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ON THE ν-TH-ORDER SOLUTIONS FOR NONLINEAR VISCOELASTIC WAVE EQUATION WITH AVERAGED DAMPING IN $\mathbb{R}^n$

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Abstract. This paper investigates the existence, uniqueness, and qualitative properties of ν-th-order solutions for a class of nonlinear viscoelastic wave equations with averaged damping in the whole space $\mathbb{R}^n, n \geq 3\nu, \nu \geq 1$. The model incorporates both linear memory effects and a time-averaged damping term, which captures a more realistic dissipation mechanism in complex media. By employing energy methods, stable set method, and an appropriate integral inequality, we establish the global well-posedness of higher-order solutions under suitable assumptions on the nonlinearity and initial data with minimal a priori mathematical restrictions on the parameters ν, q, p. The analysis extends previous results on lower-order formulations, providing a broader framework for understanding the dynamic response of viscoelastic materials with memory and damping.

Keywords. Wave equation with memory; whole space $\mathbb{R}^n$; Higher-order solutions; averaged damping; global well-posedness; energy decay rate, weighted function
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1. Introduction and model problem

Wave propagation in viscoelastic media has been a topic of significant interest due to its wide range of applications in materials science, geophysics, biomechanics, and engineering. In such media, the stress-strain relationship is influenced not only by the present state but also by the history of deformations, giving rise to memory effects. These effects are commonly modeled through convolution integrals involving relaxation kernels, leading to viscoelastic wave equations with integro-differential operators.
Over the past few decades, extensive research has been carried out on the existence, uniqueness, and asymptotic behavior of solutions to viscoelastic wave equations. Classical results for linear models were established using semigroup theory and Fourier analysis (see Dafermos, 1970s). Later, nonlinear models incorporating memory terms

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were studied using energy methods, Galerkin approximations, and fixed point techniques. The pioneering work of Dafermos and MacCamy introduced memory kernels that satisfy fading properties, leading to the decay of energy over time; see [8, 16, 18, 11].

Incorporation of damping mechanisms further enriched the mathematical structure and physical realism of these models. Particularly, averaged damping — where the damping term reflects a time-averaged dissipation — was introduced to better represent real-world damping behaviors in heterogeneous or composite materials. Compared to instantaneous damping, averaged damping introduces nonlocal-in-time effects and poses additional analytical challenges; see [3, 4, 14].

The analysis of higher-order (or ν-th-order) solutions, especially in unbounded domains such as $\mathbb{R}^n$, remains less explored. While most studies focus on weak or classical solutions, the theory of higher-regularity solutions is crucial for understanding refined qualitative behavior, such as smoothing effects, stability, and nonlinear interactions in wave propagation; see [1, 10, 15].

For $x \in \mathbb{R}^n, n > 3\nu$ and $t \in [0, +\infty)$, we consider the problem.

$$\partial_{tt}u + \Phi(x)\Delta^\nu\Big[(-1)^\nu u + \int_0^t \varpi(t-s)u(x,s)ds\Big] + \int_{\mathbb{R}^n} |\partial_t u|^{q-2}dx\partial_t u = |u|^{p-2}u, \quad (1.1)$$

where $\nu \geq 1$ is an integer and $q, p > 2$. Equipped with the following initial conditions.

$$u(x,0) = u_0(x) \in \mathcal{D}^{\nu,2}(\mathbb{R}^n), \quad \partial_t u(x,0) = u_1(x) \in L^2_\sigma(\mathbb{R}^n). \quad (1.2)$$

Here, the functional $\Phi \in C(\mathbb{R}, \mathbb{R})$ is assumed to satisfy

$$(\Phi(x))^{-1} = \sigma(x), \ \Phi(x) > 0. \quad (1.3)$$

The weighted spaces $\mathcal{D}^{\nu,2}$ and L^2_σ are defined later in Definition 2.1 and we have $\sigma \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$\sigma(x) > 0, \quad \sigma(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n), \ n \geq 2\nu, \quad (1.4)$$

here, $0 \leq \tilde{\gamma} \leq 1$ and $\sigma \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $s = \frac{n}{\nu}$.
We have

$$|\nabla^\nu u|^2 = (\Delta^{\nu/2}u)^2, \text{ for even values of } \nu,$$

and

$$|\nabla^\nu u|^2 = |\nabla(\Delta^{(\nu-1)/2}u)|^2, \text{ for odd } \nu,$$

where

$$|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2, \ \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}.$$

Assume that the relaxation function $\varpi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is nonincreasing, satisfying:

$$\varpi(0) > 0, \ 1 - \int_0^\infty \varpi(s)ds \geq \ell > 0, \quad (1.5)$$

and

$$\varpi(t) \leq -\xi\varpi'(t), \ \forall\, t \geq 0, \ \xi > 0. \quad (1.6)$$

The following notation will be used

$$(v \circ w)(t) = \int_0^t v(t-s)\int_{\mathbb{R}^n} |w(t) - w(s)|^2 dxds.$$

The current work builds upon these foundations by considering a nonlinear viscoelastic wave equation with averaged damping in $\mathbb{R}^n$, and developing a rigorous framework for the existence, uniqueness, and regularity of ν-th-order solutions. The motivation stems

from both theoretical interest in the well-posedness theory for higher-order nonlinear PDEs with memory and damping, and from practical considerations in advanced materials where higher-order effects cannot be neglected.
The next technical lemma will be used.

