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BROWDER-TYPE THEOREMS FOR GENERALIZED DRAZIN INVERTIBLE OPERATORS AND APPLICATIONS

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ABSTRACT. In this paper, we investigate the connections between certain spectra arising from Fredholm theory of a generalized Drazin invertible bounded linear operator and those of its generalized Drazin inverse. Furthermore, we analyze the transfer of Browder's theorem and its generalized form from such an operator to its corresponding generalized Drazin inverse. Applications to left, right, and multiplication operators are also presented.

Keywords. Fredholm theory, Browder's type theorems, Generalized Drazin invertible operators.
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1. Introduction and preliminaries

Let $L(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space X. For $T \in L(X)$, let T^*, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote respectively the dual, the null space and the range space of T. Let $\alpha(T)$ be the nullity of T and $\beta(T)$ its deficiency defined by $\alpha(T) = dim\mathcal{N}(T)$ and $\beta(T) = codim\mathcal{R}(T)$. For an operator $T \in L(X)$, if the range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp. *lower semi-Fredholm*). If $T \in L(X)$ is either upper or lower semi-Fredholm, then it is said to be *semi-Fredholm* and its index is defined by ind $(T) = \alpha(T) - \beta(T)$. If $T \in L(X)$ is both upper and lower semi-Fredholm, then T is said to be *Fredholm*. An operator T is said to be *Weyl* if it is Fredholm of index zero. An *upper semi-Fredholm*, with ind $(T) \leq 0$ is said to be upper semi-Weyl and a *lower semi-Fredholm*, with ind $(T) \geq 0$ is said to be lower semi-Weyl.

For an operator $T \in L(X)$, the ascent $p(T)$ and the descent $q(T)$ are defined by $p(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ and $q(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, respectively, the infimum over the empty set is taken ∞. Recall that T is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if it is Fredholm of finite ascent and descent (resp. T is upper semi-Fredholm and $p(T) < \infty$, T is lower semi-Fredholm and $q(T) < \infty$).

The concept of semi-Fredholm operators has been generalized by Berkani [9] in the following way, for every $T \in L(X)$ and $n \in \mathbb{N}$, we denote by $T_{[n]}$ the restriction of T on $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is

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closed and $T_{[n]}$ is Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm), then T is called *B-Fredholm* (resp. *upper semi-B-Fredholm*, *lower semi-B-Fredholm*), the class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass. An operator T is called *B-Weyl* operator (resp. *upper semi B-Weyl*, *lower semi B-Fredholm*) if for some integer $n \geq 0$, $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is Weyl (resp. upper semi-Weyl, lower semi-Weyl). Analogously, an operator is said to be *B-Browder* (resp. *upper semi B-Browder*, *lower semi B-Browder*) if for some integer $n \geq 0$, $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder).

An operator $T \in L(X)$ is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in L(X)$ such that

$$ST = TS,\ T^{k+1}S = T^k,\ \textit{and}\ S^2T = S.$$

It is well known that T is Drazin invertible if and only if $T = T_1 \oplus T_2$ where T_1 is an invertible operator and T_2 is a nilpotent (see [22, Corollary 2.2]). The Drazin inverse S of T will be denoted by T^D. An operator T is said to be *left Drazin* invertible if $p := p(T) < \infty$ and $\mathcal{R}(T^{p+1})$ is closed and T is said to be *right Drazin* invertible if $q := q(T) < \infty$ and $\mathcal{R}(T^q)$ is closed.

Let $acc\,\sigma(T)$ and iso $\sigma(T)$ be the set of all accumulation points of $\sigma(T)$ and the set of all isolated points of $\sigma(T)$. We recall that for every $\lambda \in \text{iso}\sigma(T)$, there corresponds a decomposition $X = H_0(T - \lambda I) \oplus K(T - \lambda I)$, where $H_0(T - \lambda I)$, the quasi-nilpotent part of $T - \lambda I$, and $K(T - \lambda I)$ the analytic core of $T - \lambda I$ are the sets

$$H_0(T - \lambda I) = \{x \in X : \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{1/n} = 0\}$$

and

$$\begin{aligned} K(T - \lambda I) = \{x \in X :\ &\text{there exists a sequence } \{x_n\} \subset X \text{ and } \delta > 0 \text{ for which} \\ &x = x_0,\ (T - \lambda I)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\|,\ \text{for all } n = 1, 2, ..\} \end{aligned}$$

An operator $T \in L(X)$ is generalized Drazin invertible if there exists a unique $S \in L(X)$ such that

$$TS = ST,\ \ STS = S \quad \text{and} \quad TST - T \ \text{ is quasi-nilpotent.}$$

The generalized Drazin inverse S of T will be denoted by T^d. Following Koliha [19], T is generalized Drazin invertible if and only if T admits on $X = K(T) \oplus H_0(T)$, the decomposition $T = T_1 \oplus T_2$ where $T_1 = T|_{K(T)}$ is invertible and $T_2 = T|_{H_0(T)}$ is quasi-nilpotent, and this is equivalent to the fact that $0 \notin acc\,\sigma(T)$. For more details about generalized Drazin invertibility, we refer the reader to [16, 19, 20].

Note that if T is Drazin invertible, then T is generalized Drazin invertible, but the converse is not true in general. Indeed, let T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, ...) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, ...),$$

then it is clear that T is quasi-nilpotent with infinite ascent. Hence T is generalized Drazin invertible but T is not Drazin invertible .

For convenience, we collect below the spectra associated with the above class of operators, as well as the notations and symbols that will be employed in what follows.

