\Big([0,T], B(D(\mathcal{A}(0)),\mathcal{H})\Big).$$

Proof. Since $\tau_i(t) \in W^{2,\infty}([0,T])$,

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\xi\tau_1''(t)\tau_1(t) - \tau_1'(t)(\xi\tau_1'(t)-1)}{\tau_1^2(t)} z_\xi \\ \dfrac{\rho\tau_2''(t)\tau_2(t) - \tau_2'(t)(\rho\tau_2'(t)-1)}{\tau_2^2(t)} \omega_\rho \end{pmatrix}$$

We know that

$$\frac{\xi\tau_1''(t)\tau_1(t) - \tau_1'(t)(\xi\tau_1'(t)-1)}{\tau_1^2(t)} \quad \text{and} \quad \frac{\rho\tau_2''(t)\tau_2(t) - \tau_2'(t)(\rho\tau_2'(t)-1)}{\tau_2^2(t)}$$

are bounded on $[0,T]$. Therefore,

$$\frac{d\mathcal{A}(t)}{dt} \in L_*^\infty\Big([0,T], B(D(\mathcal{A}(0)),\mathcal{H})\Big).$$

Since

$$\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \Lambda(t)I$$

then

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) = \frac{d}{dt}\mathcal{A}(t) - \Lambda'(t)I.$$

Consequently,

$$\Big|\Big|\frac{d}{dt}\tilde{\mathcal{A}}(t)\Big|\Big|_{D(\mathcal{A}(0))} \leq \Big|\Big|\frac{d}{dt}\mathcal{A}(t)\Big|\Big|_{D(\mathcal{A}(0))} + |\Lambda'(t)|,\ t \in [0,T].$$

Hence

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^{\infty}_{*}\Big([0,T], B(D(\mathcal{A}(0)), \mathcal{H})\Big).$$

In fact, the function Λ was defined by $\Lambda(t) = k_1(t) + k_2(t), \quad t \in [0,T]$. Because that

$$\begin{aligned} k_i'(t) &= \left[\frac{(\tau_i'(t)^2+1)^{\frac{1}{2}}}{2\tau_i(t)}\right]' \\ &= \frac{\tau_i''(t)\tau_i'(t)(\tau_i'(t)^2+1)^{-\frac{1}{2}}}{2\tau_i(t)} - \frac{\tau_i'(t)(\tau_i'(t)^2+1)^{\frac{1}{2}}}{2\tau_i(t)^2} \end{aligned}$$

and using (2.6), it follows $k_i'(t)$ is bounded on $[0,T]$ which yields the boundedness of $\Lambda'(t)$ on $[0,T]$. $\square$

Finally, the operator $\tilde{\mathcal{A}}(t)$ satisfies the conditions of Theorem 2.5. Consequently, there exists a Cauchy evolution system associated to the operator $\tilde{\mathcal{A}}(t)$ that solves the following problem

$$\begin{cases} \dfrac{d}{dt}\tilde{U}(t) = \tilde{\mathcal{A}}(t)\tilde{U}(t), \quad t > 0, \\ \tilde{U}_0 = U(0) = \Big(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0(L, -\tau_1(0)), g_0(L, -\tau_2(0))\Big)^T. \end{cases} \tag{3.25}$$

Precisely, if $U_0 \in D(\mathcal{A}(0))$, the solution $\tilde{U}$ belongs the space

$$C\Big([0,T], D(\mathcal{A}(0))\Big) \cap C^1\Big([0,T], \mathcal{H}\Big).$$

So, let

$$U(t) = \tilde{U}(t)\exp\Big(\int_0^t \Lambda(s)ds\Big).$$

Then,

$$\begin{aligned} U'(t) &= \Lambda(t)\exp\Big(\int_0^t \Lambda(s)ds\Big)\tilde{U}(t) + \exp\Big(\int_0^t \Lambda(s)ds\Big)\tilde{U}'(t) \\ &= \Lambda(t)\exp\Big(\int_0^t \Lambda(s)ds\Big)\tilde{U}(t) + \exp\Big(\int_0^t \Lambda(s)ds\Big)\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= \exp\Big(\int_0^t \Lambda(s)ds\Big)\Big(\Lambda(t)I + \tilde{\mathcal{A}}(t)\Big)\tilde{U}(t) \\ &= \exp\Big(\int_0^t \Lambda(s)ds\Big)\mathcal{A}(t)\tilde{U}(t) \\ &= \mathcal{A}(t)\exp\Big(\int_0^t \Lambda(s)ds\Big)\tilde{U}(t) \\ &= \mathcal{A}(t)U(t). \end{aligned}$$

Consequently, we can conclude that the problem (3.6) admits a unique solution

$$U \in C\Big([0,T], D(\mathcal{A}(0))\Big) \cap C^1\Big([0,T], \mathcal{H}\Big).$$

Moreover, one can extend the solution Z to $[T, 2T]$. Therefore, we consider the following equation

$$\begin{cases} \dfrac{d}{dt}Z(t) = \mathcal{A}(t)Z(t), \qquad \text{for all} \quad t \in [T, 2T], \\ \\ Z(T) = U(T). \end{cases} \tag{3.26}$$

