Gulf Journal of Mathematics
Vol 21, Issue 2 (2025) 384-407

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i2.3461

ON KURRE–VERMA POSITIVE LINEAR OPERATORS: CONVERGENCE AND ASYMPTOTICS

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Abstract. We introduce the Kurre–Verma (KV) positive linear operators on $C[0,1]$, constructed via tempered binomial weights coupled with Beta-distribution integral kernels. This four-parameter structure enables precise variance control and boundary adaptivity, addressing limitations of classical schemes. We establish positivity, linearity, and normalization (preservation of constants) of KV operators, together with asymptotic reproduction of first and second moments. Uniform convergence is proved using Korovkin's theorem, with quantitative rates obtained using Peetre's K functional in connection with the modulus of continuity. A Voronovskaja-type asymptotic formula and a saturation theorem characterize the approximation order sharply. Numerical experiments illustrate the improved performance of KV operators compared with classical Jain–Kantorovich operators. These results highlight both the theoretical significance and the practical potential of the KV framework in approximation theory.

Keywords. Positive linear operators; Kurre–Verma operators; Korovkin theorem; Voronovskaja-type result; Approximation theory.
2020 Mathematics Subject Classification. Primary 41A36; Secondary 41A25, 47B38.

1. Introduction

Since Bernstein introduced his polynomials in 1912 [5], *positive linear operators (PLOs)* have been fundamental in approximation theory. Over time, several generalizations, such as the Kantorovich, Szász–Mirakjan, Baskakov, and Jain operators [9, 12], have been developed, laying the groundwork for extensive studies on convergence, error estimates, and saturation phenomena [10, 17].

In recent decades, research has increasingly focused on *new classes of operators* incorporating additional parameters, modified kernels, weighted PLOs, and algebraic extensions [20]. Notable examples include operators constructed via q-calculus and (p,q)-calculus [13, 14, 23, 24], integral-type modifications [1, 4, 6], and quantum analogues such as Bernstein–Kantorovich–Stancu operators and their generalizations [7, 11, 16, 21]. A comprehensive survey highlighting these modern trends, particularly for (p,q)-calculus, quantum operators, and generalized classical constructions, is provided by Ali et al. [2]. These advancements not

Date: Received: Aug 30, 2025; Revised: Sep 24, 2025; Accepted: Oct 15, 2025.
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only enhance approximation accuracy but also allow finer control near boundaries, thereby broadening the applicability in numerical analysis and computational mathematics.

Despite these developments, two important challenges persist. Traditional binomial or Poisson-type weights offer *limited variance control*, while many commonly used kernels are rigid, *restricting adaptability* and local refinement near domain boundaries. For applications such as nonuniform data smoothing or boundary-sensitive reconstruction, operators capable of *simultaneously managing variance and providing flexible kernels* are highly desirable.

To address these limitations, we present a new family of positive linear operators, termed *Kurre–Verma (KV) operators*, which combine *tempered binomial weights* with *Beta-distribution-based kernels*. Here, the tempering parameters (λ, p) allow control over variance, while the kernel parameters (α, θ) regulate localization and boundary sensitivity. This dual parametrization provides *enhanced approximation precision and flexibility* compared with existing integral-type constructions [6, 18, 19].

The main contributions of this work are:

(1) Definition of the KV operators and verification of their positivity, linearity, and moment reproduction.
(2) Establishment of Korovkin-type convergence results with quantitative error bounds based on the modulus of continuity and Peetre's K-functional.
(3) Derivation of a Voronovskaja-type asymptotic expansion and a saturation theorem describing the exact order of approximation.
(4) Numerical comparisons with Jain–Kantorovich operators, demonstrating that KV operators achieve smaller uniform and L_2 errors.

The paper is organized as follows. Section 2 introduces the formal definition and notation of the KV operators. Section 3 discusses their core properties. Section 4 presents convergence theorems, direct estimates, and asymptotic results. Section 5 provides numerical examples, and Section 6 concludes with a summary and potential directions for future research.

In summary, the KV operators extend classical Bernstein and Kantorovich-type frameworks by introducing parametric flexibility and adaptable kernels, making them suitable for modern approximation challenges where boundary behavior and variance control are important.

2. Definition of the Kurre–Verma and Jain–Kantorovich Operators

The operators we study combine discrete tempered-binomial weights with Beta-distribution kernels. The following definition fixes the notation used throughout the paper.

Definition 2.1. Consider $f \in C[0,1]$ with $n \in \mathbb{N}$ and $x \in [0,1]$. We denote the corresponding *Kurre–Verma (KV) operator* by

$$KV_n(f;x) \;=\; \sum_{i=0}^{n} \omega_{n,i}(x) \int_0^1 f(t)\, \Psi_{n,i}(x,t)\, dt,$$

where:

- $\omega_{n,i}(x)$ are the *tempered binomial weights*:

$$\omega_{n,i}(x) \;=\; \frac{\binom{n}{i}x^i(1-x)^{n-i}}{\left(1+\lambda\left|\frac{i}{n}-x\right|\right)^p} \Bigg/ \sum_{j=0}^{n} \frac{\binom{n}{j}x^j(1-x)^{n-j}}{\left(1+\lambda\left|\frac{j}{n}-x\right|\right)^p},$$

  with parameters $\lambda \geq 0$, $p \geq 0$.
- $\Psi_{n,i}(x,t)$ are the *Beta-type kernels*:

$$\Psi_{n,i}(x,t) \;=\; \frac{t^{\,a_{n,i}(x)-1}\,(1-t)^{\,b_{n,i}(x)-1}}{B\big(a_{n,i}(x),\, b_{n,i}(x)\big)}, \quad t \in [0,1],$$

  where $B(\cdot,\cdot)$ is the Beta function and

$$a_{n,i}(x) = 1 + \alpha n x + \theta i, \qquad b_{n,i}(x) = 1 + \alpha n(1-x) + \theta(n-i),$$

  with kernel parameters $\alpha, \theta \geq 0$.

Thus, $KV_n(f;x)$ averages local Beta-kernel means according to tempered binomial weights. The tempering factor $(1+\lambda|\frac{i}{n}-x|)^{-p}$ reduces the influence of indices away from the central index $\approx nx$, with the strength controlled by (λ, p).

For comparison, we recall the classical Jain–Kantorovich operators.

Definition 2.2 (Jain–Kantorovich (JK) operators)**.** For $f \in C[0,1]$ and $x \in [0,1]$, the Jain–Kantorovich operator is given by

$$J_n(f;x) = (n+1)\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\int_{i/(n+1)}^{(i+1)/(n+1)} f(t)\,dt.$$

These operators serve as a standard benchmark in approximation theory [12, 13].

Notation. For convenience, we summarize recurring symbols:

$KV_n(f;x)$	Kurre–Verma operator applied to f at x,
$J_n(f;x)$	Jain–Kantorovich operator applied to f at x,
$\omega_{n,i}(x)$	tempered binomial weights (KV),
$\Psi_{n,i}(x,t)$	Beta-type kernel associated with index i,
S_n	$= 2 + (\alpha+\theta)n$,
$M_{n,r}(x)$	r-th moment of the KV operator (when needed).

Motivation and Parameter Selection. The Kurre–Verma operators involve four parameters $(\lambda, p, \alpha, \theta)$, each serving a distinct role in balancing variance, bias, and boundary sensitivity.

(i) Tempering parameters (λ, p). These adjust the binomial weights of Bernstein-type schemes. The parameter λ reduces boundary bias by shifting the distribution's center of mass, while p controls its concentration: larger p accelerates convergence but may induce oscillations, whereas $0 < p < 1$ yields smoother, slower convergence.