$\sigma(T)$:	spectrum of T	$\sigma_{bw}(T)$: B-Weyl spectrum of T
$\sigma_a(T)$:	approximate spectrum of T	$\sigma_{ubw}(T)$: upper semi B-Weyl spectrum of T
$\sigma_s(T)$:	surjective spectrum of T	$\Delta(T) := \sigma(T) \setminus \sigma_w(T)$
$\sigma_e(T)$:	essential spectrum of T	$\Delta_+(T) := \sigma(T) \setminus \sigma_{uw}(T)$
$\sigma_D(T)$:	Drazin spectrum of T	$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{uw}(T)$
$\sigma_{LD}(T)$:	Left Drazin spectrum of T	$\Delta^g(T) := \sigma(T) \setminus \sigma_{bw}(T)$
$\sigma_{RD}(T)$:	Right Drazin spectrum of T	$\Delta^g_+(T) := \sigma(T) \setminus \sigma_{ubw}(T)$
$\sigma_b(T)$:	Browder spectrum of T	$\Delta^g_a(T) := \sigma_a(T) \setminus \sigma_{ubw}(T)$
$\sigma_{ub}(T)$:	upper semi-Browder spectrum of T	$\Pi(T)$: poles of T.
$\sigma_{lb}(T)$:	lower semi-Browder spectrum of T	$\Pi^0(T)$: poles of finite rank of T
$\sigma_w(T)$:	Weyl spectrum of T	$\Pi_a(T)$: left poles of T
$\sigma_{uw}(T)$:	upper semi-Weyl spectrum of T	$\Pi^0_a(T)$: left poles of T of finite rank
$\sigma_{lw}(T)$:	lower semi-Weyl spectrum of T	

For $T \in L(X)$, we define:

$$\begin{array}{ll|ll} \Pi(T) := & \sigma(T) \setminus \sigma_D(T) & \Pi^0(T) := & \sigma(T) \setminus \sigma_b(T) \\ \Pi_a(T) := & \sigma_a(T) \setminus \sigma_{LD}(T) & \Pi^0_a(T) := & \sigma_a(T) \setminus \sigma_{ub}(T). \end{array}$$

Obviously, we have $\Pi^0(T) \subseteq \Delta(T) \subseteq \Delta_a(T)$, $\Pi^0_a(T) \subseteq \Delta_a(T)$ and

$$\Delta(T) \subseteq \Delta^g_a(T),\ \Delta(T) \subseteq \Delta^g(T) \subseteq \sigma_a(T),\ \Pi_a(T) \subseteq \Delta^g_a(T),\ \Pi_a(T) \subseteq \Delta^g(T).$$

Spectral theory of bounded linear operators on Banach spaces is a central topic in functional analysis, with the spectral mapping theorem as one of its fundamental results. This theorem ensures, for instance, that if $T \in L(X)$ is invertible, then the spectrum of T^{-1} consists of the reciprocals of the spectrum of T, and similar reciprocity holds for spectra arising from Fredholm and B-Fredholm theory. Several generalizations of invertibility have been studied, focusing on connections between the spectral properties of an operator and those of its generalized inverses. In particular, such reciprocity has been extended to Drazin invertibility by Schmoeger [27], with further developments by Aiena and Triolo [6]. Aponte et al. [8] generalize these results to framework of B-Fredholm theory.

In this paper, we extend these investigations to the setting of generalized Drazin invertibility. We show that Browder, Weyl, upper semi-Weyl, and Riesz operators are preserved under the passage from a generalized Drazin invertible operator T to its generalized Drazin inverse T^d. We further analyze the transfer of Browder-type theorems to generalized Drazin inverse and provide applications to left, right, and multiplication operators. The paper is structured as follows. Section 2 establishes reciprocity results for spectra and stability of several operator classes under generalized Drazin inversion. Section 3 deals with the transfer of some Browder-type theorems, while Section 4 presents applications to left, right, and multiplication operators.

2. Preliminary results

In the following results, we prove that if T is a generalized Drazin invertible operator with its generalized Drazin inverse T^d, then the nonzero points of certain spectra of T and T^d arising from Fredholm theory, satisfy a reciprocity relation.

Theorem 2.1. *Let $T \in L(X)$ be a generalized Drazin invertible with generalized Drazin inverse T^d. Then*

$$\sigma_*(T^d) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T) \setminus \{0\}\},$$

where $\sigma_* \in \{\sigma, \sigma_a, \sigma_s, \sigma_b, \sigma_{ub}, \sigma_{lb}, \sigma_e, \sigma_D, \sigma_{LD}, \sigma_{RD}\}$.

Proof. Suppose that T is generalized Drazin invertible.
If $0 \notin \sigma(T)$, then $f(\lambda) = \dfrac{1}{\lambda}$ is an analytic function in any open neighborhood of $\sigma(T)$ not containing 0. Hence, by Koliha [19] we have

$$T^d = T^{-1} = f(T).$$

Moreover, by [2, Chapter 3] every spectrum $\sigma_* \in \{\sigma, \sigma_a, \sigma_s, \sigma_b, \sigma_{ub}, \sigma_{lb}, \sigma_e, \sigma_D, \sigma_{LD}, \sigma_{RD}\}$ satisfies the spectral mapping theorem. Therefore,

$$\sigma_*(T^d) = \sigma_*(f(T)) = f(\sigma_*(T)) = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T)\}.$$

Now, if $0 \in \sigma(T)$, then $T \in L(K(T) \oplus H_0(T))$ admits the decomposition $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi nilpotent and $T^d = T_1^{-1} \oplus 0$. From [2, Chapter 3], it follow that

$$\sigma_*(T_1 \oplus T_2) = \sigma_*(T_1) \cup \sigma_*(T_2) = \sigma_*(T_1) \cup \{0\}$$

and

$$\sigma_*(T^d) = \sigma_*(T_1^{-1}) \cup \{0\}.$$

Since both T and T^{-1} are invertible, applying again the spectral mapping theorem for T_1, we obtain

$$\sigma_*(T^d) \setminus \{0\} = \sigma_*(T_1^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T_1)\} = \sigma_*(T) \setminus \{0\}.$$

□

As shown in [18] and [3, Chapter 3], the spectral mapping theorem does not hold for $\sigma_*(T)$, where $\sigma_* \in \{\sigma_w, \sigma_{uw}, \sigma_{ubw}, \sigma_{bw}\}$. Nevertheless, we prove below that if T is generalized Drazin invertible, then the nonzero parts of $\sigma_*(T)$ and $\sigma_*(T^d)$ are related by reciprocity.