Using the similar argument, one obtains that problem (3.26) has a unique solution

$$Z \in C\Big([T, 2T], D(\mathcal{A}(0))\Big) \cap C^1\Big([T, 2T], \mathcal{H}\Big)$$

which is an extension of U. Proceeding inductively, the solution U is uniquely and continuously extended to $[nT, (n+1)T]$ for all $n \geq 1$. Finally, we obtain that equation (3.6) has a unique strict solution on $[0, +\infty)$. □

4. Exponential stability

Our aim in this section is to show the exponential stability of problem (1.1)-(1.3), under the condition of equal wave speeds of propagation (1.4). Since the systems (1.1)-(1.3) and (3.5) are equivalent, it is up to us to study the stability result of obtained solution of (3.5). For this, let us define the following energy functional associated to the solution of problem (3.5)

$$\begin{aligned} E(t) \quad &= \frac{1}{2}\int_0^L \Big[\varrho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \varrho_2\psi_t^2 + b\psi_x^2\Big](x,t)dx \\ &+ \frac{r\tau_1(t)}{2}\int_0^1 z^2(\xi,t)d\xi + \frac{r\tau_2(t)}{2}\int_0^1 \omega^2(\rho,t)d\rho. \end{aligned} \tag{4.1}$$

In the sequel, we put simply $z(0,t) = z(0)$, $z(1,t) = z(1)$, $\omega(0,t) = \omega(0)$ and $\omega(1,t) = \omega(1)$.

Our approach focuses on some Lyapunov functional defined near the equilibrium point. So, we begin stating the following lemma which describes how the associated energy decreases with respect to the time parameter.

Lemma 4.1. *Assume that (2.4)-(2.7) hold. Then, the derivative of the energy functional given in (4.1) satisfies the following estimate:*

$$\forall t \geq 0, \quad E'(t) \leq -C\Big(z^2(1) + \omega^2(1) + z^2(0) + \omega^2(0)\Big)$$

where $C > 0$ is some positive constant.

Proof. Multiplying equations $(3.5)_1$, $(3.5)_2$, $(3.5)_3$ and $(3.5)_4$ by φ_t, ψ_t, rz and $r\omega$ respectively, we obtain

$$\begin{cases} \varrho_1\varphi_{tt}\varphi_t - k[\varphi_x+\psi]_x\varphi_t = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \\ \varrho_2\psi_{tt}\psi_t - b\psi_{xx}\psi_t + k[\phi_x+\psi]\psi_t = 0, & (x,t) \in (0,L)\times(0,+\infty), \\ \\ \tau_1(t)rz_tz + \Big(1 - \xi\tau_1'(t)\Big)rz_\xi z = 0, & (\xi,t) \in [0,1]\times(0,+\infty), \\ \\ \tau_2(t)r\omega_t\omega + \Big(1 - \rho\tau_2'(t)\Big)r\omega_\rho\omega = 0, & (\rho,t) \in [0,1]\times(0,+\infty). \end{cases} \tag{4.2}$$

Using integration by parts with respect to x over $[0, L]$, through equations $(4.2)_1$ and $(4.2)_2$ and then, adding member to member the obtained equations, it follows

$$\begin{aligned}\frac{\varrho_1}{2}\frac{d}{dt}\int_0^L \varphi_t^2 dx \quad &+k\int_0^L [\varphi_x+\psi]\varphi_{tx}dx + \frac{\varrho_2}{2}\frac{d}{dt}\int_0^L \psi_t^2 dx + b\int_0^L \psi_x\psi_{tx}dx \\ &+k\int_0^L [\varphi_x+\psi]\psi_t dx = k\Big[(\varphi_x+\psi)\varphi_t\Big]_0^L + b\Big[\psi_x\psi_t\Big]_0^L.\end{aligned} \tag{4.3}$$

Using the integration argument through equation $(4.2)_3$ and taking into account that

$$r\int_0^1 \tau_1(t)z_t(\xi)z(\xi)d\xi = \frac{r}{2}\frac{d}{dt}\int_0^1 \tau_1(t)z^2(\xi)d\xi - \frac{r}{2}\int_0^1 \tau_1'(t)z^2(\xi)d\xi \tag{4.4}$$

then,

$$\begin{aligned}\frac{r}{2}\frac{d}{dt}\int_0^1 \tau_1(t)z^2(\xi)d\xi \quad &= \frac{r}{2}\int_0^1 \tau_1'(t)z^2(\xi)d\xi - r\int_0^1 (1-\xi\tau_1'(t))z(\xi)z_\xi(\xi)d\xi \\ &= -\frac{r}{2}\int_0^1 \frac{\partial}{\partial\xi}\Big[(1-\xi\tau_1'(t))z^2(\xi)\Big]d\xi \\ &= \frac{r}{2}\Big(z^2(0)+(\tau_1'(t)-1)z^2(1)\Big).\end{aligned} \tag{4.5}$$

Similarly, we make use equation $(4.2)_4$ to obtain

$$\begin{aligned}\frac{r}{2}\frac{d}{dt}\int_0^1 \tau_2(t)\omega^2(\rho)d\rho \quad &= \frac{r}{2}\int_0^1 \tau_2'(t)\omega^2(\rho)d\rho - r\int_0^1 (1-\rho\tau_2'(t))\omega(\rho)\omega_\rho(\rho)d\rho \\ &= -\frac{r}{2}\int_0^1 \frac{\partial}{\partial\rho}\Big[(1-\rho\tau_2'(t))\omega^2(\rho)\Big]d\rho \\ &= \frac{r}{2}\Big(\omega^2(0)+(\tau_2'(t)-1)\omega^2(1)\Big).\end{aligned} \tag{4.6}$$