Lemma 1.1. [16] *For any $w \in C^1(0,T,H^\nu(\mathbb{R}^n))$, we have*

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^t \varpi(t-s)\Delta^\nu w(s)w'(t)dsdx \\ = \quad & \frac{1}{2}\frac{d}{dt}(\varpi\circ\nabla^\nu w) - \frac{1}{2}\frac{d}{dt}\int_0^t \varpi(s)\int_{\mathbb{R}^n}|\nabla^\nu w|^2dxds \\ - \quad & \frac{1}{2}(\varpi'\circ\nabla^\nu w) + \frac{1}{2}\varpi(t)\int_{\mathbb{R}^n}|\nabla^\nu w|^2dx \end{aligned}$$

For $\nu = 2$ and $x \in \Omega$ in an open bounded domain, (1.1) takes the form

$$\partial_{tt}u + \Delta^2 u + \int_{\mathbb{R}^n}|\partial_t u|^q dx\partial_t u = |u|^{p-2}u.$$

This model is studied in [20], and many results are shown. This last work extends the long-time behavior of solutions for the averaged wave equation obtained in [23]. For viscoelastic problems, it is considered in [17] as a BVP of Kirchhoff type.

$$\sigma(x)\partial_t\left(|\partial_t u|^{q-2}\partial_t u\right) - M\Big(\int_{\mathbb{R}^n}||\nabla_x u\|^2dx\Big)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds = 0, \tag{1.7}$$

where $q, n \geq 2$ and M are positive C^1 functions satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$ and the scalar function g satisfying certain conditions. The author obtained a general decay rate using a Lyapunov functional. Suitable weighted functional spaces generated by a density function are defined.
Related system is considered in [5] as

$$\partial_{tt}u + \Delta u + g\Big(\int_{\mathbb{R}^n}|\partial_t u|^2dx\Big)\partial_t u = f\Big(\int_{\mathbb{R}^n}|u|^p dx\Big)|u|^{p-2}u,$$

where a finite-time blow-up for weak solutions is obtained.
In $\mathbb{R}^n$ space, we recall the results in [9], when the authors considered

$$\partial_{tt}u + \Delta u + \alpha(x)\partial_t u + \lambda u + f\Big(\int_{\mathbb{R}^n}|u|^p dx\Big)|u|^{p-2}u = h(x), \tag{1.8}$$

and global solution of (1.8) is established by using the semigroup approach; for more related results, see [7, 8, 2].
Our approach combines energy techniques, stable set, and weighted functions arguments to handle the infinite spatial domain. Furthermore, we discuss how the interplay between spatial dimension n, the order ν of the solution, and the damping kernel influences the global behavior of solutions.
This article is organized as follows. In Section 2, we first state the basic knowledge and define the spaces in which we will study our problem. The stable set and its properties are defined in (3.3) and then the local/global solution in time is stated and proved in Section 3. Section 4 is reserved to show the decay rate properties of the solution given in Theorem 4.1 , where we found that the rate is polynomial if $p > 2$ and exponential if $p = 2$. The results obtained are based on the lemma 2.2.

2. Preliminary and tools

In this section, we introduce the mathematical framework, notations, and key assumptions used throughout the paper. The analysis is conducted in the whole Euclidean space $\mathbb{R}^n$, where $n \geq 1$.

Definition 2.1. [17] We define the function spaces and their norm as:

$$\mathcal{D}^{\nu,2}(\mathbb{R}^n) = \left\{ w \in L^r(\mathbb{R}^n) : \nabla^\nu w \in L^2(\mathbb{R}^n),\ r = \frac{2n}{n-2\nu},\ n > 3\nu \right\}, \tag{2.1}$$

and the space $L^2_\sigma(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(w, v)_{L^2_\sigma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sigma w v dx.$$

For $q \in (1, \infty)$, if w is a measurable function on $\mathbb{R}^n$, we define

$$\|w\|_{L^q_\sigma(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \sigma |w|^q dx \right)^{1/q}, \tag{2.2}$$

and

$$\|w\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |w|^q dx \right)^{1/q}. \tag{2.3}$$

Then $\mathcal{D}^{\nu,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/(n-2\nu)}(\mathbb{R}^n)$, i.e $\exists k > 0$ where

$$\int_{\mathbb{R}^n} |w|^r dx \leq k \int_{\mathbb{R}^n} |\nabla^\nu w|^2 dx, \tag{2.4}$$

where $r = \frac{2n}{n-2\nu}$.

The following inequality will be useful

$$\int_{\mathbb{R}^n} |\nabla^\nu w|^2 dx \geq \gamma^{-1} \int_{\mathbb{R}^n} \sigma |w|^2 dx,\ w \in C_0^\infty(\mathbb{R}^n),\ \sigma \in L^{n/\nu}(\mathbb{R}^n), \tag{2.5}$$

where

$$\gamma = \Big(\int_{\mathbb{R}^n} |\sigma|^{\frac{n}{\nu}} dx \Big)^{\frac{\nu}{n}} > 0.$$

The Hilbert space $L^2_\sigma(\mathbb{R}^n)$ is separable and

$$(w, w)_{L^2_\sigma(\mathbb{R}^n)} = \|w\|^2_{L^2_\sigma(\mathbb{R}^n)},$$

consist of all w where

$$\|w\|_{L^q_\sigma(\mathbb{R}^n)} < \infty,\ 1 < q < +\infty.$$

Let

$$\sigma \in L^{n/\nu}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

then the embedding $\mathcal{D}^{\nu,2}(\mathbb{R}^n) \subset L^2_\sigma(\mathbb{R}^n)$ is compact.

Lemma 2.2. [12] *For $T > 1$, let $h(t)$ be a bounded and non-negative function on $[0, T]$, where*

$$h^{1+a}(t) \leq k_0(h(t) - h(1+t)), \quad 0 \leq t \leq T,$$

here $k_0 > 0$ and a is a non-negative constant, then we have

(1) *If $a > 0$, then $h(t) \leq (h^{-a}(0) + k_0 a[t-1]^+)^{-\frac{1}{a}}$.*

(2) *If* $a = 0$, *then*

$$h(t) \leq h(0)e^{-k_1[t-1]^+},$$

where

$$[t-1]^+ = \max\{t-1, 0\}, k_1 = log(\frac{k_0}{k_0-1}).$$

Lemma 2.3. *[[17], Lemma 3.1] Let* σ *satisfy (1.4), then* $\forall w \in \mathcal{D}^{\nu,2}(\mathbb{R}^n)$

$$\Big(\int_{\mathbb{R}^n} \sigma|w|^q\Big)^{\frac{1}{q}} \leq \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{1}{s}} \Big(\int_{\mathbb{R}^n} |\nabla^\nu w|^2\Big)^{\frac{1}{2}}, \tag{2.6}$$

where

$$s = \frac{2n}{2n - qn + q\nu},$$

for all

$$\begin{cases} 2 \leq q < +\infty & if\, n = \nu, 2\nu \\ 2 \leq q \leq \frac{2n}{n-2\nu} & if, n \geq 3\nu. \end{cases} \tag{2.7}$$