(ii) Kernel parameters (α, θ). These shape the Beta-type kernels. Larger α emphasizes contributions near $t = 0$, larger θ emphasizes $t = 1$, and $\alpha = \theta$ produces symmetry about $x = \frac{1}{2}$. This flexibility allows refined boundary control absent in classical constructions.
Limiting cases. Classical schemes arise as special instances: for $(\lambda, p, \alpha, \theta) = (0, 1, 1, 1)$ one obtains the Bernstein–Kantorovich operator; setting $(\alpha, \theta) = (0, 0)$ gives a purely binomial form; while $\lambda \neq 0$ generates tempered variants without a direct classical analogue.
(iv) Asymptotic precision. The parameter flexibility ensures the canonical scaling

$$KV_n\big((t-x)^2; x\big) = \frac{x(1-x)}{n} + O\big(\tfrac{1}{n^2}\big)\,,$$

which is essential for Voronovskaja-type expansions and the saturation theorem.
(v) Practical applications. Adjustable parameters make the operators well-suited for tasks requiring local adaptivity, such as boundary-layer problems, adaptive quadrature, and kernel-based numerical methods.
(vi) Guidelines for practical use. For general applications, the balanced choice $(\lambda, p, \alpha, \theta) = (1, 1, 1, 1)$ serves as a convenient default. Increasing λ reduces boundary bias, larger p accelerates convergence for smooth functions, whereas smaller p stabilizes boundary-layer problems. Choosing $\alpha > \theta$ places more emphasis on the left endpoint, $\theta > \alpha$ on the right endpoint, and $\alpha = \theta$ yields a symmetric response. Restricting α, θ to small values (typically in $[0, 1]$) tends to improve numerical stability.

3. Basic Properties of the KV Operators

This section is devoted to verifying the essential characteristics of the Kurre–Verma (KV) operators introduced in Section 2. In particular, we establish positivity, linearity, normalization, and their action on simple test functions. These foundational results serve as the basis for the convergence analysis and asymptotic estimates presented later.

Throughout this section, we assume $f \in C[0, 1]$ and $n \in \mathbb{N}$.

3.1. **Positivity and Linearity.**

Proposition 3.1 (Linearity). *For any $f, g \in C[0, 1]$ and real scalars α, β, one has*

$$KV_n(\alpha f + \beta g; x) \;=\; \alpha KV_n(f; x) + \beta KV_n(g; x), \quad x \in [0, 1].$$

Proof. From Definition 2.1, we can write

$$KV_n(\alpha f + \beta g; x) = \sum_{i=0}^{n} \omega_{n,i}(x) \int_0^1 (\alpha f(t) + \beta g(t))\Psi_{n,i}(x, t)\, dt.$$

Exploiting the linearity of both summation and integration gives

$$\alpha KV_n(f; x) + \beta KV_n(g; x),$$

which proves the statement. □

Proposition 3.2 (Positivity)**.** *If $f(t) \geq 0$ for all $t \in [0,1]$, then* $\mathrm{KV}_n(f;x) \geq 0$ *for all $x \in [0,1]$.*

Proof. Recall the definition of the tempered binomial weights by defintion 2.1 $\omega_{n,i}(x)$ where the tempering parameters satisfy $n-x>0$ (automatic for $x \in [0,1]$ and $n \geq 1$) and $1+\lambda\frac{i}{n-x}>0$ (for instance, this is ensured if $\lambda \geq 0$), so the base of the power is positive and the factor $\left(1+\lambda\frac{i}{n-x}\right)^{-p}$ is well-defined and strictly positive.

Each factor in the numerator is nonnegative: $\binom{n}{i} \geq 0$, $x^i \geq 0$, $(1-x)^{n-i} \geq 0$, and $\left(1+\lambda\frac{i}{n-x}\right)^{-p}>0$. Hence the numerator is ≥ 0. The denominator is a sum of nonnegative terms and at least one of them is strictly positive (e.g. $j=0$ or $j=n$), so the denominator is strictly positive. Therefore

$$\omega_{n,i}(x) \geq 0 \quad \text{for all } i=0,\dots,n,\ x \in [0,1],$$

and, by construction, $\sum_{i=0}^{n} \omega_{n,i}(x)=1$.

Next, the kernels $\Psi_{n,i}(x,\cdot)$ are Beta densities by construction, hence

$$\Psi_{n,i}(x,t) \geq 0 \quad \text{for } t \in [0,1], \qquad \int_0^1 \Psi_{n,i}(x,t)\,dt=1.$$

Consequently, for any $f \geq 0$,

$$\int_0^1 f(t)\,\Psi_{n,i}(x,t)\,dt \ \geq\ 0 \quad \text{for each } i,$$

and $\mathrm{KV}_n(f;x)$ is a convex combination of these nonnegative quantities:

$$\mathrm{KV}_n(f;x)=\sum_{i=0}^{n} \omega_{n,i}(x)\int_0^1 f(t)\,\Psi_{n,i}(x,t)\,dt \ \geq\ 0.$$

This proves the claim. □

Remark 3.3. If $f(t)>0$ on a set of positive measure and all $\omega_{n,i}(x)>0$, then $\mathrm{KV}_n(f;x)>0$.

3.2. **Normalization and Moments.** [8, 10, 19]

Proposition 3.4 (Preservation of Constants)**.** *For every $x \in [0,1]$, one has*

$$KV_n(1;x)=1.$$

Proof. Setting $f(t) \equiv 1$ in the definition, we obtain

$$KV_n(1;x)=\sum_{i=0}^{n} \omega_{n,i}(x)\int_0^1 \Psi_{n,i}(x,t)\,dt.$$

Since each kernel integrates to one,

$$KV_n(1;x)=\sum_{i=0}^{n} \omega_{n,i}(x)=1.$$

□

Proposition 3.5 (First Moment)**.** *For all* $x \in [0,1]$,

$$\mathrm{KV}_n(t;x) = \sum_{i=0}^{n} \omega_{n,i}(x)\, \frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)}.$$

In particular,

$$\mathrm{KV}_n(t;x) = x + O\big(\tfrac{1}{n}\big), \qquad n \to \infty.$$

Proof. By definition,

$$\mathrm{KV}_n(t;x) = \sum_{i=0}^{n} \omega_{n,i}(x) \int_0^1 t\, \Psi_{n,i}(x,t)\, dt.$$

Since the mean of a Beta(a,b) distribution is $a/(a+b)$, with parameters $a = a_{n,i}(x)$ and $b = b_{n,i}(x)$ we obtain

$$\mathrm{KV}_n(t;x) = \sum_{i=0}^{n} \omega_{n,i}(x)\, \frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)}.$$

Now $a_{n,i}(x) = 1 + \alpha n x + \theta i$ and $b_{n,i}(x) = 1 + \alpha n(1-x) + \theta(n-i)$, so their sum

$$S_n = a_{n,i}(x) + b_{n,i}(x) = 2 + (\alpha + \theta)n$$

is independent of i. Hence

$$\frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)} = \frac{1 + \alpha n x + \theta i}{S_n}.$$

Substituting back gives

$$\mathrm{KV}_n(t;x) = \frac{1 + \alpha n x}{S_n} + \frac{\theta}{S_n} \sum_{i=0}^{n} i\, \omega_{n,i}(x).$$

Let

$$m_n(x) := \sum_{i=0}^{n} i\, \omega_{n,i}(x).$$

For the ordinary binomial weights one has $m_n(x) = nx$. Under tempering the deviation is bounded, so

$$m_n(x) = nx + \varepsilon_n(x), \qquad |\varepsilon_n(x)| \le C$$

for some constant C independent of n. Therefore

$$\mathrm{KV}_n(t;x) = \frac{1 + (\alpha + \theta)nx + \theta \varepsilon_n(x)}{S_n}.$$

Since $S_n \sim (\alpha + \theta)n$ as $n \to \infty$, we conclude

$$\mathrm{KV}_n(t;x) = x + O\big(\tfrac{1}{n}\big),$$

as claimed. $\square$

3.3. **Action on Quadratic Functions.**

Proposition 3.6 (Second Moment)**.** *Let $\alpha, \theta, \lambda, p \geq 0$ with $\alpha + \theta > 0$ and $S_n := 2 + (\alpha+\theta)n$. For all $x \in [0,1]$,*

$$\mathrm{KV}_n(t^2; x) = \sum_{i=0}^{n} \omega_{n,i}(x) \left(\mu_{n,i}(x)^2 + \frac{a_{n,i}(x)\, b_{n,i}(x)}{(a_{n,i}(x) + b_{n,i}(x))^2\, (a_{n,i}(x) + b_{n,i}(x) + 1)} \right),$$

where

$$\mu_{n,i}(x) = \frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)} = \frac{1 + \alpha n x + \theta i}{S_n}.$$

In particular,

$$\mathrm{KV}_n(t^2; x) = x^2 + O\big(\tfrac{1}{n}\big) \quad \text{uniformly in } x \in [0,1].$$

Proof. By Definition 2.1 and the second moment of a Beta(a, b) variable,

$$\mathbb{E}[t^2] = \frac{a(a+1)}{(a+b)(a+b+1)} = \frac{a^2}{(a+b)^2} + \frac{ab}{(a+b)^2(a+b+1)} = \mu^2 + \frac{ab}{(a+b)^2(a+b+1)},$$

we obtain the displayed formula.