Theorem 2.2. *Let T be a generalized Drazin invertible operator with generalized Drazin inverse T^d. Then*

$$\sigma_*(T^d) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T) \setminus \{0\}\},$$

where $\sigma_* \in \{\sigma_w, \sigma_{uw}, \sigma_{ubw}, \sigma_{bw}\}$.

Proof. Suppose that $T \in L(X)$ is generalized Drazin invertible.
If $0 \notin \sigma(T)$, then $f(\lambda) = \dfrac{1}{\lambda}$ is an analytic function on an open neighborhood of $\sigma(T)$. So by Koliha [19]

$$T^d = T^{-1} = f(T).$$

Moreover, from [23] and [26] it follows that

$$\sigma_*(T^d) = \sigma_*(T^{-1}) = \sigma_*(f(T)) \subseteq f(\sigma_*(T)) = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T)\}. \tag{2.1}$$

Conversely, let $T^{-1} = S$ and define $g(\lambda) = \dfrac{1}{\lambda}$ an analytic function on an open neighborhood of $\sigma(S)$. Then $S^d = S^{-1} = T = g(S)$ and hence

$$\sigma_*(S^d) = \sigma_*(S^{-1}) = \sigma_*(T) \subseteq g(\sigma_*(S)) = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T^{-1})\}. \tag{2.2}$$

Now, let $\lambda_0 = \dfrac{1}{\mu_0}$ with $\mu_0 \in \sigma_*(T)$. From (2.2), we obtain $\mu_0 = \dfrac{1}{\lambda_0} \in \{\dfrac{1}{\lambda} : \lambda \in \sigma_*(T^{-1})\}$. Therefore, the points of $\sigma_*(T^{-1})$ are precisely the reciprocals of the spectrum $\sigma_*(T)$. If $0 \in \sigma(T)$, then $T \in L(K(T) \oplus H_0(T))$ admits the decomposition $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi nilpotent and $T^d = T_1^{-1} \oplus 0$. Since $\sigma_*(T) \cap \{0\} = \emptyset$, it follows that :

$$\sigma_*(T_1 \oplus T_2) = \sigma_*(T_1) \cup \sigma_*(T_2) = \sigma_*(T_1) \cup \{0\}$$

and

$$\sigma_*(T^d) = \sigma_*(T_1^{-1}) \cup \{0\}.$$

As in the first part, we also have

$$\sigma_*(T^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T) \setminus \{0\}\}.$$

Hence,

$$\sigma_*(T^d) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T) \setminus \{0\}\}.$$

□

Next, we show that Browder, Weyl, upper semi-Weyl and Riesz operators are transferred from a generalized Drazin invertible operator T to its generalized Drazin inverse T^d. Recall that $T \in L(X)$ is *Riesz* if $T - \lambda I$ is Fredholm for all non-zero complex λ, and *algebraic* if $h(T) = 0$ for some nontrivial complex polynomial h.

Theorem 2.3. *Let T be a generalized Drazin invertible operator with generalized Drazin inverse T^d. Then*

(1) *T is Browder if and only if T^d is Browder.*
(2) *T is Weyl if and only if T^d is Weyl.*
(3) *T upper semi-Weyl if and only if T^d is upper semi-Weyl.*
(4) *T is Riesz if and only if T^d is Riesz.*

Proof. (1) If $0 \notin \sigma(T)$, then $f(\lambda) = \dfrac{1}{\lambda}$ is an analytic function on an open neighborhood of $\sigma(T)$. Hence, by Koliha [19]

$$T^d = T^{-1} = f(T)$$

and the assertion is trivial.

If $0 \in \sigma(T)$, then $T \in L(K(T) \oplus H_0(T))$ admits the decomposition $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi-nilpotent and $T^d = T_1^{-1} \oplus 0$. If T is Browder, then T is Fredholm with finite ascent and descent, i.e., $p(T) = q(T) < \infty$. Since T is Fredholm, we have $0 \notin \sigma_e(T) = \sigma_e(T_1) \cup \sigma_e(T_2)$. This implies that T_2 is Fredholm. As T_2 is also quasi-nilpotent, then from [21, Corollary 3.7.4], T_2 became nilpotent and $H_0(T)$ is finite dimensional. Since $ker 0 \subset H_0(T)$, we have $\alpha(T^d) = \alpha(T_1^{-1}) + \alpha(0) = \alpha(0) < \infty$. On the other hand, since the null operator 0 is quasi-nilpotent and therefore algebraic, it follows from [2, Theorem 3.83] that 0 is a pole of its resolvent. Hence,

$$p(T^d) = max\{p(T_1^{-1}), p(0)\} = p(0) < \infty,\ q(T^d) = max\{q(T_1^{-1}), q(0)\} = q(0) < \infty.$$

Thus $p(T^d) = q(T^d) < \infty$. Since moreover $\alpha(T^d) < \infty$, it follows from [2, Theorem 3.4] that $\alpha(T^d) = \beta(T^d) < \infty$. Therefore, T^d is Browder. The reverse implication follows similarly.

(2) Assume that $0 \in \sigma(T)$. Since T is generalized Drazin invertible, we have $0 \in \text{iso}\,\sigma(T)$. Moreover, if T is Weyl, it follows from [2, Theorem 3.77] that T is Browder. By the first part, we deduce that T^d is also Browder. As $0 \in \text{iso}\,\sigma(T^d)$ another application of [2, Theorem 3.77] shows that T^d is a Weyl operator. The reverse implication can be proved similarly.