For the operator $(-1)^\nu \Phi \Delta^\nu$, we consider for all $\kappa \in L^2_\sigma(\mathbb{R}^n)$

$$(-1)^\nu \Phi(x) \Delta^\nu u(x) = \kappa(x),\ x \in \mathbb{R}^n,$$

where

$$(-1)^\nu \int_{\mathbb{R}^n} \Phi \Delta^\nu w dx = \int_{\mathbb{R}^n} \nabla^\nu u \nabla^\nu w dx,$$

without any boundary condition. We define the energy of (1.1)-(1.2) by

$$2E(t) = \int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + 2J(u), \tag{2.8}$$

$$J(u) = \frac{1}{2}\Big(1 - \int_0^t \varpi(s)ds\Big) \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u) - \frac{1}{p}\int_{\mathbb{R}^n} \sigma|u|^p dx, \tag{2.9}$$

and

$$\begin{aligned} E(0) &= E(t)|_{t=0} \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \sigma|u_1|^2 dx + J(u_0). \end{aligned} \tag{2.10}$$

3. Weak and Higher-Order Solutions

Now, we state the existence of weak and higher-order solutions for (1.1)-(1.2), where the local existence is stated without proof; for more detail, please see [22].

Theorem 3.1. *If* $u_0 \in \mathcal{D}^{\nu,2}(\mathbb{R}^n)$ *and* $u_1 \in L^2_\sigma(\mathbb{R}^n)$ *such that*

$$(-1)^\nu \Delta^\nu u_0 + \int_{\mathbb{R}^n} \sigma|u_1|^{q-2} dx u_1 \in L^2(\mathbb{R}^n),$$

then $\exists t_{\max} \leq +\infty$ *where* (1.1)-(1.2)*has a unique generalized solution* u, *which also coincides with a distributional solution to the system* (1.1)-(1.2). *It can be extended to be global where*

$$E(t) - \frac{1}{2}\int_0^t \Big[(\varpi' \circ \nabla^\nu u) - \varpi(t)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx\Big] ds + \int_0^t \int_{\mathbb{R}^n} \sigma|\partial_t u|^q dx ds = E(0) \tag{3.1}$$

Proof. To begin to prove the global solution, we introduce

$$I(u) = \Big(1 - \int_0^t \varpi(s)ds\Big) \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + (\varpi \circ \nabla^\nu u) - \int_{\mathbb{R}^n} \sigma |u|^p dx, \tag{3.2}$$

and define the well-known stable set as

$$W = \{u : u \in \mathcal{D}^{\nu,2}(\mathbb{R}^n), \quad I(u) > 0, \ and \ J(u) < d\} \cup \{0\}, \tag{3.3}$$

where

$$d \ = \ \inf\{ \sup_{u \in \mathcal{D}^{\nu,2}(\mathbb{R}^n) \backslash \{0\}, \ \mu \geq 0} J\left(\mu\left(u\right)\right)\}, \tag{3.4}$$

and

$$\mathcal{N} = \left\{u \in \mathcal{D}^{\nu,2}(\mathbb{R}^n) \backslash \left\{0\right\} : I\left(u\right) = 0\right\},$$

The potential depth d is characterized by

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{3.5}$$

and shows that

$$dist\left(0, \mathcal{N}\right) = \min_{u \in \mathcal{N}} \Big(\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \Big)^{\frac{1}{2}}. \tag{3.6}$$

□

Lemma 3.2. *For all time, the potential depth d is nonnegative, i.e. $d > 0, \ \forall t > 0$.*

Proof. By (2.9), we obtain

$$J(\mu(u)) \ = \ \frac{\mu^2}{2}\Big[\Big(1 - \int_0^t \varpi(s)ds\Big) \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + (\varpi \circ \nabla^\nu u)\Big] - \frac{\mu^p}{p} \int_{\mathbb{R}^n} \sigma |u|^p dx.$$

Then, by (1.5), we have

$$J(\mu(u)) \ \geq \ K(\mu)$$

where

$$K(\mu) = \frac{\mu^2}{2} \ell \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - \frac{\mu^p}{p} \int_{\mathbb{R}^n} \sigma |u|^p dx.$$

Then

$$\frac{d}{d\mu} K(\mu) = \mu \ell \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - \mu^{p-1} \int_{\mathbb{R}^n} \sigma |u|^p dx.$$

For $\mu_1 = 0$ and

$$\mu_2 = \left(\frac{\ell \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx}{\int_{\mathbb{R}^n} \sigma |u|^p dx} \right)^{\frac{1}{p-2}},$$