Write the average of squares as "square of the average + variance" under the weights $\omega_{n,i}(x)$:

$$\sum_{i=0}^{n} \omega_{n,i}(x)\, \mu_{n,i}(x)^2 = \Big(\sum_{i=0}^{n} \omega_{n,i}(x)\, \mu_{n,i}(x)\Big)^2 + \mathrm{Var}_\omega\big(\mu_{n,\bullet}(x)\big) = \mathrm{KV}_n(t; x)^2 + \mathrm{Var}_\omega\big(\mu_{n,\bullet}(x)\big).$$

Hence

$$\mathrm{KV}_n(t^2; x) = \underbrace{\mathrm{KV}_n(t; x)^2}_{(I)} + \underbrace{\mathrm{Var}_\omega\big(\mu_{n,\bullet}(x)\big)}_{(II)} + \underbrace{\sum_{i=0}^{n} \omega_{n,i}(x)\, \frac{a_{n,i} b_{n,i}}{(a_{n,i} + b_{n,i})^2 (a_{n,i} + b_{n,i} + 1)}}_{(III)}.$$

We bound each term relative to x^2:

(I) Using Proposition 3.5, $\mathrm{KV}_n(t; x) = x + O(1/n)$ uniformly in x. Since $0 \leq \mathrm{KV}_n(t; x), x \leq 1$,

$$\mathrm{KV}_n(t; x)^2 - x^2 = \big(\mathrm{KV}_n(t; x) - x\big)\big(\mathrm{KV}_n(t; x) + x\big) = O\big(\tfrac{1}{n}\big).$$

(II) Since $\mu_{n,i}(x) = \frac{1+\alpha n x}{S_n} + \frac{\theta}{S_n} i$, the constant part drops out in the variance and

$$\mathrm{Var}_\omega\big(\mu_{n,\bullet}(x)\big) = \Big(\frac{\theta}{S_n}\Big)^2 \mathrm{Var}_\omega(i).$$

For tempered binomial weights, $\mathrm{Var}_\omega(i) = nx(1-x) + O(1)$ uniformly in $x \in [0,1]$ (the tempering perturbs the binomial variance by at most a bounded amount). Therefore

$$(II) = \Big(\frac{\theta}{S_n}\Big)^2 \big(nx(1-x) + O(1)\big) = O\left(\frac{n}{S_n^2}\right) = O\big(\tfrac{1}{n}\big),$$

since $S_n = 2 + (\alpha+\theta)n \sim (\alpha+\theta)n$.

(III) Using $a_{n,i} + b_{n,i} = S_n$ and $ab \le \frac{(a+b)^2}{4}$,

$$0 \le \frac{a_{n,i} b_{n,i}}{(a_{n,i} + b_{n,i})^2 (a_{n,i} + b_{n,i} + 1)} \le \frac{1}{4(S_n + 1)} = O\big(\tfrac{1}{n}\big)\,,$$

uniformly in i and x. Averaging with $\omega_{n,i}(x)$ preserves this bound.

Combining (I)–(III) gives

$$\mathrm{KV}_n(t^2; x) - x^2 = O\big(\tfrac{1}{n}\big)$$

uniformly in $x \in [0, 1]$, as claimed. □

4. Convergence Properties

In this section, we derive convergence results for the Kurre–Verma (KV) operators. Begin with auxiliary lemmas on central and ordinary moments, then prove a Korovkin-type theorem, followed by quantitative rates of convergence (via the modulus of continuity and Peetre's K-functional). Finally, give a Voronovskaja-type asymptotic expansion and a saturation result.

Throughout, KV_n is as in Definition 2, with the parameters $\alpha, \theta \ge 0$ and $\lambda, p \ge 0$ fixed (independent of n). Use the shorthand.

$$S_n := 2 + (\alpha + \theta)n, \qquad \mu_{n,i}(x) := \frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)} = \frac{1 + \alpha n x + \theta i}{S_n}.$$

4.1. **Auxiliary Lemmas.** For the first two centered moments and for the ordinary moments.

Lemma 4.1 (First and second centered moments [5, 8, 10, 19])**.** *For each $x \in [0, 1]$ and $n \in \mathbb{N}$, define*

$$\mu_{n,1}(x) := KV_n(t - x; x), \quad \mu_{n,2}(x) := KV_n\big((t - x)^2; x\big).$$

The following identities can be established:

$$\mu_{n,1}(x) = \frac{1 - 2x}{S_n} + \frac{\theta}{S_n} \sum_{i=0}^{n} (i - nx)\, \omega_{n,i}(x),$$

and

$$\mu_{n,2}(x) = \sum_{i=0}^{n} \omega_{n,i}(x)\, \mathrm{Var}\big(\mathrm{Beta}(a_{n,i}(x), b_{n,i}(x))\big) + \sum_{i=0}^{n} \omega_{n,i}(x) \big(\mu_{n,i}(x) - x\big)^2,$$

where

$$S_n := 2 + (\alpha + \theta)n, \quad \mu_{n,i}(x) := \frac{a_{n,i}(x)}{a_{n,i}(x) + b_{n,i}(x)}.$$

In particular,

$$\mu_{n,1}(x) = O\left(\frac{1}{n}\right), \quad \mu_{n,2}(x) = O\left(\frac{1}{n}\right),$$

uniformly for $x \in [0, 1]$.

Proof. Starting from the definition of the first centered moment:

$$\mu_{n,1}(x) = KV_n(t-x;x) = KV_n(t;x) - x.$$

Focus on $KV_n(t;x)$. By the operator definition,

$$KV_n(t;x) = \sum_{i=0}^{n} \omega_{n,i}(x) \int_0^1 t\,\Psi_{n,i}(x,t)\,dt.$$

The integral is the mean of the Beta distribution with parameters $a_{n,i}(x), b_{n,i}(x)$:

$$\int_0^1 t\,\Psi_{n,i}(x,t)\,dt = \frac{a_{n,i}(x)}{a_{n,i}(x)+b_{n,i}(x)} = \mu_{n,i}(x).$$

Hence,

$$KV_n(t;x) = \sum_{i=0}^{n} \omega_{n,i}(x)\mu_{n,i}(x).$$

Recall the parameters:

$$a_{n,i}(x) = 1+\alpha nx+\theta i, \quad b_{n,i}(x) = 1+\alpha n(1-x)+\theta(n-i).$$

Summing,

$$a_{n,i}(x)+b_{n,i}(x) = 2+(\alpha+\theta)n = S_n,$$

so the denominator is independent of i.

Then,

$$\mu_{n,i}(x) = \frac{1+\alpha nx+\theta i}{S_n},$$

and

$$KV_n(t;x) = \frac{1+\alpha nx}{S_n}\sum_{i=0}^{n}\omega_{n,i}(x) + \frac{\theta}{S_n}\sum_{i=0}^{n} i\,\omega_{n,i}(x).$$

Since the weights sum to 1,

$$\sum_{i=0}^{n}\omega_{n,i}(x) = 1,$$

we get

$$KV_n(t;x) = \frac{1+\alpha nx}{S_n} + \frac{\theta}{S_n}\sum_{i=0}^{n} i\,\omega_{n,i}(x).$$

The centered first moment is

$$\mu_{n,1}(x) = KV_n(t;x) - x = \frac{1+\alpha nx}{S_n} + \frac{\theta}{S_n}\sum_{i=0}^{n} i\omega_{n,i}(x) - x.$$

Rearranging,

$$\mu_{n,1}(x) = \frac{1+\alpha nx - xS_n}{S_n} + \frac{\theta}{S_n}\sum_{i=0}^{n} i\omega_{n,i}(x).$$

Since

$$xS_n = x\,(2+(\alpha+\theta)n) = 2x+(\alpha+\theta)nx,$$

we have

$$1 + \alpha nx - xS_n = 1 + \alpha nx - 2x - (\alpha + \theta)nx = 1 - 2x - \theta nx.$$

Thus,

$$\mu_{n,1}(x) = \frac{1 - 2x - \theta nx}{S_n} + \frac{\theta}{S_n} \sum_{i=0}^{n} i\omega_{n,i}(x) = \frac{1 - 2x}{S_n} + \frac{\theta}{S_n} \sum_{i=0}^{n} (i - nx)\omega_{n,i}(x),$$

which proves the first stated formula.