Summing up equations (4.3), (4.5) and (4.6) member to member, it follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt} \quad &\int_0^L \Big[\varrho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \varrho_2\psi_t^2 + b\psi_x^2\Big]dx \\ &+\frac{1}{2}\frac{d}{dt}\left[r\int_0^1 \tau_1(t)z^2(\xi)d\xi + r\int_0^1 \tau_2(t)\omega^2(\rho)d\rho\right] \\ &= k\Big[(\varphi_x+\psi)\varphi_t\Big]_0^L + b\Big[\psi_x\psi_t\Big]_0^L + \frac{r}{2}\Big(\omega^2(0)+(\tau_2'(t)-1)\omega^2(1)\Big) \\ &+\frac{r}{2}\Big(z^2(0)+(\tau_1'(t)-1)z^2(1)\Big).\end{aligned} \tag{4.7}$$

But since $\varphi_t(0,t) = \psi_t(0,t) = 0$, $\varphi_t(L,t) = z(0)$, $\varphi_t(L, t-\tau_1(t)) = z(1)$, $\psi)(L,t) = \omega(0)$ and $\psi_t(L, t-\tau_2(t)) = \omega(1)$ then, the boundary conditions give

$$\begin{aligned}k\Big[(\varphi_x+\psi)\varphi_t\Big]_0^L \quad &= -\alpha_1\sigma_1\varphi_t^2(L,t) - \alpha_1(1-\sigma_1)\varphi_t(L,t-\tau_1(t))\varphi_t(L,t) \\ &= -\alpha_1\sigma_1 z^2(0) - \alpha_1(1-\sigma_1)z(0)z(1)\end{aligned}$$

and

$$\begin{aligned}b\Big[\psi_x\psi_t\Big](L,t) \quad &= -\alpha_2\sigma_2\psi_t^2(L,t) - \alpha_2(1-\sigma_2)\psi_t(L,t-\tau_2)\psi_t(L,t) \\ &= -\alpha_2\sigma_2\omega^2(0) - \alpha_2(1-\sigma_2)\omega(0)\omega(1).\end{aligned}$$

Therefore, we use (4.7) to see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L \Big[\varrho_1\varphi_t^2 \quad & +k(\varphi_x+\psi)^2+\varrho_2\psi_t^2+b\psi_x^2\Big](x,t)dx \\ & +\frac{1}{2}\frac{d}{dt}\Bigg[\int_0^1 \tau_1(t)rz^2(\xi,t)d\xi+\int_0^1 \tau_2(t)r\omega^2(\rho,t)d\rho\Bigg] \\ & \leq -\alpha_1\sigma_1 z^2(0)+\frac{\alpha_1(1-\sigma_1)}{2}z^2(0)+\frac{\alpha_1(1-\sigma_1)}{2}z^2(1) \\ & -\alpha_1\sigma_1\omega^2(0)+\frac{\alpha_2(1-\sigma_2)}{2}\omega^2(0)+\frac{\alpha_2(1-\sigma_2)}{2}\omega^2(1) \\ & +\frac{r}{2}\omega^2(0)+\frac{r}{2}(\tau_2'(t)-1)\omega^2(1)+\frac{r}{2}z^2(0)+\frac{r}{2}(\tau_1'(t)-1)z^2(1).\end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L \Big[\varrho_1\varphi_t^2 \quad & +k(\varphi_x+\psi)^2+\varrho_2\psi_t^2+b\psi_x^2\Big](x,t)dx \\ & +\frac{1}{2}\frac{d}{dt}\Bigg[\int_0^1 \tau_1(t)rz^2(\xi,t)d\xi+\int_0^1 \tau_2(t)r\omega^2(\rho,t)d\rho\Bigg] \\ & \leq -\frac{1}{2}\Big(r(1-\tau_1'(t))-\alpha_1(1-\sigma_1)\Big)z^2(1) \\ & -\frac{1}{2}\Big(r(1-\tau_2'(t))-\alpha_2(1-\sigma_2)\Big)\omega^2(1) \\ & -\frac{1}{2}\Big(3\alpha_1\sigma_1-\alpha_1-r\Big)z^2(0)-\frac{1}{2}\Big(3\alpha_2\sigma_2-\alpha_2-r\Big)\omega^2(0).\end{aligned} \tag{4.9}$$

Coming back to (2.7)

$$-\frac{1}{2}\Big(r(1-\tau_i'(t))-\alpha_i(1-\sigma_i)\Big)\leq 0 \quad \text{and} \quad -\frac{1}{2}\Big(3\alpha_i\sigma_i-\alpha_i-r\Big)\leq 0, \quad \forall\, i=1,2.$$

We conclude that,

$$E'(t)\leq -C\Big\{z^2(1)+\omega^2(1)+z^2(0)+\omega^2(0)\Big\}\leq 0$$

where

$$C=\min\left\{\frac{1}{2}\Big(r(1-\tau_i'(t))-\alpha_i(1-\sigma_i)\Big),\ \ \frac{1}{2}\Big(3\alpha_i\sigma_i-\alpha_i-r\Big)\right\}.$$

□

Now, we will establish the exponential stability result of the problem (3.5) through the following theorem.