we have

$$\frac{d}{d\mu} K(\mu) = 0.$$

Since

$$\frac{d}{d\mu} K(\mu_2) = 0, K(\mu_1) = 0,$$

and

$$\frac{d^2}{d\mu^2} K(\mu)\big|_{\mu = \mu_2} < 0.$$

We have

$$\begin{aligned}
\sup_{\mu\geq 0} j(\mu) &\geq \sup_{\mu\geq 0} K(\mu) = K(\mu_2) \\
&\geq \frac{\ell}{2}\left(\frac{\ell\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx}{\int_{\mathbb{R}^n}\sigma|u|^pdx}\right)^{\frac{2}{p-2}}\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx \\
&- \frac{1}{p}\left(\frac{\ell\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx}{\int_{\mathbb{R}^n}\sigma|u|^pdx}\right)^{\frac{p}{p-2}}\int_{\mathbb{R}^n}\sigma|u|^pdx \\
&\geq \frac{1}{2}\left(\left(\ell\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{p}{p-2}}\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{-2}{p-2}}\right) \\
&- \frac{1}{p}\left(\left(\ell\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{p}{p-2}}\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{-2}{p-2}}\right) \\
&\geq \frac{\ell^{\frac{2p}{p-2}}}{2}\frac{\int_{\mathbb{R}^n}|\nabla^\nu u|^{\frac{2p}{p-2}}dx}{\int_{\mathbb{R}^n}\sigma|u|^{\frac{2p}{p-2}}dx} - \frac{\ell^{\frac{2p}{p-2}}}{p}\frac{\int_{\mathbb{R}^n}|\nabla^\nu u|^{\frac{2p}{p-2}}dx}{\int_{\mathbb{R}^n}\sigma|u|^{\frac{2p}{p-2}}dx} \\
&\geq \left(\frac{p-2}{2p}\right)\left(\frac{\left(\ell\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{1}{p}}}\right)^{\frac{2p}{p-2}}.
\end{aligned}$$

By generalized Poincaré's inequality in Lemma 2.3, we obtain

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{R}^n}|\sigma|^sdx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{1}{2}} \\
&\leq \ell\left(\int_{\mathbb{R}^n}|\sigma|^sdx\right)^{\frac{1}{s}}\left(\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{1}{2}},
\end{aligned}\tag{3.7}$$

for $s=\frac{2n}{2n-pn+\nu p}$, which implies that

$$\frac{\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{1}{p}}}{\ell\left(\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{1}{2}}} \leq \left(\int_{\mathbb{R}^n}|\sigma|^sdx\right)^{\frac{1}{s}}. \tag{3.8}$$

Then

$$\left(\frac{\ell\left(\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx\right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^n}\sigma|u|^pdx\right)^{\frac{1}{p}}}\right)^{\frac{2p}{p-2}} \geq \left(\int_{\mathbb{R}^n}|\sigma|^sdx\right)^{\frac{-2p}{s(p-2)}}. \tag{3.9}$$

As $p>2$, we have

$$\sup_{\mu\geq 0} j(\mu) \geq \frac{p-2}{2p}\left(\int_{\mathbb{R}^n}|\sigma|^sdx\right)^{\frac{-2p}{s(p-2)}} = d > 0. \tag{3.10}$$

□

Lemma 3.3. *The stable set W is a bounded neighborhood of 0 in $\mathcal{D}^{\nu,2}$.*

Proof. Let $u \in W$, then for $u \neq 0$, we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u) - \frac{1}{p}\int_{\mathbb{R}^n} \sigma|u|^p dx \\ &= \left(\frac{p-2}{2p}\right)\Big[\Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u)\Big] + \frac{1}{p}I(u) \\ &\geq \left(\frac{p-2}{2p}\right)\Big[\Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u)\Big]. \end{aligned} \tag{3.11}$$

Then by (1.5), we have

$$\begin{aligned} \ell \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx &\leq \Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \\ &\leq \left(\frac{2p}{p-2}\right) J(u) \end{aligned}$$

Then

$$\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx < \frac{1}{\ell}\left(\frac{2p}{p-2}\right) d = R. \tag{3.12}$$

Thus, $\forall u \in W$, we have $u \in B$, where

$$B = \left\{ u \in \mathcal{D}^{\nu,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx < R \right\}. \tag{3.13}$$

This completes the proof. $\square$

Lemma 3.4. *Let u be a solution of (1.1)-(1.2). If $u_0 \in W$ and $u_1 \in L^2_\sigma(\mathbb{R}^n)$ such that*

$$r = \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{p}{s}} \Big(\frac{1}{\ell}\frac{2p}{p-2}E(0)\Big)^{\frac{p-2}{2}} < 1, \tag{3.14}$$

for $s = \frac{2n}{2n-pn+\nu p}$. Then $u \in W$ for all t.
In addition, $\exists B > 0$ and $B_1 > 0$ such that

$$\int_{\mathbb{R}^n} \sigma|u|^p dx \leq BE(t), \quad \ell\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \leq BE(t), \tag{3.15}$$

and

$$\ell \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \leq B_1 I(u). \tag{3.16}$$

Proof. Since $u_0 \in W$, then $I(u_0) > 0$ and owing to the continuity of u, we have

$$I(u) \geq 0, \quad t \in [0, t_1), \tag{3.17}$$

where t_1 is the maximal time (possibly $t_1 = t_{\max}$) when (3.17) holds for all time t. As $I(u) > 0$, and if $0 \leq t < t_1$, we get by (2.8), (2.9), and (3.1)

$$\begin{aligned} E(0) &\geq E(t) \\ &\geq J(u) \\ &= \frac{1}{p}I(u) + \frac{p-2}{2p}\Big[\Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u)\Big] \\ &\geq \frac{p-2}{2p}\Big[\Big(1 - \int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + \frac{1}{2}(\varpi \circ \nabla^\nu u)\Big] \\ &\geq \frac{p-2}{2p}\ell\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx, \quad 0 \leq t < t_1. \end{aligned} \tag{3.18}$$

by (1.5) and since $\int_0^t \varpi(s)ds \leq \int_0^\infty \varpi(s)ds$. Then

$$\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \leq \frac{1}{\ell}\frac{2p}{p-2}E(0).$$

By generalized Poincaré's inequality in Lemma 2.3 and (3.18), we obtain for $s = \frac{2n}{2n-pn+p\nu}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \sigma|u|^p dx &\leq \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{p}{s}} \Big(\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx\Big)^{\frac{p}{2}} \\
&\leq \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{p}{s}} \Big(\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx\Big)^{\frac{p-2}{2}} \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \\
&\leq \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{p}{s}} \Big(\frac{1}{\ell}\frac{2p}{p-2}E(0)\Big)^{\frac{p-2}{2}} \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \\
&\leq r\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx, \qquad (3.19)
\end{aligned}$$

and then

$$\begin{aligned}
I(u) &= \Big(1-\int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + (\varpi \circ \nabla^\nu u) - \int_{\mathbb{R}^n} \sigma|u|^p dx \\
&\geq \ell\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - \int_{\mathbb{R}^n} \sigma|u|^p dx \\
&\geq (\ell - r)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx > 0. \qquad (3.20)
\end{aligned}$$