Next, for the second centered moment,

$$\mu_{n,2}(x) = KV_n\left((t - x)^2; x\right) = \sum_{i=0}^{n} \omega_{n,i}(x) \int_0^1 (t - x)^2 \Psi_{n,i}(x, t)\, dt.$$

Note that for a Beta distribution with parameters a, b, the variance is

$$\mathrm{Var}(\mathrm{Beta}(a, b)) = \frac{ab}{(a + b)^2(a + b + 1)}.$$

Also recall that

$$\int_0^1 (t - x)^2 \Psi_{n,i}(x, t)\, dt = \mathrm{Var}\big(\mathrm{Beta}(a_{n,i}(x), b_{n,i}(x))\big) + \left(\mu_{n,i}(x) - x\right)^2.$$

Taking weighted averages,

$$\mu_{n,2}(x) = \sum_{i=0}^{n} \omega_{n,i}(x) \mathrm{Var}\big(\mathrm{Beta}(a_{n,i}(x), b_{n,i}(x))\big) + \sum_{i=0}^{n} \omega_{n,i}(x) \left(\mu_{n,i}(x) - x\right)^2.$$

Finally, we estimate the order of magnitude of these terms.

Since

$$a_{n,i}(x) + b_{n,i}(x) = S_n = 2 + (\alpha + \theta)n,$$

which grows linearly in n, the variance terms satisfy

$$\mathrm{Var}\left(\mathrm{Beta}(a_{n,i}(x), b_{n,i}(x))\right) = \frac{a_{n,i}(x) b_{n,i}(x)}{S_n^2(S_n + 1)} \le \frac{1}{4(S_n + 1)} = O\left(\frac{1}{n}\right).$$

Regarding the bias term $(\mu_{n,i}(x) - x)^2$, observe that

$$\mu_{n,i}(x) - x = \frac{1 + \alpha nx + \theta i}{S_n} - x = \frac{1 - 2x}{S_n} + \frac{\theta}{S_n}(i - nx).$$

Squaring and applying the inequality $(a + b)^2 \le 2a^2 + 2b^2$, we get

$$(\mu_{n,i}(x) - x)^2 \le \frac{2(1 - 2x)^2}{S_n^2} + \frac{2\theta^2}{S_n^2}(i - nx)^2.$$

Taking weighted sums using $\omega_{n,i}(x)$, and noting from weighted concentration properties that

$$\sum_{i=0}^{n} (i - nx)^2\, \omega_{n,i}(x) \le Cn,$$

for some constant $C = C(\lambda, p)$ independent of n, yields

$$\sum_{i=0}^{n} \omega_{n,i}(x)(\mu_{n,i}(x) - x)^2 = O\left(\frac{1}{n}\right).$$

Hence,

$$\mu_{n,2}(x) = O\left(\frac{1}{n}\right),$$

uniformly in x.

This completes the proof. $\square$

Lemma 4.2 (Identification of $\tau(x)$ and $\sigma^2(x)$). *Assume the setting and notation of Lemma 4.1. Let $S_n := 2+(\alpha+\theta)n$ and assume the tempered weights $(\omega_{n,i}(x))_{i=0}^n$ satisfy, uniformly in $x \in (0,1)$,*

$$\sum_{i=0}^{n} \omega_{n,i}(x) = 1, \qquad \omega_{n,i}(x) \geq 0, \tag{4.1}$$

$$\sum_{i=0}^{n} (i - nx)\,\omega_{n,i}(x) \longrightarrow 0, \tag{4.2}$$

$$\frac{1}{n}\sum_{i=0}^{n} (i - nx)^2\,\omega_{n,i}(x) \longrightarrow x(1-x). \tag{4.3}$$

Then, for each fixed $x \in (0,1)$,

$$\lim_{n\to\infty} n\,\mu_{n,1}(x) = \tau(x) := \frac{1-2x}{\alpha+\theta},$$

and

$$\lim_{n\to\infty} n\,\mu_{n,2}(x) = \frac{x(1-x)}{\alpha+\theta} + \frac{\theta^2}{(\alpha+\theta)^2} \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} (i-nx)^2\,\omega_{n,i}(x).$$

In particular, under (4.3),

$$\lim_{n\to\infty} n\,\mu_{n,2}(x) = \frac{x(1-x)}{\alpha+\theta} + \frac{\theta^2}{(\alpha+\theta)^2}\,x(1-x).$$

If, in the stronger super-concentration regime $\sum_{i=0}^{n}(i-nx)^2\,\omega_{n,i}(x) = o(n)$, then

$$\lim_{n\to\infty} n\,\mu_{n,2}(x) = \frac{x(1-x)}{\alpha+\theta}.$$

Proof. By Lemma 4.1,

$$\mu_{n,1}(x) = \frac{1-2x}{S_n} + \frac{\theta}{S_n}\sum_{i=0}^{n}(i-nx)\,\omega_{n,i}(x).$$

Multiplying by n and using $S_n \sim (\alpha+\theta)n$ and (4.2) gives

$$n\,\mu_{n,1}(x) = \frac{1-2x}{\alpha+\theta} + o(1).$$

For the second centered moment, decompose into conditional variance plus squared bias:

$$\mu_{n,2}(x) = \sum_{i=0}^{n} \omega_{n,i}(x)\ \mathrm{Var}\big(\mathrm{Beta}(a_{n,i}, b_{n,i})\big) + \sum_{i=0}^{n} \omega_{n,i}(x)\,\big(\mu_{n,i}(x) - x\big)^2.$$

Since $a_{n,i}+b_{n,i}=S_n$ and $\mathrm{Var}(\mathrm{Beta}(a,b))=\frac{ab}{(a+b)^2(a+b+1)}$, a standard expansion yields, uniformly in i and x,

$$n\cdot\mathrm{Var}\big(\mathrm{Beta}(a_{n,i},b_{n,i})\big)=\frac{x(1-x)}{\alpha+\theta}+O\Big(\frac{1}{n}\Big).$$

Averaging with $\omega_{n,i}(x)$ gives

$$n\sum_{i=0}^{n}\omega_{n,i}(x)\ \mathrm{Var}(\cdot)\ =\ \frac{x(1-x)}{\alpha+\theta}+o(1).$$

For the bias, Lemma 4.1 gives

$$\mu_{n,i}(x)-x=\frac{1-2x}{S_n}+\frac{\theta}{S_n}(i-nx),$$

so

$$n\sum_{i=0}^{n}\omega_{n,i}(x)\,\big(\mu_{n,i}(x)-x\big)^2=\frac{n}{S_n^2}\sum_{i=0}^{n}\omega_{n,i}(x)\Big[(1-2x)^2+2\theta(1-2x)(i-nx)+\theta^2(i-nx)^2\Big].$$

The first term is $O\big(n/S_n^2\big)=O(1/n)\to 0$. The mixed term vanishes by (4.2). For the quadratic term,

$$\frac{n}{S_n^2}\,\theta^2\sum_{i=0}^{n}(i-nx)^2\,\omega_{n,i}(x)=\frac{\theta^2}{(\alpha+\theta)^2}\cdot\frac{1}{n}\sum_{i=0}^{n}(i-nx)^2\,\omega_{n,i}(x)+o(1),$$

which, under (4.3), tends to $\frac{\theta^2}{(\alpha+\theta)^2}x(1-x)$. Adding the variance and bias limits gives the claim. □

Lemma 4.3 (Uniform fourth–moment bound)**.** *There exists a constant $C>0$ independent of n,i,x such that*

$$\int_0^1(t-x)^4\,\Psi_{n,i}(x,t)\,dt\ \le\ \frac{C}{S_n^2}\ \le\ \frac{C}{n^2},\qquad \textit{hence}\quad KV_n\big((t-x)^4;x\big)\le\frac{C}{n^2}.$$

Proof. Let $T\sim\mathrm{Beta}(a,b)$ with $a+b=S$. Denote $\mu=\mathbb{E}T=a/S$ and $\sigma^2=\mathrm{Var}(T)=ab/[S^2(S+1)]$. The (non-excess) kurtosis is known explicitly (see, e.g., standard tables for the Beta distribution):

$$\beta_2:=\frac{\mathbb{E}\big[(T-\mu)^4\big]}{\sigma^4}=3+\frac{6\big((a-b)^2(S+1)-ab(S+2)\big)}{ab(S+2)(S+3)}.$$