(3) If $0 \notin \sigma(T)$, then $f(\lambda) = \dfrac{1}{\lambda}$ is an analytic function on an open neighborhood of $\sigma(T)$. Hence, by Koliha [19]

$$T^d = T^{-1} = f(T)$$

and the assertion is trivial in this case. If $0 \in \sigma(T)$, since T is generalized Drazin invertible, then $0 \in \text{iso}\,\sigma(T)$. Since T is upper semi-Weyl, it follows from [2, Theorem 3.77] that T is Browder. From the first part we deduce that that T^d is Browder and since $0 \in \text{iso}\,\sigma(T^d)$. It follows from [2, Theorem 3.77] that T^d is upper semi-Weyl operator. The converse may be proved in a similar way.

(4) Suppose that $T \in L(X)$ is Riesz. Then $\sigma_e(T) = \{0\}$. Assume, for contradiction, that T^d is not Riesz. Then there exist $\lambda \neq 0$ such that $\lambda \in \sigma_e(T^d)$. By Theorem 2.1, we deduce that $0 \neq \frac{1}{\lambda} \in \sigma_e(T)$ which contradicts the fact that $\sigma_e(T) = \{0\}$. Hence, T^d must be Riesz. The converse may be proved in a similar way.

□

Remark 2.4. (1) If T is an algebraic non-invertible operator, then by [2, Theorem 3.83], the spectrum of T is a finite set of poles of the resolvent. Then 0 is an isolated point of $\sigma(T)$, so T is generalized Drazin invertible and its generalized Drazin inverse $T^D = T^d$ is also algebraic.

(2) If T is a Riesz operator and generalized Drazin invertible, then $\sigma(T)$ must be finite. Indeed, if T is Riesz and generalized Drazin invertible, it follows from [1, Theorem 3.21] that the spectrum $\sigma(T)$ is either finite or a sequence converging to 0. The second case cannot occur, since $0 \in acc\,\sigma(T)$ would contradict the assumption that T is generalized Drazin invertible. Then the spectrum must be finite.

(3) If T is Riesz and generalized Drazin invertible, then $\sigma(T)$ is finite. By [14, Theorem 2.3], T admits the decomposition $T = F + Q$, with $Q,\ F \in L(X)$, $QF = FQ = 0$, $\sigma(Q) = \{0\}$ and F is a finite rank operator. In this case, by [15, Theorem 2.2], the generalized Drazin inverse of T is given explicitly by $T^d = (F+Q)^d = \sum_{n=0}^{\infty} Q^n (F^d)^{n+1}$.

Lemma 2.5. *Let $T \in L(X)$ be generalized Drazin invertible with generalized Drazin inverse T^d. We have*

(1) $0 \in \Pi(T) \Leftrightarrow 0 \in \Pi(T^d)$. *If* $\lambda \neq 0$, *then* $\lambda \in \Pi(T) \Leftrightarrow \lambda^{-1} \in \Pi(T^d)$.
(2) $0 \in \Pi_a(T) \Leftrightarrow 0 \in \Pi_a(T^d)$. *If* $\lambda \neq 0$, *then* $\lambda \in \Pi_a(T) \Leftrightarrow \lambda^{-1} \in \Pi_a(T^d)$.
(3) $0 \in \Pi^0(T) \Leftrightarrow 0 \in \Pi^0(T^d)$. *If* $\lambda \neq 0$, *then* $\lambda \in \Pi^0(T) \Leftrightarrow \lambda^{-1} \in \Pi^0(T^d)$.
(4) $0 \in \Pi_a^0(T) \Leftrightarrow 0 \in \Pi_a^0(T^d)$. *If* $\lambda \neq 0$, *then* $\lambda \in \Pi_a^0(T) \Leftrightarrow \lambda^{-1} \in \Pi_a^0(T^d)$.

Proof. (1) If $0 \in \Pi(T)$, then T is Drazin invertible. By [27], $T^D = T^d$ and T^d is also Drazin invertible. Hence, $0 \in \Pi(T^d)$. Similarly, we obtain the other implication.
Let $\lambda \neq 0$ and assume that $\lambda \in \Pi(T) = \sigma(T) \setminus \sigma_D(T)$. By Theorem 2.1, it follows that $\lambda^{-1} \in \sigma(T^d) \setminus \sigma_D(T^d) = \Pi(T^d)$. Similarly, we obtain the other implication. Hence $\lambda \in \Pi(T) \Leftrightarrow \lambda^{-1} \in \Pi(T^d)$.

(2) Since T is generalized Drazin invertible, then $T \in L(K(T) \oplus H_0(T))$ admits the decomposition $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi nilpotent and $T^d = T_1^{-1} \oplus 0$. Let $0 \in \Pi_a(T)$. By definition, $0 \in \sigma_a(T) \setminus \sigma_{LD}(T)$, hence $0 \in \sigma_a(T)$ and $0 \notin \sigma_{LD}(T) = \sigma_{LD}(T_1) \cup \sigma_{LD}(T_2)$, it follows that $0 \in \sigma_a(T_2)$ and $0 \notin \sigma_{LD}(T_2)$. From [4, Theorem 2.8], we have $\sigma_{LD}(T_2) = \sigma_{RD}(T_2^*)$. Thus T_2^* is quasi-nilpotent with finite descent, and by [28, Corollary 10.6] T_2^* is nilpotent, hence T_2 is nilpotent. Consequently, T is Drazin invertible, i.e., $0 \in \Pi(T)$. By assertion (1) this yields $0 \in \Pi(T^d)$, and since we have always $\Pi(T^d) \subset \Pi_a(T^d)$, it follows that $0 \in \Pi_a(T^d)$. Similarly, we obtain the other implication.
Now, let $\lambda \neq 0$, and assume $\lambda \in \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T)$. By Theorem 2.1 $\lambda^{-1} \in \sigma_a(T^d) \setminus \sigma_{LD}(T^d) = \Pi_a(T^d)$. Again, the converse implication follows in the same way.