Then, $I(u) > 0$, for all $0 \leq t < t_1$.
We repeat this process, and since

$$\lim_{t\to t_1} \Big(\int_{\mathbb{R}^n} |\sigma|^s dx\Big)^{\frac{p}{s}} \left(\frac{1}{\ell}\frac{2p}{p-2}E(t)\right)^{\frac{p-2}{2}} \leq r < \ell,$$

we find that t_1 can be extended to $t_{\max}$. Furthermore, letting $B = \frac{\ell(p-2)}{2p}$ and $B_1 = \frac{1}{\ell - r}$, we get (3.15) from (3.18)and (3.19), and (3.16) from (3.20). □

Theorem 3.5. *(Global existence) Assume that the condition of Lemma 3.4 holds. Then the solution of (1.1)-(1.2) is global in time.*

Proof. It is sufficient to show that

$$\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx,$$

is bounded independently of t. To do this we should use (2.8), (2.9), (3.1), and the fact that $I(u) > 0$ to obtain

$$\begin{aligned}
E(0) \geq E(t) &= \frac{1}{2}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + J(u) \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + \frac{1}{p}I(u) + \frac{p-2}{2p}\Big[\Big(1-\int_0^t \varpi(s)ds\Big)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx + (\varpi \circ \nabla^\nu u)\Big] \\
&\geq \frac{1}{2}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + \frac{\ell(p-2)}{2p}\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \\
&\geq c\Big(\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dx + \int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx\Big), \ \forall t \in [0, +\infty),
\end{aligned}$$

where

$$c = \min\left\{\frac{1}{2}, \frac{\ell(p-2)}{2p}\right\} > 0,\ p > 2, \ell > 0.$$

This completes the proof. □

4. Qualitative Results

In this section, we summarize the main qualitative properties of ν-th-order solutions to the nonlinear viscoelastic wave equation with averaged damping. These results are obtained under the assumptions stated in the previous section.

Theorem 4.1. *Let the conditions of Lemma 3.4 hold. Then the global solution of (1.1)-(1.2) grows at the following decay rate*

(1) $E(t) \leq E(0)e^{-k_1[t-1]^+}$ *if* $q = 2$,

(2) $E(t) \leq \left(E(0)^{-\frac{q-2}{2}} + k_2[t-1]^+\right)^{-\frac{2}{q-2}}$ *if* $q > 2$,

where k_1, k_2 are positive constants.

Proof. By multiplication of (1.1) by $\sigma\partial_t u$ and integrating over $\mathbb{R}^n$, we get

$$\frac{d}{dt}E(t) + \int_{\mathbb{R}^n} \sigma|\partial_t u|^q dx = \frac{1}{2}\Big[(\varpi' \circ \nabla^\nu u) - \varpi(s)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx\Big].$$

Then, integrate over $[t, t+1]$ to get

$$\begin{aligned} E(t) - E(t+1) &= \int_t^{t+1}\int_{\mathbb{R}^n} \sigma|\partial_t u|^q dxds \\ &+ \frac{1}{2}\int_t^{t+1} \varpi(s)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dxds - \frac{1}{2}\int_t^{t+1}(\varpi' \circ \nabla^\nu u)ds. \end{aligned} \tag{4.1}$$

We put

$$F^2(t) = \int_t^{t+1}\int_{\mathbb{R}^n} \sigma|\partial_t u|^q dxds. \tag{4.2}$$

This choice is the key in our discussion in the last paragraph of this proof. The mean value theorem, we find that $\exists t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\int_{\mathbb{R}^n} \sigma|\partial_t u(t_i)|^q dxds \leq kF^2(t),\ k > 0,\ i = 1, 2. \tag{4.3}$$

Exploiting Hölder's inequality, we obtain

$$\begin{aligned} \int_t^{t+1}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dxds &\leq \left(\int_t^{t+1} 1^{\frac{q}{q-2}} ds\right)^{\frac{q-2}{q}}\left(\int_t^{t+1}\int_{\mathbb{R}^n} \sigma(|\partial_t u|^2)^{\frac{q}{2}} dxds\right)^{\frac{2}{q}} \\ &\leq \left(\int_t^{t+1}\int_{\mathbb{R}^n} \sigma|\partial_t u|^q dx\right)^{\frac{2}{q}}, \end{aligned} \tag{4.4}$$

and then

$$\int_t^{t+1}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dxds \leq F^{\frac{4}{q}}(t), \tag{4.5}$$

Multiplying (1.1) by σu and integrating over $[t_1, t_2] \times \mathbb{R}^n$ to get

$$
\begin{aligned}
&\int\limits_{t_1}^{t_2} \int_{\mathbb{R}^n} \left[\left(1 - \int\limits_0^t \varpi(\tau) d\tau \right) |\nabla^\nu u|^2 - \sigma |u|^p \right] dx ds \\
= &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \sigma \partial_{tt} u u dx ds - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\partial_t u|^{q-2} dx \int_{\mathbb{R}^n} \sigma u \partial_t u dx ds \\
+ &\int\limits_{t_1}^{t_2} \int\limits_0^s \varpi(s-\tau) \int\limits_{\mathbb{R}^n} \nabla^\nu u(s) \left[\nabla^\nu u(\tau) - \nabla^\nu u(s) \right] dx d\tau ds.
\end{aligned}
$$