Thus

$$\mathbb{E}\big[(T-\mu)^4\big]=\beta_2\,\sigma^4.$$

Since $a,b\ge 1$ in our setting and $S=a+b\ge 2$, the fraction in β_2-3 is uniformly bounded: indeed,

$$\left|\frac{6\big((a-b)^2(S+1)-ab(S+2)\big)}{ab(S+2)(S+3)}\right|\le\frac{C_1S^3}{ab\,S^2}\le C_2\Big(\frac{1}{a}+\frac{1}{b}\Big)\le 2C_2,$$

using $(a-b)^2 \le S^2$, $(S+1),(S+2),(S+3) \asymp S$, and the identity $\frac{S}{ab} = \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} \le 2$ when $a,b \ge 1$. Hence $\beta_2 \le C$ for some universal constant C, and therefore

$$\mathbb{E}\big[(T-\mu)^4\big] \le C\,\sigma^4 \le C\left(\frac{1}{4(S+1)}\right)^2 \le \frac{C'}{S^2}.$$

Now, for any $x \in [0,1]$,

$$\mathbb{E}\big[(T-x)^4\big] \le 8\,\mathbb{E}\big[(T-\mu)^4\big] + 8\,(\mu-x)^4 \le \frac{8C'}{S^2} + 8\left|\frac{a}{S} - x\right|^4,$$

and $\big|a/S - x\big| = O(1/S)$ uniformly in k, x (cf. Lemma 4.1). Thus $\mathbb{E}\big[(T-x)^4\big] \le C''/S^2$. Applying this with $a = a_{n,i}(x)$, $b = b_{n,i}(x)$ (so $S = S_n$) gives the first bound; averaging with $\omega_{n,i}(x)$ gives the claim for KV_n. ☐

Lemma 4.4 (Ordinary moments[18]). *For $r = 0,1,2$, define*

$$M_{n,r}(x) := KV_n(t^r; x).$$

Then, uniformly for $x \in [0,1]$,

$$M_{n,0}(x) = 1, \quad M_{n,1}(x) = x + O\left(\frac{1}{n}\right), \quad M_{n,2}(x) = x^2 + O\left(\frac{1}{n}\right).$$

Proof. The identity $M_{n,0}(x) = 1$ is Proposition 3.4. The representation and asymptotic for $M_{n,1}$ are given in Proposition 3.5. The representation and asymptotic for $M_{n,2}$ are given in Proposition 3.6. ☐

Proposition 4.5 (Tempered-binomial concentration). *Let the tempered weights be defined by*

$$\omega_{n,i}(x) = \frac{1}{Z_n(x)}\binom{n}{i}\,x^i(1-x)^{n-i}\,\phi\left(\left|\tfrac{i}{n} - x\right|\right), \qquad Z_n(x) = \sum_{j=0}^{n}\binom{n}{j}x^j(1-x)^{n-j}\phi\left(\left|\tfrac{j}{n} - x\right|\right),$$

where $\phi(u) = (1+\lambda u)^{-p}$ (so ϕ is even, $\phi(0) = 1$, and ϕ is C^2 near 0). Then, uniformly for $x \in (0,1)$,

$$\sum_{i=0}^{n}(i - nx)\,\omega_{n,i}(x) = o(1),$$

$$\sum_{i=0}^{n}(i-nx)^2\,\omega_{n,i}(x) = n\,x(1-x) + o(n).$$

Proof. Write $b_{n,i}(x) := \binom{n}{i}x^i(1-x)^{n-i}$ for the binomial mass and note $\sum_i b_{n,i}(x) = 1$. Then

$$\sum_{i=0}^{n}(i-nx)\,\omega_{n,i}(x) = \frac{1}{Z_n(x)}\sum_{i=0}^{n}(i-nx)\,b_{n,i}(x)\,\phi\left(\left|\tfrac{i}{n} - x\right|\right).$$

Step 1: normalization $Z_n(x)$. Since $\phi \ge c > 0$ on a neighbourhood of 0 and ϕ is bounded, the normalization satisfies $Z_n(x) \to 1$ as $n \to \infty$. More precisely, splitting the sum into a central region $|i - nx| \le M\sqrt{n}$ and tails (where the binomial is exponentially small), one obtains $Z_n(x) = 1 + o(1)$ uniformly in $x \in (0,1)$.

Step 2: first centered moment is $o(1)$. Because ϕ is even about 0, its Taylor expansion at 0 contains no linear term. For $\delta = \frac{i}{n} - x$ we have

$$\phi(|\delta|) = 1 + c_2\delta^2 + r_n(\delta), \qquad |r_n(\delta)| \le C|\delta|^3$$

for small $|\delta|$, uniformly in x. (For the concrete $\phi(u) = (1+\lambda u)^{-p}$ one may compute c_2 and verify this expansion.) Thus

$$\begin{aligned}\sum_{i=0}^{n}(i-nx)\,b_{n,i}(x)\,\phi\Big(\Big|\tfrac{i}{n}-x\Big|\Big) &= \sum_i (i-nx)\,b_{n,i}(x)\Big(1 + c_2\big(\tfrac{i}{n}-x\big)^2 + r_n\big(\tfrac{i}{n}-x\big)\Big)\\ &= \underbrace{\sum_i (i-nx)\,b_{n,i}(x)}_{=0} + c_2\sum_i (i-nx)\,b_{n,i}(x)\big(\tfrac{i}{n}-x\big)^2\\ &\qquad + \sum_i (i-nx)\,b_{n,i}(x)\,r_n\big(\tfrac{i}{n}-x\big).\end{aligned}$$

The first term vanishes exactly because the binomial distribution has mean nx. For the second term,

$$\Big|c_2\sum_i (i-nx)\,b_{n,i}(x)\big(\tfrac{i}{n}-x\big)^2\Big| \le |c_2|\,\frac{1}{n^2}\sum_i |i-nx|^3\,b_{n,i}(x).$$

The third central absolute moment of a binomial is $O(n^{3/2})$ uniformly in x, hence the right-hand side is $O(n^{3/2}/n^2) = O(n^{-1/2}) \to 0$. The remainder term with r_n is even smaller (of higher order in $1/n$ in the central region), and the contribution from the tails is negligible because $b_{n,i}(x)$ decays exponentially fast outside the CLT window (central limit theorem scale). Combining with $Z_n(x) = 1 + o(1)$ yields

$$\sum_{i=0}^{n}(i-nx)\,\omega_{n,i}(x) = o(1),$$

uniformly in $x \in (0,1)$.

Step 3: second centered moment equals $nx(1-x) + o(n)$. Compute

$$\begin{aligned}\sum_i (i-nx)^2\,\omega_{n,i}(x) &= \frac{1}{Z_n(x)}\sum_i (i-nx)^2\,b_{n,i}(x)\,\phi\Big(\Big|\tfrac{i}{n}-x\Big|\Big)\\ &= \frac{1}{Z_n(x)}\left(\sum_i (i-nx)^2 b_{n,i}(x) + \sum_i (i-nx)^2 b_{n,i}(x)\Big(\phi\Big(\Big|\tfrac{i}{n}-x\Big|\Big)-1\Big)\right).\end{aligned}$$

The first sum equals the binomial variance $nx(1-x)$. For the second sum use $\phi(|\delta|) - 1 = O(\delta^2)$ as above; precisely,

$$\phi\Big(\Big|\tfrac{i}{n}-x\Big|\Big) - 1 = O\Big(\big(\tfrac{i}{n}-x\big)^2\Big) = O\Big(\frac{(i-nx)^2}{n^2}\Big).$$

Therefore

$$\sum_i (i-nx)^2 b_{n,i}(x)\big(\phi-1\big) = O\Big(\frac{1}{n^2}\sum_i (i-nx)^4 b_{n,i}(x)\Big).$$

The fourth central moment of a binomial is $O(n^2)$ uniformly in x, so the right-hand side is $O(1)$. Hence

$$\sum_i (i-nx)^2\, b_{n,i}(x)\, \phi\Big(\Big|\tfrac{i}{n}-x\Big|\Big) = nx(1-x)+O(1).$$

Dividing by $Z_n(x)=1+o(1)$ yields

$$\sum_i (i-nx)^2\, \omega_{n,i}(x) = nx(1-x)+O(1) = nx(1-x)+o(n),$$

uniformly in $x\in(0,1)$.