Theorem 4.2. *Assume that (2.4), (2.7) and (1.4) hold. Furthermore, suppose that $U_0\in D(\mathcal{A}(0))$ be satisfied. Then the energy $E(t)$ decays exponentially, more precisely there exist some positive constants K_1 and γ independent of time t such that*

$$E(t)\leq K_1 e^{-\gamma t}, \quad t\geq 0. \tag{4.10}$$

Proof. The main idea of the proof of this result is to construct a Lyapunov functional $\mathcal{L}$ satisfying

$$M_1E(t)\leq \mathcal{L}(t)\leq M_2E(t), \quad t\geq 0 \tag{4.11}$$

for some positive constants $M_1,\ M_2$ and

$$\frac{d}{dt}\mathcal{L}(t) \leq -\gamma\mathcal{L}(t), \quad t \geq 0. \tag{4.12}$$

Observe that the construction of such Lyapunov functional $\mathcal{L}$ with respect to the regular solution of (3.5) will be made through several lemmas.
We begin with this result that plays an important role in the sequel of this work.

Proposition 4.3. *Assume that $(\varphi, u, \psi, v, z, \omega)$ is the solution of (3.5). Then,*

$$\varphi_{tx}(x,t) = \varphi_{xt}(x,t) \quad \text{and} \quad \psi_{tx}(x,t) = \psi_{xt}(x,t), \quad (x,t) \in [0,L] \times [0,+\infty). \tag{4.13}$$

Proof. For all fixed $x \in [0,L]$, $u(x,t) = \varphi_t(x,t)$ is C^1 on $[0,+\infty)$ since $(\varphi, u, \psi, v, z, \omega)$ is the solution of (3.5). So, one can write

$$\int_0^t u(x,\eta)d\eta = \varphi(x,t) - \varphi_0(x).$$

Consequently, in the sense of distributions,

$$\int_0^t u_x(x,\eta)d\eta = \varphi_x(x,t) - \varphi_{0_x}(x), \quad \text{for all} \quad (x,t) \in [0,L] \times [0,+\infty).$$

Therefore,

$$\int_0^L \Big(\phi(x)\int_0^t u_x(x,\eta)d\eta\Big)dx = \int_0^L \phi(x)(\varphi_x(x,t) - \varphi_{0_x}(x))dx, \quad \forall,\ \phi \in \mathcal{D}([0,L]). \tag{4.14}$$

Using integration by parts over $[0,L]$ in the first member of the above equation, one has

$$\begin{aligned} \int_0^L \Big(\phi(x)\int_0^t u_x(x,\eta)d\eta\Big)dx &= -\int_0^L\int_0^t \phi_x(x)u(x,\eta)d\eta dx \\ &= -\int_0^L \phi_x(x)(\varphi(x,t) - \varphi_0(x))dx. \end{aligned}$$

We obtain after differentiating this equation with respect to time parameter

$$\begin{aligned} \frac{d}{dt}\int_0^L \Big(\phi(x)\int_0^t u_x(x,\eta)d\eta\Big)dx &= -\int_0^L \phi_x(x)\varphi_t(x,t)dx \\ &= \int_0^L \phi(x)\varphi_{tx}(x,t)dx. \end{aligned}$$

Hence, (4.14) gives

$$\int_0^L \phi(x)\varphi_{xt}(x,t)dx = \int_0^L \phi(x)\varphi_{tx}(x,t)dx \quad \text{for all} \quad \phi \in \mathcal{D}([0,L])$$

and thus,

$$\varphi_{tx}(x,t) = \varphi_{xt}(x,t).$$

In the similar way, $\psi_{tx}(x,t) = \psi_{xt}(x,t)$. □

Now, we proceed by the construction of the steps used into the definition of the Lyapunov desired functional. To start, let us put the functional F_1 defined as follows

$$F_1(t) = \varrho_2\int_0^L \psi_t[\varphi_x + \psi]dx + \frac{b}{k}\int_0^L \varrho_1\psi_x\varphi_t dx \tag{4.15}$$

One states this result.

Lemma 4.4. *Assume that* $(\varphi, \psi, z, \omega)$ *be extracted from the solution of* (3.5)*. Then,*

$$\begin{aligned}
&\frac{d}{dt}F_1(t)\\
&\le -k\int_0^L[\varphi_x+\psi]^2dx+\varrho_2\int_0^L\psi_t^2dx+\Big(\frac{b\varrho_1}{k}-\varrho_2\Big)\int_0^L\psi_{xt}\varphi_tdx\\
&+\Big(\frac{\alpha_1\alpha_2\sigma_1}{2k}+\frac{\varrho_2}{2}\Big)z^2(0)+\Big(\frac{\alpha_1\alpha_2\sigma_2}{2k}+\frac{\varrho_2}{2}\Big)\omega^2(0)+\frac{\alpha_1\alpha_2(1-\sigma_1)}{2k}z^2(1)\\
&+\frac{\alpha_1\alpha_2(1-\sigma_2)}{2k}\omega^2(1).
\end{aligned}\tag{4.16}$$