Obviously

$$
\begin{aligned}
\int_{t_1}^{t_2} I(u) ds \;=\; &- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \sigma \partial_{tt} u u dx ds \\
&- \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\partial_t u|^{q-2} dx \int_{\mathbb{R}^n} \sigma u \partial_t u dx ds + \int\limits_{t_1}^{t_2} (\varpi \circ \nabla^\nu u) ds \\
&+ \int\limits_{t_1}^{t_2} \int\limits_0^s \varpi(s-\tau) \int\limits_{\mathbb{R}^n} \nabla^\nu u(s) \left[\nabla^\nu u(\tau) - \nabla^\nu u(s) \right] dx d\tau ds. \qquad (4.6)
\end{aligned}
$$

Then, the first term in the RHS of (4.6) takes the form

$$
\begin{aligned}
\left| \int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^n} \sigma \partial_{tt} u u dx ds \right| &= \left| \left[\int_{\mathbb{R}^n} \sigma \partial_t u u dx \right]_{t_1}^{t_2} - \int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx ds \right| \\
&= \Big| \int_{\mathbb{R}^n} \sigma |\partial_t u(t_2) u(t_2) dx - \int_{\mathbb{R}^n} \sigma \partial_t u(t_1) u(t_1) dx \\
&- \int_t^{t+1} \int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx ds \Big|.
\end{aligned}
$$

Thanks to Holder's inequalities and Lemma 2.3, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\sigma u \partial_t u| \, dx &\leq \left(\int_{\mathbb{R}^n} \sigma |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx \right)^{\frac{1}{2}} \\
&\leq \gamma \left(\int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \sigma \partial_{tt} u u dx ds \right| &\leq \gamma \left(\int_{\mathbb{R}^n} \sigma |\partial_t u(t_2)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla^\nu u(t_2)|^2 dx \right)^{\frac{1}{2}} \\
&+ \gamma \left(\int_{\mathbb{R}^n} \sigma |\partial_t u(t_1)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla^\nu u(t_1)|^2 dx \right)^{\frac{1}{2}} \\
&+ \int_t^{t+1} \int_{\mathbb{R}^n} \sigma |\partial_t u^2| dx ds. \qquad (4.7)
\end{aligned}
$$

We have

$$\begin{aligned}
E(0) &\geq E(t)\\
&\geq J(u)\\
&\geq \frac{1}{p}I(u)+\frac{p-2}{2p}\Big[\left(1-\int_0^t \varpi(\tau)d\tau\right)\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx+(\varpi\circ\nabla^\nu u)\Big]\\
&\geq \frac{\ell(p-2)}{2p}\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx,
\end{aligned}$$

then

$$\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx \leq \frac{2p}{\ell(p-2)}E(t). \tag{4.8}$$

Furthermore, by (4.8), we have

$$\left(\int_{\mathbb{R}^n}\sigma|\partial_t u(t_2)|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\nabla^\nu u(t_2)|^2dx\right)^{\frac{1}{2}} \leq 2\gamma^{\frac{1}{2}}B^{\frac{-1}{2}}F^{\frac{2}{q}}(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}. \tag{4.9}$$

Therefore, by (4.5), then (4.7) becomes

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\sigma\partial_{tt}uudxds\right| \leq 4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}}F^{\frac{2}{q}}(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}+F^{\frac{4}{q}}(t). \tag{4.10}$$

Since $J(u)\geq 0$ and $E(0)\geq E(t)$, by (2.8) together with Hölder's inequality, (3.15), (4.5), and (4.8), we get

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n}|u|^{q-2}dx\int_{\mathbb{R}^n}\sigma\partial_t uudsdx \leq (2E(0))^{\frac{1}{2}}\gamma BF^{\frac{2}{q}}(t)\sup_{t\leq s\leq t+1}E^{\frac{1}{2}}(s). \tag{4.11}$$

Then, by (4.10) and (4.11), we get from (4.6) that

$$\begin{aligned}
\int_{t_1}^{t_2}I(u)ds &\leq \Big[4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}}+(2E(0))^{\frac{1}{2}}\gamma B\Big]F^{\frac{2}{q}}(t)\sup_{t\leq s\leq t+1}E^{\frac{1}{2}}(s)+F^{\frac{4}{q}}(t)\\
&+\int_{t_1}^{t_2}\int_0^s\varpi(s-\tau)\int_{\mathbb{R}^n}\nabla^\nu u(s)\left[\nabla^\nu u(\tau)-\nabla^\nu u(s)\right]dxd\tau ds\\
&+\int_{t_1}^{t_2}(\varpi\circ\nabla^\nu u)ds.
\end{aligned}$$

We then exploit Young's inequality to estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_0^t\varpi(t-\tau)\nabla^\nu u(t)\left[\nabla^\nu u(t)-\nabla^\nu u(\tau)\right]d\tau dxdt\\
&\leq \delta\int_{t_1}^{t_2}\int_0^t\varpi(s)ds\int_{\mathbb{R}^n}|\nabla^\nu u|^2dxdt+\frac{1}{4\delta}\int_{t_1}^{t_2}\left(\varpi\circ\nabla^\nu u\right)(t)dt,\ \forall\delta>0.
\end{aligned} \tag{4.12}$$

Then

$$\begin{aligned}\int_{t_1}^{t_2} I(u)ds &\le \Big[4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}} + (2E(0))^{\frac{1}{2}}\gamma B\Big] F^{\frac{2}{q}}(t) \sup_{t\le s\le t+1} E^{\frac{1}{2}}(s) + F^{\frac{4}{q}}(t) \\ &+ \Big(1+\frac{1}{4\delta}\Big)\int_{t_1}^{t_2} (\varpi \circ \nabla^{\nu} u)ds + \delta \int_{t_1}^{t_2}\int_0^t \varpi(s)ds \int_{\mathbb{R}^n} |\nabla^{\nu} u|^2 dxdt. \quad (4.13)\end{aligned}$$

Moreover, from (2.8), (3.16), we get

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx + J(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx + \frac{1}{p} I(u) \\ &+ \frac{p-2}{2p}\Big[\Big(1-\int_0^t \varpi(s)ds\Big)\|\nabla^{\nu} u\|_2^2 + (\varpi \circ \nabla^{\nu} u)\Big]. \end{aligned} \quad (4.14)$$