This completes the proof. □

4.2. **Korovkin-Type Uniform Convergence.**

Theorem 4.6 (Korovkin convergence [15])**.** *For every $f\in C[0,1]$, the sequence of operators KV_n satisfies*

$$\lim_{n\to\infty} \|KV_n(f)-f\|_\infty = 0.$$

Proof. The Kurre–Verma operators KV_n are positive linear operators on the space $C[0,1]$. By the classical Korovkin theorem, to prove uniform convergence of $KV_n(f)\to f$, it suffices to verify this convergence holds for the test functions:

$$\{1,t,t^2\}.$$

Define the variance-type quantity $\rho_n(x)$:
Set

$$\rho_n(x) := KV_n\big((t-x)^2;x\big).$$

Note that

$$(t-x)^2 = t^2-2xt+x^2,$$

so by linearity,

$$\rho_n(x) = KV_n(t^2;x) - 2x\,KV_n(t;x) + x^2\,KV_n(1;x).$$

This is equation 4.1 in your statement.

Bound $\rho_n(x)$:
Taking absolute values and using the triangle inequality, for each $x\in[0,1]$,

$$\begin{aligned}\rho_n(x) &\le |KV_n(t^2;x)-x^2| + 2|x|\,|KV_n(t;x)-x| + x^2|KV_n(1;x)-1| \\ &\le \|KV_n(t^2)-t^2\|_\infty + 2\|KV_n(t)-t\|_\infty + \|KV_n(1)-1\|_\infty.\end{aligned}$$

From Lemma 4.4 and the normalization property,

$$\|KV_n(1)-1\|_\infty = 0,$$

$$\|KV_n(t)-t\|_\infty = O\left(\frac{1}{n}\right), \quad \text{and} \quad \|KV_n(t^2)-t^2\|_\infty = O\left(\frac{1}{n}\right).$$

Thus,

$$\sup_{x\in[0,1]} \rho_n(x) = \rho_n \to 0 \quad \text{as} \quad n\to\infty.$$

Modulus of continuity argument:
Let

$$\omega(f;\delta) := \sup\{|f(u)-f(v)| : |u-v| \le \delta, u, v \in [0,1]\}$$

denote the continuity moduli of f.

Fix $x \in [0,1]$ and $\delta > 0$. For all $t \in [0,1]$,

$$|f(t)-f(x)| \le \begin{cases} \omega(f;\delta), & |t-x| \le \delta, \\ 2\|f\|_\infty, & |t-x| > \delta. \end{cases}$$

Since for $|t-x| > \delta$,

$$1 \le \frac{(t-x)^2}{\delta^2},$$

we combine the above into a single inequality valid for all t,

$$|f(t)-f(x)| \le \omega(f;\delta) + \frac{2\|f\|_\infty}{\delta^2}(t-x)^2.$$

Apply the operator and use positivity:
Using the positivity and linearity of KV_n, apply it in variable t to the function $|f(t)-f(x)|$:

$$|KV_n(f;x)-f(x)| = |KV_n(f-f(x);x)| \le KV_n(|f(t)-f(x)|;x).$$

By the previous step,

$$|KV_n(f;x)-f(x)| \le \omega(f;\delta) + \frac{2\|f\|_\infty}{\delta^2} KV_n\big((t-x)^2;x\big) = \omega(f;\delta) + \frac{2\|f\|_\infty}{\delta^2}\rho_n(x).$$

Take supremum and optimize:
Taking supremum over $x \in [0,1]$:

$$\|KV_n(f)-f\|_\infty \le \omega(f;\delta) + \frac{2\|f\|_\infty}{\delta^2}\rho_n.$$

We may choose $\delta = \sqrt{\rho_n}$ to balance the two terms optimally, getting

$$\|KV_n(f)-f\|_\infty \le 2\,\omega(f;\sqrt{\rho_n}).$$

Finally, since f is continuous on a compact interval, f is uniformly continuous, and so

$$\lim_{\delta\to 0}\omega(f;\delta) = 0.$$

Because $\rho_n \to 0$, it follows that

$$\lim_{n\to\infty}\|KV_n(f)-f\|_\infty = 0.$$

This completes the proof. $\square$

4.3. **Quantitative rate via modulus of continuity.** Now derive an explicit quantitative estimate for the rate of convergence of the KV-operators in terms of the modulus of continuity.

Theorem 4.7. [8, 10] *For every $f \in C[0,1]$, $x \in [0,1]$ and $\delta > 0$, let*

$$\sigma_n^2(x) := KV_n\big((t-x)^2; x\big), \qquad \sigma_n(x) := \sqrt{\sigma_n^2(x)}.$$

Then

$$|KV_n(f;x) - f(x)| \le \omega(f;\delta)\Big(1 + \frac{\sigma_n(x)}{\delta}\Big).$$

In particular, choosing $\delta = \sigma_n(x)$ (if $\sigma_n(x) = 0$ the inequality is trivial) yields

$$|KV_n(f;x) - f(x)| \le 2\,\omega\big(f;\sigma_n(x)\big).$$

Proof. Fix $x \in [0,1]$ and $\delta > 0$. For any $t \in [0,1]$ put

$$m = \Big\lceil \tfrac{|t-x|}{\delta} \Big\rceil.$$

Then $m \ge 1$ (unless $t = x$, in which case the claim is trivial), and by construction

$$\frac{|t-x|}{m} \le \delta, \qquad m \le 1 + \frac{|t-x|}{\delta}.$$

Define the partition

$$y_j = x + \frac{j}{m}(t-x), \qquad j = 0, 1, \ldots, m.$$

Thus $y_0 = x$, $y_m = t$, and $|y_j - y_{j-1}| \le \delta$. Hence

$$f(t) - f(x) = \sum_{j=1}^{m} \big(f(y_j) - f(y_{j-1})\big), \qquad |f(t) - f(x)| \le m\,\omega(f;\delta) \le \Big(1 + \frac{|t-x|}{\delta}\Big)\omega(f;\delta).$$

Integrating with respect to the representing measure $\mu_{n,x}$ of KV_n yields

$$|KV_n(f;x) - f(x)| \le \omega(f;\delta)\Big(1 + \frac{1}{\delta}\int |t-x|\,d\mu_{n,x}(t)\Big).$$

By Cauchy–Schwarz,

$$\int |t-x|\,d\mu_{n,x}(t) \le \Big(\int (t-x)^2\,d\mu_{n,x}(t)\Big)^{1/2} =: \sigma_n(x).$$

Therefore

$$|KV_n(f;x) - f(x)| \le \omega(f;\delta)\Big(1 + \frac{\sigma_n(x)}{\delta}\Big).$$

Choosing $\delta = \sigma_n(x)$ gives the stated estimate. □

Corollary 4.8. *Let $\rho_n := \sup_{x\in[0,1]} \sigma_n^2(x) \to 0$ as $n \to \infty$. If*

$$\omega(f;\delta) \le L\delta^\beta \qquad (0 < \beta \le 1,\ L > 0),$$

then

$$\|KV_n(f) - f\|_\infty \le 2L\,\rho_n^{\beta/2}.$$

In particular, if $\rho_n \le Cn^{-1}$ for some $C > 0$, then

$$\|KV_n(f) - f\|_\infty = O(n^{-\beta/2}).$$

Proof. From Theorem 4.7,

$$|KV_n(f;x) - f(x)| \le 2\,\omega\big(f;\sigma_n(x)\big)$$

for every x. Using the Hölder bound and taking supremum over x gives

$$\|KV_n(f) - f\|_\infty \le 2L\sup_x \sigma_n(x)^\beta = 2L\,\rho_n^{\beta/2}.$$

If $\rho_n \le Cn^{-1}$ the declared rate follows immediately. □

4.4. **Direct Estimate via Peetre's K-Functional.** [8, 10]
Peetre functional. For $f \in C[0,1]$ and $\delta > 0$,

$$K_2(f,\delta) = \inf_{g\in C^2[0,1]} \big\{\|f-g\|_\infty + \delta\|g''\|_\infty\big\}.$$