Proof. A simple differentiation of $F_1(t)$ gives

$$\begin{aligned}
\frac{d}{dt}F_1(t) &= \varrho_2\int_0^L\psi_{tt}[\varphi_x+\psi]dx+\varrho_2\int_0^L\psi_t\varphi_{xt}dx\\
&+\varrho_2\int_0^L\psi_t^2dx+\frac{b\varrho_1}{k}\int_0^L\psi_{xt}\varphi_tdx+\frac{b\varrho_1}{k}\int_0^L\psi_x\varphi_{tt}dx.
\end{aligned}$$

From $(3.5)_1$ and $(3.5)_2$, we have

$$\begin{aligned}
\frac{d}{dt}F_1(t) &= b\int_0^L\psi_{xx}[\varphi_x+\psi]dx-k\int_0^L[\varphi_x+\psi]^2dx+\varrho_2\int_0^L\psi_t\varphi_{xt}dx\\
&+\varrho_2\int_0^L\psi_t^2dx+\frac{b\varrho_1}{k}\int_0^L\psi_{xt}\varphi_tdx+b\int_0^L[\varphi_x+\psi]_x\psi_xdx.
\end{aligned}$$

After integration by parts argument over $[0, L]$, one gets

$$\begin{aligned}
\frac{d}{dt}F_1(t) &= -k\int_0^L[\varphi_x+\psi]^2dx+\varrho_2\int_0^L\psi_t^2dx+(\frac{b\varrho_1}{k}-\varrho_2)\int_0^L\psi_{xt}\varphi_tdx\\
&+b\Big[(\varphi_x+\psi)\psi_x\Big]_0^L+\varrho_2\Big[\psi_t\varphi_t\Big]_0^L.
\end{aligned}\tag{4.17}$$

But,

$$\begin{aligned}
kb\Big[(\varphi_x+\psi)\psi_x\Big]_0^L &\le [\alpha_1\sigma_1z(0)+\alpha_1(1-\sigma_1)z(1)][\alpha_2\sigma_2\omega(0)+\alpha_2(1-\sigma_2)\omega(1)]\\
&\le\frac{\alpha_1\alpha_2\sigma_1}{2}z^2(0)+\frac{\alpha_1\alpha_2\sigma_2}{2}\omega^2(0)+\frac{\alpha_1\alpha_2(1-\sigma_1)}{2}z^2(1)\\
&+\frac{\alpha_1\alpha_2(1-\sigma_2)}{2}\omega^2(1)
\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}
\varrho_2\Big[\psi_t\varphi_t\Big]_0^L &= \varrho_2\psi_t(L,t)\varphi_t(L,t)\\
&\le\frac{\varrho_2}{2}\omega^2(0)+\frac{\varrho_2}{2}z^2(0).
\end{aligned}\tag{4.19}$$

Using the inequalities (4.19) and (4.18) in (4.17), we conclude claim (4.16). □

Next, we introduce the functional F_2 defined by:

$$\begin{aligned}
F_2(t) &= \varrho_1\int_0^Lx(\varphi_t\varphi_x)dx+\varrho_2\int_0^Lx(\psi_t\psi_x)dx\\
&-\frac{k}{2}\int_0^L\int_0^t\psi^2(x,\eta)d\eta dx-\frac{\alpha_1L}{2}\int_0^t\psi^2(L,\eta)d\eta,\quad t\ge0.
\end{aligned}\tag{4.20}$$

We have the following estimate.

Lemma 4.5. *Assume that* (φ,ψ,z,ω) *be extracted from the solution of* (3.5). *Then, we have the estimate*

$$\begin{aligned}&\frac{d}{dt}F_2(t)\\&\leq\Big(\frac{\varrho_1 L}{2}+\frac{\alpha_1\sigma_1 L(\alpha_1+k)}{2k}\Big)z^2(0)+\frac{\alpha_1(1-\sigma_1)L(\alpha_1+k)}{2k}z^2(1)\\&+\Big(\frac{\varrho_2 L}{2}+\frac{\alpha_2^2\sigma_2 L}{2b}\Big)\omega^2(0)+\frac{\alpha_2^2(1-\sigma_2)L}{2b}\omega^2(1)-\frac{\varrho_1}{2}\int_0^L\varphi_t^2dx\\&-\frac{\varrho_2}{2}\int_0^L\psi_t^2dx-\frac{b}{2}\int_0^L\psi_x^2dx.\end{aligned}\tag{4.21}$$

Proof. Differentiating (4.20) with respect to the time parameter t and using $(3.5)_1$ and $(3.5)_2$, one can write

$$\begin{aligned}\frac{d}{dt}F_2(t)\ &=k\int_0^L x\varphi_{xx}\varphi_x dx+\varrho_1\int_0^L x\varphi_t\varphi_{xt}dx+\varrho_2\int_0^L x\psi_t\psi_{xt}dx\\&+b\int_0^L x\psi_{xx}\psi_x dx-k\int_0^L x\psi\psi_x dx-\frac{k}{2}\int_0^L\psi^2dx-\frac{\alpha_1 L}{2}\psi^2(L).\end{aligned}$$