By, (3.20), we have

$$\Big(1-\int_0^t \varpi(s)ds\Big)\|\nabla^{\nu} u\|_2^2 \le B_1 I(u),$$

we have

$$\begin{aligned} E(t) &\le \frac{1}{2}\int_{\mathbb{R}^n} \sigma |\partial_t u|^2 dx + \Big(\frac{1}{p} + \frac{B_1(p-2)}{2p}\Big) I(u) \\ &+ \frac{p-2}{2p}(\varpi \circ \nabla^{\nu} u). \end{aligned} \quad (4.15)$$

By integrating (4.15) over $[t_1, t_2]$, by (4.4) and (4.13), we have

$$\begin{aligned} \int_{t_1}^{t_2} E(t)dt &= \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \sigma|\partial_t u|^2 dxdt + \Big(\frac{1}{p} + \frac{(p-2)B_1}{2p}\Big)\int_{t_1}^{t_2} I(u)dt \\ &+ \frac{p-2}{2p}\int_{t_1}^{t_2} (\varpi \circ \nabla^{\nu} u)dt \quad (4.16) \\ &\le \frac{3}{2}F^{\frac{4}{q}}(t) + \Big(\frac{1}{p} + \frac{(p-2)B_1}{2p}\Big)\Big[4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}} + (2E(0))^{\frac{1}{2}}\gamma B\Big] F^{\frac{2}{q}}(t) \sup_{t\le s\le t+1} E^{\frac{1}{2}}(s) \\ &+ \Big[\Big(1+\frac{1}{4\delta}\Big)\Big(\frac{1}{p} + \frac{(p-2)B_1}{2p}\Big) + \frac{p-2}{2p}\Big]\int_{t_1}^{t_2} (\varpi \circ \nabla^{\nu} u)dt \\ &+ \delta\Big(\frac{1}{p} + \frac{(p-2)B_1}{2p}\Big)\int_{t_1}^{t_2}\int_0^t \varpi(s)ds \int_{\mathbb{R}^n} |\nabla^{\nu} u|^2 dxdt. \end{aligned}$$

The fourth term in (4.13) can be treated as

$$\int_0^t \varpi(s)ds \, \|\nabla^{\nu} u\|_2^2 \le \frac{2p(1-\ell)}{\ell(p-2)} E(t).$$

Thus,

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t \varpi(s)ds\,\|\nabla^\nu u\|_2^2\,dt &\leq \frac{2p\,(1-\ell)}{\ell\,(p-2)}\int_{t_1}^{t_2} E(t)dt\\ &\leq \frac{p\,(1-\ell)}{\ell\,(p-2)}E(t_1)\\ &\leq \frac{p\,(1-\ell)}{\ell\,(p-2)}E(t). \qquad (4.17)\end{aligned}$$

Then

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt &\leq \frac{3}{2}F^{\frac{4}{q}}(t)+\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)\Big[4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}}+(2E(0))^{\frac{1}{2}}\gamma B\Big]F^{\frac{2}{q}}(t)\sup_{t\leq s\leq t+1}E^{\frac{1}{2}}(s)\\ &+\Big[\Big(1+\frac{1}{4\delta}\Big)\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)+\frac{p-2}{2p}\Big]\int_{t_1}^{t_2}(\varpi\circ\nabla^\nu u)dt\\ &+\delta\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)\frac{p\,(1-\ell)}{\ell\,(p-2)}E(t). \qquad (4.18)\end{aligned}$$

Again from (3.1), we have for $s\in[0,t_2]$

$$\begin{aligned}&E(s)=E(t_2)\\ &+\frac{1}{2}\int_s^{t_2}[\varpi(t)\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx-(\varpi'\circ\nabla^\nu u)]dt+\int_s^t\int_{\mathbb{R}^n}\sigma|\partial_t u|^q dxdr. \qquad (4.19)\end{aligned}$$

Using the fact that $t_2-t_1\geq\dfrac{1}{2}$, we obtain

$$\int_{t_1}^{t_2} E(s)ds\geq\int_{t_1}^{t_2}E(t_2)ds\geq\frac{1}{2}E(t_2). \qquad (4.20)$$

By (4.19), (4.20), we have

$$\begin{aligned}E(t) &\leq 3F^{\frac{4}{q}}(t)+kF^2(t)+c_1F^{\frac{2}{q}}(t)\sup_{t\leq s\leq t+1}E^{\frac{1}{2}}(s)\\ &+c_2\int_{t_1}^{t_2}(\varpi\circ\nabla^\nu u)dt+c_3E(t)\\ &+\frac{1}{2}\int_s^{t_2}[\varpi(t)\int_{\mathbb{R}^n}|\nabla^\nu u|^2dx-(\varpi'\circ\nabla^\nu u)]dt,\end{aligned}$$

where

$$c_1=2\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)\Big[4\Big(\frac{\gamma}{B}\Big)^{\frac{1}{2}}+(2E(0))^{\frac{1}{2}}\gamma B\Big]>0,$$

$$c_2=2\Big[\Big(1+\frac{1}{4\delta}\Big)\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)+\frac{p-2}{2p}\Big]>0,$$

$$c_3=2\delta\Big(\frac{1}{p}+\frac{(p-2)B_1}{2p}\Big)\frac{p\,(1-\ell)}{\ell\,(p-2)}.$$

Using (1.6), we can write

$$\int_{t_1}^{t_2} (\varpi \circ \nabla^\nu u)(t)dt \leq -\xi \int_{t_1}^{t_2} (\varpi' \circ \nabla^\nu u)(t)dt.$$

With an appropriate use of Young's inequality, we can choose δ so that $1 - c_3 > 0$ and then there exist $K_1 > 0$ such that

$$\begin{aligned} E(t) &\leq K_1\Big[F^{\frac{4}{q}}(t) + F^2(t)\Big] \\ &+ \frac{1}{2}\int_s^{t_2} [\varpi(t)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - (2c_2\xi + 1)(\varpi' \circ \nabla^\nu u)]dt, \end{aligned} \tag{4.21}$$