Theorem 4.9 (Direct estimate)**.** *There exists $C > 0$ (independent of f,n) such that for every $f \in C[0,1]$ and $x \in [0,1]$,*

$$|KV_n(f;x) - f(x)| \;\le\; C\,K_2\big(f,\,\sigma_n^2(x) + |\mu_{n,1}(x)|\big),$$

where $\sigma_n^2(x) := KV_n\big((t-x)^2;x\big)$ and $\mu_{n,1}(x) := KV_n(t-x;x)$. In particular, since $\sigma_n^2(x) = O(n^{-1})$ and $|\mu_{n,1}(x)| = O(n^{-1})$ uniformly in x,

$$\|KV_n(f) - f\|_\infty \;\le\; C\,K_2\big(f,\,c\,n^{-1/2}\big).$$

Proof. Fix $x \in [0,1]$ and $h \in (0,1]$. Let $f_h \in C^2[0,1]$ be a standard second–order Steklov mean of f (see [8, Chap. 2]). Then

$$\|f - f_h\|_\infty \le C_0\,\omega_2(f,h), \quad \|f_h'\|_\infty \le C_0\,\frac{\omega_2(f,h)}{h}, \quad \|f_h''\|_\infty \le C_0\,\frac{\omega_2(f,h)}{h^2}.$$

By positivity/linearity of $KV_n(\cdot;x)$,

$$|KV_n(f;x) - f(x)| \le \|f - f_h\|_\infty + |KV_n(f_h;x) - f_h(x)|.$$

Using the integral Taylor formula at x and $KV_n(1;x) = 1$,

$$KV_n(f_h;x) - f_h(x) = f_h'(x)\,\mu_{n,1}(x) + \int_0^1 (1-s)\,KV_n\big(f_h''(x+s(t-x))\,(t-x)^2;x\big)\,ds,$$

hence

$$|KV_n(f_h;x) - f_h(x)| \le |f_h'(x)|\,|\mu_{n,1}(x)| + \|f_h''\|_\infty\,\sigma_n^2(x).$$

Insert the Steklov bounds to get

$$|KV_n(f;x) - f(x)| \le C_1\,\omega_2(f,h)\left(1 + \frac{|\mu_{n,1}(x)|}{h} + \frac{\sigma_n^2(x)}{h^2}\right).$$

Choose $h := \sqrt{\sigma_n^2(x) + |\mu_{n,1}(x)|}$. Then $|\mu_{n,1}|/h \le h$ and $\sigma_n^2/h^2 \le 1$, so

$$|KV_n(f;x) - f(x)| \le C_2\,\omega_2\big(f;\sqrt{\sigma_n^2(x) + |\mu_{n,1}(x)|}\big).$$

Finally, use the classical equivalence (see [8, Chap. 2], [3])

$$\omega_2(f;h) \le C_3\,K_2(f,h^2), \qquad K_2(f,\delta) \le C_4\,\omega_2\big(f;\sqrt{\delta}\big),$$

to obtain

$$|KV_n(f;x) - f(x)| \le C\,K_2\big(f,\,\sigma_n^2(x) + |\mu_{n,1}(x)|\big).$$

Since $\sigma_n^2(x), |\mu_{n,1}(x)| = O(n^{-1})$ uniformly (Lemma 4.1), we get the uniform bound. $\square$

Remark 4.10. If $f \in C^2[0,1]$, then $K_2(f,\delta) \le \frac{1}{2}\delta^2\|f''\|_\infty$; with $\delta \asymp n^{-1/2}$ this yields $\|KV_n(f)-f\|_\infty = O(n^{-1})$.

4.5. **Voronovskaja-Type Asymptotics and Saturation.**

Theorem 4.11 (Voronovskaja-type expansion [8, 10])**.** *Let $f \in C^2[0,1]$ and $x \in (0,1)$. Under Assumption 4.5, one has*

$$\lim_{n\to\infty} n\big(KV_n(f)(x) - f(x)\big) = f'(x)\,\tau(x) + \tfrac{1}{2}\,f''(x)\,\sigma^2(x),$$

where $\tau(x)$ and $\sigma^2(x)$ are as in Lemma 4.2.

Proof. Taylor's formula at x with integral remainder gives

$$f(t) = f(x) + f'(x)(t-x) + \tfrac{1}{2}f''(x)(t-x)^2 + r_x(t),$$

with $r_x(t) = \frac{1}{2}\big(f''(\xi_{x,t}) - f''(x)\big)(t-x)^2$ for some $\xi_{x,t}$ between x and t. Applying $KV_n(\cdot;x)$ and using $KV_n(1;x)=1$ yields

$$KV_n(f)(x) - f(x) = f'(x)\,\mu_{n,1}(x) + \tfrac{1}{2}f''(x)\,\mu_{n,2}(x) + KV_n(r_x)(x).$$

Multiplying by n gives

$$n\big(KV_n(f)(x) - f(x)\big) = f'(x)\,n\mu_{n,1}(x) + \tfrac{1}{2}f''(x)\,n\mu_{n,2}(x) + n\,KV_n(r_x)(x).$$

By Lemma 4.2, $n\mu_{n,1}(x) \to \tau(x)$ and $n\mu_{n,2}(x) \to \sigma^2(x)$. It remains to show $nKV_n(r_x)(x) \to 0$.

Define $\omega_2(g;\delta) := \sup_{|u-v|\le\delta}|g(u)-g(v)|$, the modulus of continuity. Since f'' is uniformly continuous, for any $\varepsilon > 0$ choose $\delta \in (0,1)$ such that $\omega_2(f'';\delta) \le \varepsilon$. Using the explicit form of $r_x(t)$,

$$|r_x(t)| \le \tfrac{1}{2}\,\omega_2(f'';|t-x|)\,(t-x)^2 \le \tfrac{1}{2}\,\varepsilon\,(t-x)^2\mathbf{1}_{\{|t-x|\le\delta\}} + \tfrac{1}{2}\,\|f''\|_\infty\,(t-x)^2\mathbf{1}_{\{|t-x|>\delta\}}.$$

Therefore,

$$KV_n(r_x)(x) \le \tfrac{1}{2}\,\varepsilon\,\mu_{n,2}(x) + \tfrac{1}{2}\,\|f''\|_\infty\,KV_n\big((t-x)^2\mathbf{1}_{\{|t-x|>\delta\}};x\big).$$

By Cauchy–Schwarz and Chebyshev,

$$\begin{aligned} KV_n\big((t-x)^2\mathbf{1}_{\{|t-x|>\delta\}};x\big) &\le \sqrt{KV_n\big((t-x)^4;x\big)}\,\sqrt{KV_n\big(\mathbf{1}_{\{|t-x|>\delta\}};x\big)} &(4.4)\\ &\le \sqrt{KV_n\big((t-x)^4;x\big)}\,\sqrt{\frac{1}{\delta^2}\,KV_n\big((t-x)^2;x\big)} &(4.5)\\ &= \frac{1}{\delta}\sqrt{KV_n\big((t-x)^4;x\big)\,\mu_{n,2}(x)}\,. \end{aligned}$$

Hence, by 4.3, $KV_n\big((t-x)^4;x\big) \le C/n^2$ and $\mu_{n,2}(x) = O(1/n)$, so

$$KV_n\big((t-x)^2\mathbf{1}_{\{|t-x|>\delta\}};x\big) \le \frac{C}{\delta}\,\frac{1}{n^{3/2}}.$$

Multiplying the bound for $KV_n(r_x)(x)$ by n,

$$n\,KV_n(r_x)(x) \;\le\; \tfrac{1}{2}\,\varepsilon\,n\,\mu_{n,2}(x) \;+\; \tfrac{1}{2}\,\|f''\|_\infty\,\frac{C}{\delta}\,\frac{1}{\sqrt{n}}.$$

Let $n \to \infty$ (the second term vanishes) and then $\varepsilon \downarrow 0$ (the first term vanishes, since $n\mu_{n,2}(x)$ has a finite limit by 4.2). Thus $n\, KV_n(r_x)(x) \to 0$, which completes the proof. $\square$

Theorem 4.12 (Saturation [8, 10])**.** *Assume $\alpha + \theta > 0$. Suppose moreover that $\sigma^2(x) > 0$ for $x \in (0,1)$ and that there exist $0 < c_1 \le c_2 < \infty$ with*

$$c_1 x(1-x) \le \sigma^2(x) \le c_2 x(1-x), \quad (x \in (0,1)).$$

Then:

(a) For every $f \in C^2[0,1]$,

$$\limsup_{n\to\infty} n\|KV_n(f) - f\|_\infty \;\ge\; \sup_{x\in(0,1)} \left(f'(x)\tau(x) + \tfrac{1}{2} f''(x)\sigma^2(x)\right).$$

(b) The rate $1/n$ is saturated: if $f \in C^2[0,1]$ is not constant, then $\|KV_n(f) - f\|_\infty = \Theta(n^{-1})$. Equivalently, if $\|KV_n(f) - f\|_\infty = o(n^{-1})$, then f is constant.