(3) Recall that $\Pi^0(T) = \sigma(T) \setminus \sigma_b(T)$. If $0 \in \Pi^0(T)$, then $0 \in \sigma(T)$ and $0 \notin \sigma_b(T)$. By Theorem 2.3 it follows that T^d is Browder, hence $0 \in \Pi^0(T^d)$. Similarly, we obtain the other implication. If $\lambda \neq 0$, and $\lambda \in \Pi^0(T) = \sigma(T) \setminus \sigma_b(T)$, from Theorem 2.1, we obtain $\lambda^{-1} \in \sigma(T^d) \setminus \sigma_b(T^d) = \Pi^0(T^d)$. Similarly, we obtain the other implication.

(4) Since T is generalized Drazin invertible, then $T \in L(K(T) \oplus H_0(T))$ admits the decomposition $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasi nilpotent and $T^d = T_1^{-1} \oplus 0$. Let $0 \in \Pi_a^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$. This means $0 \in \sigma_a(T)$ and $0 \notin \sigma_{ub}(T) = \sigma_{ub}(T_1) \cup \sigma_{ub}(T_2)$, it follows that $0 \in \sigma_a(T_2)$ and $0 \notin \sigma_{ub}(T_2) = \sigma_{lb}(T_2^*)$. Since T_2 is quasi-nilpotent, its adjoint T_2^* is also quasi-nilpotent with finite descent. By [28, Corollary 10.6], T_2^* is nilpotent, hence T_2 is nilpotent. Therefore, $T^d = T^D$, and by [6, Lemma 4.4], $0 \in \Pi_a^0(T^d)$. Similarly, we obtain the other implication. Let $\lambda \neq 0$, and suppose $\lambda \in \Pi_a^0(T) = \sigma_a(T) \setminus \sigma_{ub}(T)$. From Theorem 2.1, we deduce $\lambda^{-1} \in \sigma_a(T^d) \setminus \sigma_{ub}(T^d) = \Pi_a^0(T^d)$. Again, the converse is obtained similarly.

□

The following proposition is an immediate consequence of the proof of Theorem 2.3 and Lemma 2.5

Proposition 2.6. *Let $T \in L(X)$. Then the following assertions hold :*

(1) *If T is Fredholm, then T is generalized Drazin invertible if and only if T is Drazin invertible. In this case, $T^D = T^d$*
(2) *If T is polaroid (i.e., $iso\sigma(T) \subset \Pi(T)$), then T is generalized Drazin invertible if and only if T is Drazin invertible. hence, $T^D = T^d$.*
(3) *If T or T^* is of finite descent, then T is generalized Drazin invertible if and only if T is Drazin invertible. Consequently, $T^D = T^d$.*

Proof. (1) Follows from the proof of the first part of Theorem 2.3
(2) If T is polaroid, then $iso\sigma(T) = \Pi(T)$. In particular, T is Drazin invertible.
(3) Follows from the proof of the second part of Lemma 2.5

□

3. Browder's and generalized Browder's type Theorems

Browder-type theorems deal with the spectral structure of certain classes of operators. In this section, we establish that Browder's theorem and its generalized version are preserved when passing from a generalized Drazin invertible operator to its generalized Drazin inverse. For this purpose, we first recall the following definitions.

Definition 3.1. Let $T \in L(X)$. We say that T satisfies

(1) Browder's theorem ((B) for brevity) if $\Delta(T) = \Pi^0(T)$.
(2) a-Browder's theorem ((aB) for brevity) if $\Delta_a(T) = \Pi_a^0(T)$.
(3) Property (b) if $\Delta_a(T) = \Pi^0(T)$.
(4) Property (ab) if $\Delta(T) = \Pi_a^0(T)$.

Clearly, a-Browder's theorem for T entails Browder's theorem. In addition, property (b) yields a-Browder's theorem, whereas property (ab) yields Browder's theorem for T.

Now, we recall several spectral properties which are new strong versions of Browder-type theorems, recently introduced by H. Zariouh [29, 30] and J. Sanabria et al. [25, 24].

Definition 3.2. Let $T \in L(X)$. We say that T satisfies:

(1) Property (az) [29] if $\Delta_+(T) = \Pi_a^0(T)$.
(2) Property (ah) [30] if $\Delta_+(T) = \Pi^0(T)$.
(3) Property (V_Π) [24] if $\Delta_+(T) = \Pi(T)$
(4) Property (V_{Π_a}) [25] if $\Delta_+(T) = \Pi_a(T)$.

The following diagram summarizes the implications among the Browder-type theorems introduced above.

$$(V_\Pi) \Leftrightarrow (V_{\Pi_a}) \Leftrightarrow (az) \Leftrightarrow (ah) \Rightarrow (b) \Rightarrow (ab) \Rightarrow (B)$$

Theorem 3.3. *Let T be a generalized Drazin invertible operator with generalized Drazin inverse T^d. Then*

(1) *T satisfies (B) if and only if T^d satisfies (B).*
(2) *T satisfies (aB) if and only if T^d satisfies (aB).*
(3) *T satisfies property (b) if and only if T^d satisfies property (b).*
(4) *T satisfies property (ab) if and only if T^d satisfies property (ab).*
(4) *T satisfies property (ah) if and only if T^d satisfies property (ah).*

Proof. (1) We show that $\Delta(T^d) = \Pi^0(T^d)$. First, note that $\Pi^0(T^d) \subseteq \Delta(T^d)$, since $\sigma_w(T^d) \subseteq \sigma_b(T^d)$. Conversely, let $0 \in \Delta(T^d)$. Then $0 \in \sigma(T^d)$ and $0 \notin \sigma_w(T^d)$ which implies that T^d is Weyl. By Theorem 2.3 this is equivalent to T is Weyl. Since T satisfies Browder's theorem, we have $\sigma_w(T) = \sigma_b(T)$, so T is Browder. Again by Theorem 2.3, this is equivalent to T^d being Browder, hence $0 \in \Pi^0(T^d)$.