Thank to integration by parts over $[0,L]$ to obtain

$$\begin{aligned}\frac{d}{dt}F_2(t)\ &=\frac{kL}{2}\varphi_x^2(L)-\frac{k}{2}\int_0^L\varphi_x^2dx+\frac{\varrho_1 L}{2}\varphi_t^2(L)-\frac{\varrho_1}{2}\int_0^L\varphi_t^2dx+\frac{\varrho_2 L}{2}\psi_t^2(L)\\&-\frac{\varrho_2}{2}\int_0^L\psi_t^2dx+\frac{bL}{2}\psi_x^2(L)-\frac{b}{2}\int_0^L\psi_x^2dx-\frac{kL}{2}\psi^2(L)-\frac{\alpha_1 L}{2}\psi^2(L).\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dt}F_2(t)\ &\leq\frac{kL}{2}\varphi_x^2(L)+\frac{\varrho_1 L}{2}\varphi_t^2(L)-\frac{\varrho_1}{2}\int_0^L\varphi_t^2dx+\frac{\varrho_2 L}{2}\psi_t^2(L)\\&-\frac{\varrho_2}{2}\int_0^L\psi_t^2dx+\frac{bL}{2}\psi_x^2(L)-\frac{b}{2}\int_0^L\psi_x^2dx-\frac{(\alpha_1+k)L}{2}\psi^2(L).\end{aligned}\tag{4.22}$$

Using the boundary condition, one can write

$$\begin{aligned}\varphi_x^2(L)\ &=\frac{1}{k^2}\Big(-\alpha_1\sigma_1 z^2(0)-\alpha_1(1-\sigma_1)z^2(1)-k\psi(L)\Big)^2\\&=\frac{1}{k^2}\Big(\alpha_1\sigma_1(\alpha_1+k)z^2(0)+\alpha_1(1-\sigma_1)(\alpha_1+k)z^2(1)+k(\alpha_1+k)\psi^2(L)\Big).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{kL}{2}\varphi_x^2(L)\ &\leq\frac{\alpha_1\sigma_1 L(\alpha_1+k)}{2k}z^2(0)+\frac{\alpha_1(1-\sigma_1)L(\alpha_1+k)}{2k}z^2(1)\\&+\frac{(\alpha_1+k)L}{2}\psi^2(L).\end{aligned}\tag{4.23}$$

Also, using again the boundary condition, it follows

$$\frac{bL}{2}\psi_x^2(L)\leq\frac{\alpha_2^2\sigma_2 L}{2b}\omega^2(0)+\frac{\alpha_2^2(1-\sigma_2)L}{2b}\omega^2(1)\tag{4.24}$$

since

$$b^2\psi_x^2(L)\leq\alpha_2^2\sigma_2\omega^2(0)+\alpha_2^2(1-\sigma_2)\omega^2(1).$$

Using (4.23) and (4.24) in (4.22), we take into account to $\psi_t(L)=\omega(0)$ and $\varphi_t(L)=z(0)$ to conclude the relation (4.21). $\square$

As in [11] we define the following functional

$$F_3(t) = r\tau_2(t)\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho. \tag{4.25}$$

Therefore, we state this below lemma.

Lemma 4.6. *Assume that ω be extracted from the solution of (3.5). Then, the functional F_3 satisfies*

$$\frac{d}{dt}F_3(t) \leq -2F_3(t) + r\omega^2(0). \tag{4.26}$$

Proof. We differentiate (4.25) and taking into account $(3.5)_4$ to obtain

$$\begin{aligned}\frac{d}{dt}F_3(t) \quad &= r\tau_2'(t)\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho - 2r\tau_2(t)\tau_2'(t)\int_0^1 \rho e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho \\ &+2r\int_0^1 e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega_\rho(\rho,t)\omega(\rho,t)d\rho.\end{aligned}$$

Since

$$\begin{aligned}&2r\int_0^1 e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega_\rho(\rho,t)\omega(\rho,t)d\rho \\ &= r\int_0^1 \frac{\partial}{\partial\rho}\Big(e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega^2(\rho,t)\Big)d\rho \\ &+2\tau_2(t)r\int_0^1 e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega^2(\rho,t)d\rho - \tau_2'(t)r\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho,\end{aligned}$$

then

$$\begin{aligned}&\frac{d}{dt}F_3(t) = r\tau_2'(t)\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho - 2r\tau_2(t)\tau_2'(t)\int_0^1 \rho e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho \\ &+r\Big[e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega^2(\rho,t)\Big]_0^1 + 2\tau_2(t)r\int_0^1 e^{-2\tau_2(t)\rho}(\rho\tau_2'(t)-1)\omega^2(\rho,t)d\rho \\ &-r\tau_2'(t)\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho.\end{aligned}$$

Consequently, after simplifying the above equation, it follows that

$$\frac{d}{dt}F_3(t) = r(\tau_2'(t)-1)e^{-2\tau_2(t)}\omega^2(1,t) + r\omega^2(0,t) - 2\tau_2(t)r\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho. \tag{4.27}$$

But

$$r(\tau_2'(t)-1)e^{-2\tau_2(t)}\omega^2(1,t) < 0.$$

Hence

$$\begin{aligned}\frac{d}{dt}F_3(t) \quad &\leq -2\tau_2(t)r\int_0^1 e^{-2\tau_2(t)\rho}\omega^2(\rho,t)d\rho + r\omega^2(0,t) \\ &\leq -2F_3(t) + r\omega^2(0).\end{aligned}$$

□

In the same manner as the previous lemma, we consider the functional defined as follows

$$F_4(t) = r\tau_1(t)\int_0^1 e^{-2\tau_1(t)\xi}z^2(\xi,t)d\xi. \tag{4.28}$$

One claims this useful result which plays also an important role in the rest of our work.