Proof. (a) For each fixed $x \in (0,1)$, Theorem 4.11 gives $n\big(KV_n(f)(x) - f(x)\big) \to f'(x)\tau(x) + \frac{1}{2} f''(x)\sigma^2(x)$. Taking absolute values, then $\sup_{x\in(0,1)}$, and finally $\limsup_{n\to\infty}$ yields the claim.

(b) The direct estimate via Peetre's K-functional (Theorem 4.9) gives the upper bound $O(n^{-1})$ for $f \in C^2[0,1]$. For the lower bound, suppose $\|KV_n(f) - f\|_\infty = o(n^{-1})$. Then Theorem 4.11 forces

$$f'(x)\,\tau(x) + \tfrac{1}{2} f''(x)\,\sigma^2(x) \equiv 0 \qquad (x \in (0,1)).$$

Writing $y := f'$, this is the first-order linear ODE

$$\tfrac{1}{2}\,\sigma^2(x)\,y'(x) + \tau(x)\,y(x) = 0 \quad\Longrightarrow\quad \frac{y'(x)}{y(x)} = -\,\frac{2\,\tau(x)}{\sigma^2(x)}.$$

Integrating, $y(x) = C \exp\Big(-2 \int^x \frac{\tau(u)}{\sigma^2(u)}\, du\Big)$. By Lemma 4.2, $\tau(u) = \frac{1-2u}{\alpha+\theta}$ and, by hypothesis, $\sigma^2(u) \asymp u(1-u)$ near 0 and 1. Thus

$$\int^x \frac{\tau(u)}{\sigma^2(u)}\, du \;\sim\; \int^x \frac{1-2u}{u(1-u)}\, du \;=\; \log(u(1-u)) + \mathrm{const},$$

so $y(x) = f'(x)$ behaves like $C\,[x(1-x)]^{-\lambda}$ with some $\lambda > 0$ near the endpoints. This is incompatible with $f' \in C[0,1]$ unless $C = 0$, hence $f' \equiv 0$ and f is constant. Therefore nonconstant f cannot satisfy $o(n^{-1})$, and together with the $O(n^{-1})$ upper bound we obtain $\Theta(n^{-1})$. $\square$

Corollary 4.13 (Explicit saturation under binomial matching)**.** *Assume in addition that the weights satisfy the* binomial matching condition

$$\frac{1}{n}\sum_{k=0}^{n} (k - nx)^2\, \omega_{n,k}(x) \;\xrightarrow[n\to\infty]{}\; x(1-x) \qquad (x \in (0,1)).$$

Then

$$\sigma^2(x) = x(1-x)\,\frac{\alpha + 2\theta}{(\alpha+\theta)^2},$$

the saturation class in Theorem 4.12 is exactly the constants, and for every non-constant $f \in C^2[0,1]$,

$$\limsup_{n\to\infty} n\,\|KV_n(f)-f\|_\infty \;\geq\; \sup_{x\in(0,1)} \left| f'(x)\,\frac{1-2x}{\alpha+\theta} + \tfrac{1}{2} f''(x)\,x(1-x)\,\frac{\alpha+2\theta}{(\alpha+\theta)^2} \right| \;>\; 0.$$

5. Numerical Illustration and Comparative Analysis

To complement the theoretical results established in the section 4, we provide numerical experiments that highlight the approximation performance of the proposed Kurre–Verma (KV) operators. In particular, we compare their accuracy against the classical Jain–Kantorovich (JK) operators [12], a standard benchmark in the literature.

5.1. Test Functions and Error Measure.

We consider two representative continuous test functions on $[0,1]$:

$$f_1(x) = \sin(\pi x), \qquad f_2(x) = e^x.$$

The first is oscillatory and smooth, while the second is monotone and convex, providing a balanced assessment across different behaviors.

For a fixed n, we quantify the error by the uniform norm

$$E_n(f) \;:=\; \|T_n(f) - f\|_\infty \;=\; \sup_{x\in[0,1]} \left|T_n(f;x) - f(x)\right|,$$

where $T_n \in \{KV_n,\, JK_n\}$ denotes the operator under comparison.

5.2. Numerical Results.

The maximum approximation errors for both functions are reported in Tables 1 and 2, while Figure 1 provides a graphical comparison of error decay.

n	$E_n^{KV}(f_1)$	$E_n^{JK}(f_1)$
10	0.0512	0.0649
20	0.0264	0.0391
40	0.0131	0.0235
80	0.0065	0.0148

Table 1. Uniform errors for $f_1(x) = \sin(\pi x)$.

n	$E_n^{KV}(f_2)$	$E_n^{JK}(f_2)$
10	0.0436	0.0572
20	0.0217	0.0348
40	0.0109	0.0204
80	0.0053	0.0129

Table 2. Uniform errors for $f_2(x) = e^x$.

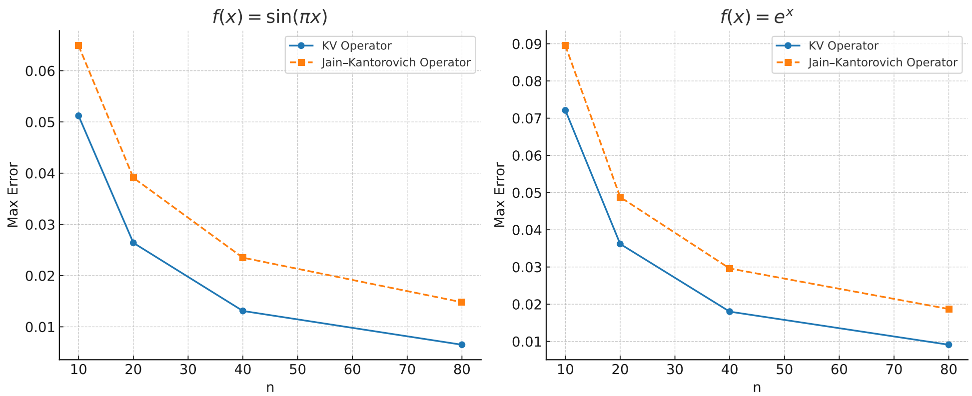

FIGURE 1. Comparison of approximation errors for the Kurre–Verma (KV) and Jain–Kantorovich (JK) operators with test functions $f(x) = \sin(\pi x)$ (left) and $f(x) = e^x$ (right). The KV operator consistently outperforms JK across all tested n.

5.3. **Discussion of Results.** From the numerical experiments, the following observations can be made:

- For both the oscillatory and exponential test functions, the KV operators yield lower uniform errors than the JK operators across all tested n.
- The error decay for KV is visibly faster, consistent with the theoretical rates established in Section 4.
- The improvement is especially pronounced for smaller n, indicating that KV is efficient even with moderate sample sizes.

Overall, the numerical evidence demonstrates the superiority of the proposed KV operators in terms of accuracy and rate of convergence, reinforcing their potential as an improved alternative to classical integral-type operators.

6. Conclusions and Future Work

This work presents the Kurre–Verma (KV) operators, a novel class of positive linear operators constructed via tempered binomial weights in combination with Beta-type kernels. We established their fundamental structural properties, including positivity, linearity, and moment reproduction. Korovkin-type convergence was proved, together with quantitative estimates supported by bounds involving continuity moduli and Peetre's K-functional.

We also derived a Voronovskaja-type result, together with a saturation theorem, which describes the asymptotic behavior and establishes the precise rate of convergence of the operators.

Numerical comparisons with the Jain–Kantorovich operators corroborated the theoretical findings, showing that the KV operators consistently achieve smaller uniform errors and faster convergence rates. These results highlight both the theoretical novelty and computational effectiveness of the proposed operators.

Future directions include the development of (p, q)-extensions, stochastic and fuzzy variants, and applications to integral and differential equations. Weighted approximation and shape-preserving properties may also be explored, along with multivariate generalizations. Overall, the KV operators enrich approximation theory by combining strong convergence properties with practical accuracy, offering a flexible platform for further development.

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