Let $\lambda \neq 0$. By Theorem 2.1, Theorem 2.2 and Lemma 2.5, we have the following equivalences:

$$\begin{aligned}\lambda \in \Delta(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta(T)\\ &\Leftrightarrow \lambda^{-1} \in \Pi^0(T)\\ &\Leftrightarrow \lambda \in \Pi^0(T^d).\end{aligned}$$

Therefore, T^d satisfies Browder's theorem. The proof of the converse is analogous.

(2) Suppose that T satisfies a-Browder's theorem and we show the equality $\Delta_a(T^d) = \Pi_a^0(T^d)$. First, note that $\Pi_a^0(T^d) \subseteq \Delta_a(T^d)$, since $\sigma_{uw}(T^d) \subseteq \sigma_{ub}(T^d)$. Conversely, let $0 \in \Delta_a(T^d)$, then $0 \in \sigma_a(T^d)$ and T^d is an upper semi-Weyl operator. By Theorem 2.3, this is equivalent to saying that T is an upper semi-Weyl operator. Since T satisfies a-Browder's theorem, we have $\sigma_{uw}(T) = \sigma_{ub}(T)$, so T is upper semi-Browder. Again, from Theorem 2.3, this implies that T^d is

upper semi-Browder, hence $0 \in \Pi_a^0(T^d)$.
If $\lambda \neq 0$, by Theorem 2.1, Theorem 2.2 and Lemma 2.5, we have the following equivalences

$$\begin{aligned}\lambda \in \Delta_a(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta_a(T)\\ &\Leftrightarrow \lambda^{-1} \in \Pi_a^0(T)\\ &\Leftrightarrow \lambda \in \Pi_a^0(T^d).\end{aligned}$$

Therefore, T^d satisfies a-Browder's theorem. The converse is similarly.

(3) Since T satisfies property (b), it follows that T satisfies a-Browder's theorem. Hence, by Theorem 2.3, the generalized Drazin inverse T^d also satisfies a-Browder's theorem, and therefore $\Delta_a(T^d) = \Pi_a^0(T^d)$. So $\Pi^0(T^d) \subset \Delta_a(T^d)$. Conversely, let $0 \in \Delta_a(T^d)$. Then $0 \in \sigma_a(T^d)$ and $0 \notin \sigma_{uw}(T^d)$ which means that T^d is upper semi-Weyl. By Theorem 2.3 this is equivalent to T is an upper semi-Weyl operator. Since $0 \in iso\sigma(T)$, it follows from [**?**, Theorem 3.77] that T is Browder. By Theorem 2.3, T^d is Browder, hence $0 \in \Pi^0(T^d)$.
If $\lambda \neq 0$, by Theorems 2.1, Theorem 2.2 and Lemma 2.5, we obtain the following equivalences

$$\begin{aligned}\lambda \in \Delta_a(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta_a(T)\\ &\Leftrightarrow \lambda^{-1} \in \Pi^0(T)\\ &\Leftrightarrow \lambda \in \Pi^0(T^d).\end{aligned}$$

Therefore, T^d satisfies property (b). The converse may be proved in a similar way.

(4) Since T satisfies property (ab), it follows that T satisfies Browder's theorem. Hence, T^d also satisfies Browder's theorem, and therefore $\Delta(T^d) = \Pi^0(T^d)$. Clearly, $\Delta(T^d) \subset \Pi_a^0(T^d)$. Conversely, let $0 \in \Pi_a^0(T^d)$. Then $0 \in \sigma_a(T^d)$ and $0 \notin \sigma_{ub}(T^d)$ this implies that T^d is upper semi-Browder, and hence upper semi-Weyl. By Theorem 2.3 again, we deduce that T is upper semi-Weyl. Since $0 \in iso\sigma(T)$, it follows from [2, Theorem 3.77] that T is Browder. Then T^d also is Browder. Consequently, T^d is Weyl and $0 \in \Delta(T^d)$.
If $\lambda \neq 0$, by Theorems 2.1, Theorem 2.2 and Lemma 2.5, we have the following equivalences

$$\begin{aligned}\lambda \in \Delta(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta(T)\\ &\Leftrightarrow \lambda^{-1} \in \Pi_a^0(T)\\ &\Leftrightarrow \lambda \in \Pi_a^0(T^d).\end{aligned}$$

Therefore, T^d satisfies property (ab). The converse may be proved in a similar way.

(5) Suppose that T satisfies property (ah). Then T satisfies property (b), so T^d also satisfies property (b). Hence, $\Delta_a(T^d) = \Pi^0(T^d)$. Since $\Delta_a(T) \subset \Delta_+(T)$ holds for every operator, we obtain $\Pi^0(T^d) \subset \Delta_+(T^d)$. Now, let $0 \in \Delta_+(T^d)$. Then $0 \in \sigma(T^d)$ and $0 \notin \sigma_{uw}(T^d)$,it follows that T^d is upper semi-Weyl, hence by Theorem 2.3, T is also upper semi-Weyl. Since $0 \in iso\sigma(T)$, then from [2, Theorem 3.77] T is Browder. Consequently, by Theorem 2.3, T^d is Browder as well. Therefore, $0 \in \Pi^0(T^d)$.
If $\lambda \neq 0$, by Theorems 2.1, Theorem 2.2 and Lemma 2.5, we have the following

equivalences

$$\begin{aligned}\lambda \in \Delta_+(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta_+(T)\\ &\Leftrightarrow \lambda^{-1} \in \Pi^0(T)\\ &\Leftrightarrow \lambda \in \Pi^0(T^d).\end{aligned}$$

Therefore, T^d satisfies property (ah). The converse is similarly.

□

In what follows, we extend the previously cited properties to their generalized versions.

Definition 3.4. Let $T \in L(X)$. We say that T satisfies :

(1) Generalized Browder's theorem [10]((gB) for brevity), if $\Delta^g(T) = \Pi(T)$.
(2) Generalized a-Browder's theorem [10]((gaB) for brevity), if $\Delta_a^g(T) = \Pi_a(T)$.
(3) Property (gb) [11] if $\Delta_a^g(T) = \Pi(T)$.
(4) Property (gab) [12] if $\Delta^g(T) = \Pi_a(T)$.
(5) Property (gah) [30] if $\Delta_+^g(T) = \Pi(T)$.
(6) Property (gaz) [29] if $\Delta_+^g(T) = \Pi_a(T)$.