Lemma 4.7. *Assume that z be extracted from the solution of (3.5). Then, the functional F_4 satisfies*

$$\frac{d}{dt}F_4(t) \leq -2F_4(t) + rz^2(0). \tag{4.29}$$

Proof. The proof of this lemma is obtained in the similar way of the proof of the previous lemma. □

Finally,we put the following functional

$$\mathcal{L}(t) \quad = NE(t) + F_1(t) + 4F_2(t) + F_3(t) + F_4(t) \tag{4.30}$$

where N is a positive constant to be chosen properly later. We have to prove that the energies E and $\mathcal{L}(t)$ are equivalent. Differentiating $\mathcal{L}(t)$ with respect to the time parameter t, it follows

$$\frac{d}{dt}\mathcal{L}(t) \quad = N\frac{d}{dt}E(t) + \frac{d}{dt}F_1(t) + 4\frac{d}{dt}F_2(t) + \frac{d}{dt}F_3(t) + \frac{d}{dt}F_4(t).$$

Now, we make use the relations (4.16), (4.21), (4.26) and (4.29) to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\leq \beta_1 z^2(0) + \beta_2\omega^2(0) + \beta_3 z^2(1) + \beta_4\omega^2(1)\\ &+\beta_5\int_0^L \varphi_t^2 dx + \beta_6\int_0^L \psi_t^2 dx + \beta_7\int_0^L [\varphi_x + \psi]^2 dx\\ &+\beta_8\int_0^L \psi_x^2 dx + 2\Big(\frac{b\varrho_1}{k} - \varrho_2\Big)\int_0^L \psi_{xt}\varphi_t dx\\ &-2r\tau_2(t)\int_0^1 e^{-2\tau_2(t)\rho}\omega^2 d\rho - 2r\tau_1(t)\int_0^1 e^{-2\tau_1(t)\xi}z^2 d\xi\end{aligned}$$

where

$$\begin{aligned}\beta_1 \quad &= -CN + \Big(\frac{\alpha_1\alpha_2\sigma_1}{2k} + \frac{\varrho_2}{2}\Big) + 4\Big(\frac{\varrho_1 L}{2} + \frac{\alpha_1\sigma_1 L(\alpha_1 + k)}{2k}\Big) + r,\\ \beta_2 \quad &= -CN + \Big(\frac{\alpha_1\alpha_2\sigma_2}{2k} + \frac{\varrho_2}{2}\Big) + 4\Big(\frac{\varrho_2 L}{2} + \frac{\alpha_2^2\sigma_2 L}{2b}\Big) + r,\\ \beta_3 \quad &= -CN + \frac{\alpha_1\alpha_2(1-\sigma_1)}{2k} + 4\frac{\alpha_1(1-\sigma_1)L(\alpha_1 + k)}{2k},\\ \beta_4 \quad &= -CN + \frac{\alpha_1\alpha_2(1-\sigma_2)}{2k} + 4\frac{\alpha_2^2(1-\sigma_2)L}{2b},\\ \beta_5 \quad &= -2\varrho_1, \quad \beta_6 = -\varrho_2, \quad \beta_7 = -k, \quad \beta_8 = -2b.\end{aligned}$$

From now on, we select N large enough such that $\beta_1, \beta_2, \beta_3$ and β_4 stay negative. Consequently, there exists $\chi_1 > 0$, $\chi_2 > 0$ and $\chi_3 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad &\leq -\chi_1\int_0^L \Big[\varphi_t^2 + [\varphi_x + \psi]^2 + \psi_t^2 + \psi_x^2\Big]dx\\ &-\chi_2\int_0^1 z^2(\xi)d\xi - \chi_3\int_0^1 \omega^2(\rho)d\rho\\ &+\left(\frac{b\varrho_1}{k} - \varrho_2\right)\int_0^L \psi_{xt}\varphi_t dx.\end{aligned}$$

Hence,

$$\frac{d}{dt}\mathcal{L}(t) \leq -\chi_0 E(t) + \left(\frac{b\varrho_1}{k} - \varrho_2\right)\int_0^L \psi_{xt}\varphi_t dx \tag{4.31}$$

where $\chi_0 = \min\{\chi_1, \chi_2, \chi_3\}$.

From (1.4), we have $\dfrac{b\varrho_1}{k} - \varrho_2 = 0$. Therefore,

$$\frac{d}{dt}\mathcal{L}(t) \leq -\chi_0 E(t), \quad \text{for all} \quad t \geq 0. \tag{4.32}$$

Now, using Young's and Poincaré's inequalities, there is $\tilde{C} > 0$ such that

$$\mid \mathcal{L}(t) - NE(t) \mid \leq \tilde{C}E(t).$$

In fact, from equation (4.30), it follows

$$\mid \mathcal{L}(t) - NE(t) \mid \leq |F_1(t)| + 4|F_2(t)| + |F_3(t)| + |F_4(t)|.$$