We deduce from (4.21) that

$$\begin{aligned} E(t) &\leq K_1\Big[F^{\frac{4}{q}}(t) + F^2(t)\Big] \\ &+ \frac{1}{2}(2c_2\xi + 1)\int_s^{t_2} [\varpi(t)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - (\varpi' \circ \nabla^\nu u)]dt, \end{aligned} \tag{4.22}$$

At this stage, we can distinguish two cases:

(1) If $q = 2$: By (4.22) and (4.2), we get for K_2

$$E(t) \leq K_2(E(t) - E(t+1)), \ \forall t \geq 0.$$

Then, using Lemma 2.2 to get

$$E(t) \leq E(0)e^{-k_1[t-1]^+},$$

where

$$k_1 = log(\frac{K_2}{K_2 - 1}).$$

(2) If $q > 2$: We will use the algebraic inequality

$$(a+b)^{\frac{m}{2}} \leq 2^{\frac{m}{2}}\left(a^{\frac{m}{2}} + b^{\frac{m}{2}}\right), \quad m \geq 2. \tag{4.23}$$

We have by (4.22)

$$\begin{aligned} E(t)^{\frac{q}{2}} &\leq 2^{\frac{q}{2}} K_1\Big[(1 + F^{\frac{2(q-2)}{q}}(t))\Big]^{\frac{q}{2}} F^2(t) \\ &+ 2^{\frac{q-2}{2}}(2c_2\xi + 1)\Big[\int_s^{t_2} [\varpi(t)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - (\varpi' \circ \nabla^\nu u)]dt\Big]^{\frac{q}{2}} \\ &\leq 2^{\frac{q}{2}} K_1\Big[(1 + F^{\frac{2(q-2)}{q}}(t))\Big]^{\frac{q}{2}} (E(t) - E(t+1)) \\ &+ 2^{\frac{q-2}{2}}(2c_2\xi + 1)^{\frac{m}{2}}\Big[\int_s^{t_2} [\varpi(t)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dx - (\varpi' \circ \nabla^\nu u)]dt\Big]^{\frac{q}{2}}. \end{aligned}$$

In the other hand, we have by (4.1)

$$\begin{aligned} &\left(\int_t^{t+1} \varpi(s)\int_{\mathbb{R}^n} |\nabla^\nu u|^2 dxds - \int_t^{t+1} (\varpi' \circ \nabla^\nu u)(s)ds\right)^{\frac{q}{2}} \\ &\leq \ (2(E(t) - E(t+1)))^{\frac{q}{2}}. \end{aligned}$$

Then

$$\begin{aligned} E(t)^{\frac{q}{2}} &\leq 2^{\frac{q}{2}} K_1 \Big[(1 + F^{\frac{2(q-2)}{q}}(t)) \Big]^{\frac{q}{2}} (E(t) - E(t+1)) \\ &+ 2^{\frac{q-2}{2}} (2c_2\xi + 1)\, (2(E(t) - E(t+1)))^{\frac{q}{2}} \\ &\leq \Big[2^{\frac{q}{2}} K_1 \Big((1 + F^{\frac{2(q-2)}{q}}(t)) \Big)^{\frac{q}{2}} + 2^{\frac{2(q-1)}{2}} (2c_2\xi + 1)\, (E(t) - E(t+1))^{\frac{q-2}{2}} \Big] \\ &\times (E(t) - E(t+1)) \end{aligned}$$

Since $E(t)$ is positive, non-increasing functional, we have by (1.5), (3.1), and (4.2)

$$F^2(t) \leq E(0),$$

and

$$F^{\frac{2(q-2)}{q}}(t) \leq E(0)^{\frac{(q-2)}{q}},$$

and

$$(E(t) - E(t+1))^{\frac{q-2}{2}} \leq (E(0))^{\frac{q-2}{2}} \tag{4.24}$$

Then

$$\begin{aligned} E(t)^{\frac{q}{2}} &\leq \Big[2^{\frac{q}{2}} K_1 \Big((1 + E(0)^{\frac{(q-2)}{q}}) \Big)^{\frac{q}{2}} + 2^{\frac{2(q-1)}{2}} (2c_2\xi + 1)\, (E(0))^{\frac{q-2}{2}} \Big] \\ &\times (E(t) - E(t+1)). \end{aligned}$$

There exist $K_0 > 0$ such that

$$E^{\frac{q}{2}}(t) \quad \leq \quad K_0(E(t) - E(t+1)),$$

Again, using Lemma 2.2, we conclude

$$E(t) \leq \left[E(0)^{-r} + k_2 \, [t-1]^+ \right]^s. \tag{4.25}$$

with $r = \frac{q-2}{2} > 0, s = \dfrac{2}{2-q}$ and K_0 is some given positive constant.
This completes the proof.

$\square$

Conclusion

We proposed the model with two different types (natures) of damping terms, "Viscoelasticity" and "averaged damping," with the presence of an algebraic source to discuss the impact of each on the stability. It is known that the viscoelastic term under a minimal requirement ensures stability in the absence of a source with many approaches, see [11] and in the presence of a nonlinear source, under certain conditions. The stability of the wave equation with averaged damping (without viscoelasticity) is treated in [3] and the quantitative/qualitative results are obtained under certain suitable conditions with classical methods.

The question was: what happens when these two damping terms appear at the same time, especially in unbounded domains and higher order spaces (As a result of the many factors that hinder the achievement of potential stability). This is the main subject of our studies, where we found a special interaction between viscoelasticity, averaged damping, and nonlinear source to guarantee the stability of the solution.

In many physical systems, two damping terms are often introduced in the governing equations to capture different mechanisms of energy dissipation. Especially when it comes with two different natures (types) of dissipation as in our case. One handles global/low-frequency damping, and the other one captures localized or high-frequency dissipation.

Conflict of interest. The authors declare that there is no conflict of interest.

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