From [7, Theorem 2.1, Theorem 2.2] (B) (resp. (aB)) theorem for T is equivalent to (gB) (resp. (gaB)). The diagram below shows the implications between the generalized Browder-type theorems introduced above

$$(az) \Leftrightarrow (gaz) \Leftrightarrow (gah) \Rightarrow (gb) \Rightarrow (gab) \Rightarrow (gB) \Leftrightarrow (B)$$

.

Proposition 3.5. *Let $T \in L(X)$ be a generalized Drazin invertible operator with its generalized Drazin inverse T^d. Then,*

(1) *T satisfies (gB) (resp. (gaB)) if and only if T^d satisfies (gB) (resp. (gaB)).*
(2) *T satisfies property (gah) if and only if T^d satisfies property (gah).*
(3) *T satisfies property (gaz) if and only if T^d satisfies property (gaz).*

Proof. (1) Recall from [7, Theorem 2.1, Theorem 2.2] that Browder's (resp. a-Browder's) theorem for T is equivalent to Generalized Browder's (resp. generalized a-Browder's) theorem. Consequently, by Theorem 3.3 the result follows immediately.
(2) Recall from [30] that T satisfies property (gah) if and only if T satisfies property (ah). Therefore, by Theorem 3.3, the desired result follows.
(3) From [29], it is known that an operator T satisfies property (gaz) if and only if it satisfies property (az). Hence, by Theorem 3.3, the result follows.

□

Before stating the next theorem, we first recall the following definition.

Definition 3.6. An operator $T \in L(X)$ is said to have the single-valued extension property at $\lambda \in \mathbb{C}$ (abbreviated SVEP at λ,) if for open disc $\mathbb{D}_\lambda$, the only analytic function $f : \mathbb{D}_\lambda \to X$ which satisfies the equation

$$(T - \mu I)f(\mu) = 0$$

is the function $f \equiv 0$. The operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Theorem 3.7. *Let $T \in L(X)$ be a generalized Drazin invertible operator with its generalized Drazin inverse T^d. Then,*

1. *T satisfies property (gb) if and only if T^d satisfies property (gb).*
2. *T satisfies property (gab) if and only if T^d satisfies property (gab).*

Proof. (1) Suppose that T satisfies property (gb) and let $0 \in \Delta_a^g(T^d)$. Then T^d is an upper semi B-Weyl operator and $0 \in iso\sigma(T^d)$. Since $\sigma(T^d) = \sigma((T^d)^*)$, it follows that $0 \in iso\sigma((T^d)^*)$. Hence, $(T^d)^*$ has SVEP at zero and by [5, Theorem 2.5] T^d is Drazin invertible, so $0 \in \Pi(T^d)$. On the other hand, by [5, Lemma 3.1], we have $\Pi_a(T^d) \subseteq \Delta_a^g(T^d)$ and since the inclusion $\Pi(T^d) \subseteq \Pi_a(T^d)$ always holds. It follows that $0 \in \Pi(T^d) \Rightarrow 0 \in \Delta_a^g(T^d)$.

If $\lambda \neq 0$, by Theorem 2.1, Theorem 2.2 and Lemma 2.5, we have the following equivalences

$$\begin{aligned} \lambda \in \Delta_a^g(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta_a^g(T) \\ &\Leftrightarrow \lambda^{-1} \in \Pi(T) \\ &\Leftrightarrow \lambda \in \Pi(T^d). \end{aligned}$$

Therefore, T^d satisfies property (gb). The converse may be proved in a similar way.

(2) Suppose that T satisfies property (gab) and let $0 \in \Pi_a(T^d)$. Then T^d is left Drazin invertible, hence T^d is upper semi B-Weyl. Since $0 \in iso\sigma(T^d)$ and $\sigma(T^d) = \sigma((T^d)^*)$, it follows that $0 \in iso\sigma((T^d)^*)$.Therefore, $(T^d)^*$ has SVEP at zero and by [5, Theorem 2.5], T^d is Drazin invertible. Hence $0 \in \Pi(T^d) \subseteq \Delta^g(T^d)$. Therefore, $0 \in \Pi_a(T), \Rightarrow 0 \in \Delta^g(T^d)$.

Conversely, Suppose that $0 \in \Delta^g(T^d)$. Then T is B-Weyl. Since $0 \in iso\sigma(T^d)$ and $\sigma(T^d) = \sigma((T^d)^*)$, it follows that $0 \in iso\sigma((T^d)^*)$. Thus both T^d and $(T^d)^*$ have SVEP at zero and again by [5, Theorem 2.5] T^d is Drazin invertible. Consequently, $0 \in \Pi(T^d)$, so $0 \in \Pi_a(T^d)$.

If $\lambda \neq 0$, by Theorem 2.1, Theorem 2.2 and Lemma 2.5, we have the following equivalences

$$\begin{aligned} \lambda \in \Delta^g(T^d) &\Leftrightarrow \lambda^{-1} \in \Delta^g(T) \\ &\Leftrightarrow \lambda^{-1} \in \Pi_a(T) \\ &\Leftrightarrow \lambda \in \Pi_a(T^d). \end{aligned}$$

Therefore, T^d satisfies property (gab). The converse may be proved in a similar way.

□

4. Applications

The left and the right multiplication operators induced by A and B are denoted by L_A and R_B, respectively and defined by

$$L_A(Y) = AY \quad \text{and} \quad R_B(Y) = YB, \quad \text{for all} \quad Y \in L(X).$$

The multiplication operator, denoted by $M_{A,B}$ and defined as

$$M_{A,B}(Y) = AYB = L_A R_B(Y), \quad \text{for all } Y \in L(X).$$

This section begins with the transfer of the generalized Drazin inverse from A and B to the left multiplication L_A, right multiplication R_A and multiplication operator

$M_{A,B}$. We then use the results of Sections 2 and 3 to characterize when multiplication operators and their generalized Drazin inverses satisfy Browder's and generalized Browder's theorems.