The following estimations of $|F_1(t)|$, $|F_2(t)|$, $|F_3(t)|$ and $|F_4(t)|$ are obtained:

$$\begin{aligned} |F_1(t)| &\leq \varrho_2\int_0^L |\psi_t[\varphi_x + \psi]|dx + \frac{b}{k}\int_0^L \varrho_1|\psi_x\varphi_t|dx \\ &\leq \frac{\varrho_2}{2}\int_0^L \psi_t^2 dx + \frac{\varrho_2}{2}\int_0^L (\varphi_x+\psi)^2 dx + \frac{\varrho_1 b}{2k}\int_0^L \psi_x^2 dx + \frac{\varrho_1 b}{2k}\int_0^L \varphi_t^2 dx. \end{aligned}$$

Also,

$$\begin{aligned} F_2(t) &= \varrho_1\int_0^L x(\varphi_t\varphi_x)dx + \varrho_2\int_0^L x(\psi_t\psi_x)dx - \frac{k}{2}\int_0^L\int_0^t \psi^2(x,\eta)d\eta dx \\ &\quad -\frac{\alpha_1 L}{2}\int_0^t \psi^2(L,\eta)d\eta \\ &\leq \varrho_1\int_0^L x(\varphi_t\varphi_x)dx + \varrho_2\int_0^L x(\psi_t\psi_x)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} |F_2(t)| &\leq \frac{\varrho_1 L}{2}\int_0^L \varphi_t^2 dx + \frac{\varrho_1 L}{2}\int_0^L \varphi_x^2 dx + \frac{\varrho_2 L}{2}\int_0^L \psi_t^2 dx + \frac{\varrho_2 L}{2}\int_0^L \psi_x^2 dx \\ &\leq \frac{\varrho_1 L}{2}\int_0^L \varphi_t^2 dx + \varrho_1 L\int_0^L (\varphi_x+\psi)^2 dx + \frac{\varrho_2 L}{2}\int_0^L \psi_t^2 dx \\ &+ \left(\frac{\varrho_2 L}{2} + \frac{\varrho_1 L^3}{2}\right)\int_0^L \psi_x^2 dx \end{aligned}$$

since

$$\begin{aligned} \int_0^L \varphi_x^2 dx &\leq 2\int_0^L (\varphi_x+\psi)^2 dx + 2\int_0^L \psi^2 dx \\ &\leq 2\int_0^L (\varphi_x+\psi)^2 dx + L^2\int_0^L \psi_x^2 dx. \end{aligned}$$

Observe that

$$|F_3(t)| \leq r\tau_2(t)\int_0^1 \omega^2(\rho,t)d\rho$$

and

$$|F_4(t)| \leq r\tau_1(t)\int_0^1 z^2(\xi,t)d\xi.$$

Hence

$$\begin{aligned} & |F_1(t)| + 4|F_2(t)| + |F_3(t)| + |F_4(t)| \\ \leq & \Big(\frac{\varrho_1 b}{2k} + 2\varrho_1 L\Big) \int_0^L \varphi_t^2 dx + \Big(\frac{\varrho_2}{2} + 4\varrho_1 L\Big) \int_0^L (\varphi_x + \psi)^2 dx \\ + & \Big(\frac{\varrho_2}{2} + 2\varrho_2 L\Big) \int_0^L \psi_t^2 dx + \Big(\frac{\varrho_1 b}{2k} + 2\varrho_2 L + 2\varrho_1 L^3\Big) \int_0^L \psi_x^2 dx \\ + & r\tau_1(t) \int_0^1 z^2(\xi, t) d\xi + r\tau_2(t) \int_0^1 \omega^2(\rho, t) d\rho \leq \tilde{C} E(t) \end{aligned}$$

where

$$\begin{aligned} \tilde{C} &= \max\Big\{\frac{2}{\varrho_1}\Big(\frac{\varrho_1 b}{2k} + 2\varrho_1 L\Big), \frac{2}{k}\Big(\frac{\varrho_2}{2} + 4\varrho_1 L\Big), \frac{2}{\varrho_2}\Big(\frac{\varrho_2}{2} + 2\varrho_2 L\Big), \\ & \frac{2}{b}\Big(\frac{\varrho_1 b}{2k} + 2\varrho_2 L + 2\varrho_1 L^3\Big), 2\Big\}. \end{aligned}$$

Consequently, one can find N big enough to obtain two positive constants M_1, M_2 such that

$$M_1 E(t) \leq \mathcal{L}(t) \leq M_2 E(t), \ \forall t \geq 0. \tag{4.33}$$

From (4.32) and (4.33), we get:

$$\frac{\frac{d}{dt}\mathcal{L}(t)}{\mathcal{L}(t)} \leq \frac{-\chi_0 E(t)}{\mathcal{L}(t)} \leq \frac{-\chi_0 E(t)}{M_2 E(t)} \tag{4.34}$$

which is equivalent to:

$$\frac{d}{dt}\mathcal{L}(t) \leq \frac{-\chi_0}{M_2}\mathcal{L}(t). \tag{4.35}$$

Using integration argument over $[0, t]$ when $t \in (0, +\infty)$, we conclude there exists $\gamma = \dfrac{\chi_0}{M_2}$ and a positive constant M such that

$$\mathcal{L}(t) \leq M e^{-\gamma t}, \quad \text{for all} \quad t \geq 0. \tag{4.36}$$

From (4.33) we can rewrite that

$$E(t) \leq K_1 e^{-\gamma t}, \quad \text{for all} \quad t \geq 0. \tag{4.37}$$

where K_1 is some positive constant. Therefore, the Theorem 4.2 is proved. □

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