Theorem 4.1. *For $T \in L(X)$ the following statements are equivalent*

(1) *T is generalized Drazin invertible.*
(2) *L_T is generalized Drazin invertible.*
(3) *R_T is generalized Drazin invertible.*

Moreover in this case, if T^d is the generalized Drazin inverse of T, then L_{T^d} (resp. R_{T^d}) is the generalized Drazin inverse of L_T (resp. R_T)

Proof. Since $\sigma(T) = \sigma(L_T)$ and $\sigma(T) = \sigma(R_T)$, then we obtain the equivalence. On the other hand, observe that $L_{T^d}L_T = L_{T^dT} = L_{TT^d} = L_T L_{T^d}$ and $L_{T^d}L_T L_{T^d} = L_{T^d}$. Moreover, we have

$$\sigma(L_T L_{T^d} L_T - L_T) = \sigma(L_{(TT^dT-T)}) = \sigma(TT^dT - T) = \{0\}.$$

Hence L_{T^d} is the generalized Drazin inverse of L_T. Similarly, one shows that R_{T^d} is the generalized Drazin inverse of R_T. $\square$

Theorem 4.2. *Let $A,\ B \in L(X)$ be generalized Drazin invertible operators. Then $M_{A,B} = L_A R_B$ is generalized Drazin invertible and $M^d_{A,B} = M_{A^d,B^d}$.*

Proof. If A and B are generalized Drazin invertible, then by Theorem 4.1, both L_A and R_B are generalized Drazin invertible, with L_{A^d} and R_{B^d} being their respective generalized Drazin inverses. Since the operators L_A and R_B commutes, it follows from [19, Theorem 5.5] that

$$M^d_{A,B} = (L_A R_B)^d = L^d_A R^d_B = M_{A^d,B^d}.$$

$\square$

Theorem 4.3. *Let $T \in L(X)$, then the following statements are equivalent*

(1) *T is not invertible algebraic.*
(2) *L_T and R_T are not invertible algebraic.*
(3) *$L^D_T = L^d_T$ and $R^D_T = R^d_T$ are algebraic.*

Proof. Since $\sigma(T) = \sigma(L_T)$ and $\sigma(T) = \sigma(R_T)$, it follows that (1) $\Leftrightarrow$ (2).
Moreover, the equivalence (2) $\Leftrightarrow$ (3) follows directly from Remark 2.4 $\square$

Proposition 4.4. *Let $A,\ B \in L(X)$.*

(1) *if A and B are algebraic, then $M_{A,B}$ and $M^D_{A,B} = M^d_{A,B} = M_{A^D,B^D} = M_{A^d,B^d}$ are algebraic.*
(2) *Let A and B be generalized Drazin invertible, if L_A and R_B are Riesz operators, then $M_{A,B}$ is Riesz generalized Drazin operator if and only if $M^d_{A,B} = M_{A^d,B^d}$ is Riesz*

Proof. (1) Follows from [17, Corollary 2.5] and Theorem 4.2
(2) Follows from [2, Theorem 4.7], Theorem 2.3 and Remark 2.4

$\square$

Theorem 4.5. *If $T \in L(X)$ is generalized Drazin invertible, then L_T, R_T, L^d_T, R^d_T satisfies (B), (gB), (aB), (gaB).*

Proof. Apply [13, Theorem 3.2, Theorem 3.7] and Theorem 4.1. $\square$

Theorem 4.6. *Let $A,\ B \in L(X)$ be generalized Drazin operators.*
(a) Suppose that A, B^* satisfy (gB) . Then the following statements are equivalent

(1) *(gB) holds for $M_{A,B}$.*
(2) *(gB) holds for M_{A^d,B^d}.*
(3) $\sigma_w(M_{A,B}) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$.
(4) $\sigma_w(M_{A^d,B^d}) = \sigma(A^d)\sigma_w(B^d) \cup \sigma_w(A^d)\sigma(B^d)$.

(b) Suppose that A, B^* satisfy (gaB). Then the following statements are equivalent

(1) *(gaB) holds for $M_{A,B}$.*
(2) *(gaB) holds for M_{A^d,B^d}.*
(3) $\sigma_{uw}(M_{A,B}) = \sigma(A)\sigma_{uw}(B) \cup \sigma_w(A)\sigma(B)$.
(4) $\sigma_{uw}(M_{A^d,B^d}) = \sigma(A^d)\sigma_{uw}(B^d) \cup \sigma_{uw}(A^d)\sigma(B^d)$.

Proof. Apply [13, Theorem 4.5], Theorem 4.2 and Proposition 3.5 □

Theorem 4.7. *Let $A,\ B \in L(X)$ two operators that satisfy (gB). Suppose that A is not invertible algebraic and not nilpotent and B is generalized Drazin invertible not Drazin invertible. Then the following statements are equivalent*

(1) *(gB) holds for $M_{A,B}$.*
(2) *(gB) holds for M_{A^d,B^d}.*
(3) $\sigma_{bw}(M_{A,B}) = \sigma(A)\sigma_{bw}(B) \cup \sigma_{bw}(A)\sigma(B)$.
(4) $\sigma_w(M_{A,B}) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)$.
(5) $\sigma_{bw}(M_{A^d,B^d}) = \sigma(A^d)\sigma_{bw}(B^d) \cup \sigma_{bw}(A^d)\sigma(B^d)$.
(6) $\sigma_w(M_{A^d,B^d}) = \sigma(A^d)\sigma_w(B^d) \cup \sigma_w(A^d)\sigma(B^d)$.

Proof. Apply [13, Theorem 4.7], Theorem 4.2, Remark 2.4 and Proposition 3.5 □

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