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NEW BI-UNIVALENT FUNCTION FAMILIES ASSOCIATED WITH LAGUERRE POLYNOMIALS: COEFFICIENT ESTIMATES AND FEKETE–SZEGÖ INEQUALITY

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Abstract. This study presents and explores subclasses of bi-univalent analytic functions defined on the open unit disk, constructed in connection with Laguerre polynomials. By employing appropriate coefficient techniques and subordination principles. Upper estimates are established for the early coefficients $|\mathfrak{a}_2|$ and $|\mathfrak{a}_3|$ for functions belonging to the proposed classes. Furthermore, we establish Fekete–Szegö type inequalities involving the third coefficient in terms of the second. Several known results are obtained as special cases of our findings. The results presented here generalize and extend various earlier works in the context of bi-univalent function theory.

Keywords. Fekete–Szegö inequalities,Bi-univalent functions, Coefficient estimates, q-calculus, Laguerre polynomials, Subordination.
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1. Introduction

Recent developments in mathematics and physics have increasingly emphasized quantum calculus, or q-calculus, as a significant and versatile tool. Originating from Jacksons pioneering work in the late 19th century [19], which introduced the q-difference operator and its integral analogue, this approach extends classical calculus by removing the reliance on limit processes. Subsequently, Aral and Gupta [2] generalized standard mathematical structures into their q-deformed counterparts, leading to important applications in geometric function theory.

Fundamentally, q-calculus generalizes classical analysis through the q-difference operator, providing a powerful framework to study families of analytic functions, notably those with starlike and convex properties. The deformation parameter $q \in (0, 1)$ plays a crucial role in maintaining convergence and structural consistency. Consequently, q-calculus offers novel viewpoints for both theoretical investigations and practical applications.

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Definition 1.1 ([4, 17])**.** The q-bracket $\lceil\kappa\rfloor_{\mathfrak{q}}$ is defined by

$$\lceil\kappa\rfloor_{\mathfrak{q}} = \begin{cases} \dfrac{1-q^{\kappa}}{1-q}, & 0<q<1,\ \kappa\in\mathbb{C}\setminus\{0\},\\ 1, & q\to 0^{+},\ \kappa\in\mathbb{C}\setminus\{0\},\\ \kappa, & q\to 1^{-},\ \kappa\in\mathbb{C}\setminus\{0\},\\ \sum\limits_{s=0}^{\gamma-1} q^{s}, & 0<q<1,\ \kappa=\gamma\in\mathbb{N}. \end{cases}$$

with the identity

$$\lceil\kappa+1\rfloor_{\mathfrak{q}} = \lceil\kappa\rfloor_{\mathfrak{q}} + q^{\kappa}.$$

This q-bracket notation plays a crucial role in defining the q-derivative operator, which generalizes the classical derivative in the context of q-calculus. Accordingly, we proceed with the following definition:

Definition 1.2 ([4, 17])**.** The q-derivative (or q-difference operator) of an analytic function f is given by

$$\mathcal{D}_{\mathfrak{q}}\langle\vartheta(z)\rangle = \begin{cases} \dfrac{\vartheta(z)-\vartheta(qz)}{z(1-q)}, & z\neq 0,\ 0<q<1,\\ \vartheta'(0), & z=0,\\ \vartheta'(z), & q\to 1^{-}. \end{cases}$$

Remark 1.3. Let $\vartheta\in\mathcal{A}$ be of the form

$$\vartheta(z) = z + \sum_{s=2}^{\infty}\mathfrak{a}_s z^s. \tag{1.1}$$

Then

$$\mathcal{D}_{\mathfrak{q}}\langle\vartheta(z)\rangle = 1 + \sum_{s=2}^{\infty}\lceil s\rfloor_{\mathfrak{q}}\mathfrak{a}_s z^{s-1}, \quad z\in\mathbb{D}.$$

For the inverse function $\Theta(\mathfrak{w}) := \vartheta^{-1}(\mathfrak{w}) = \mathfrak{w}+\sum_{n=2}^{\infty} c_n\mathfrak{w}^n$, the q-derivative expands as

$$\mathcal{D}_{\mathfrak{q}}\langle\Theta(\mathfrak{w})\rangle = 1 - \lceil 2\rfloor_{\mathfrak{q}}\mathfrak{a}_2\mathfrak{w} + \lceil 3\rfloor_{\mathfrak{q}}(2\mathfrak{a}_2^2-\mathfrak{a}_3)\mathfrak{w}^2 - \lceil 4\rfloor_{\mathfrak{q}}(5\mathfrak{a}_2^3-5\mathfrak{a}_2\mathfrak{a}_3+\mathfrak{a}_4)\mathfrak{w}^3 + \cdots. \tag{1.2}$$

see for example [8].

Recall that a function $f\in\mathcal{S}$, of the form

$$\vartheta(z) = z + \sum_{k=2}^{\infty}\mathfrak{a}_k z^k,$$

is univalent in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and admits an inverse function $\Theta := \vartheta^{-1}$, analytic in $|\mathfrak{w}|<r_0(\vartheta)$, where $r_0(\vartheta)\geq\frac{1}{4}$. The inverse function admits the following series representation:

$$\Theta(\mathfrak{w}) = \mathfrak{w} - \mathfrak{a}_2\mathfrak{w}^2 + (2\mathfrak{a}_2^2-\mathfrak{a}_3)\mathfrak{w}^3 - (5\mathfrak{a}_2^3-5\mathfrak{a}_2\mathfrak{a}_3+\mathfrak{a}_4)\mathfrak{w}^4 + \cdots.$$

A function ϑ belonging to the class $\mathcal{S}$ is termed *bi-univalent* within the unit disk $\mathbb{D}$ if both ϑ and its inverse Θ are univalent in $\mathbb{D}$. We denote by Π the collection of all such bi-univalent functions.

Typical members of Π include:

$$\vartheta_1(z) = \frac{z}{1-z}, \qquad \Theta_1(\mathfrak{w}) = \frac{\mathfrak{w}}{1+\mathfrak{w}},$$
$$\vartheta_2(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \qquad \Theta_2(\mathfrak{w}) = \frac{e^{2\mathfrak{w}}-1}{e^{2\mathfrak{w}}+1},$$
$$\vartheta_3(z) = -\log(1-z), \qquad \Theta_3(\mathfrak{w}) = \frac{e^{\mathfrak{w}}-1}{e^{\mathfrak{w}}}.$$

The foundational concept of the class $\mathcal{S}$ was first introduced by Lewin, who demonstrated that $|\mathfrak{a}_2| < 1.51$ for all $f \in \mathcal{S}$ [20]. Subsequently, Styer and Wright confirmed the existence of functions in $\mathcal{S}$ with $|\mathfrak{a}_2| > \frac{4}{3}$ [22]. Since its inception, considerable attention has been given to uncovering the relationship between the geometric features of functions within $\mathcal{S}$ and their associated coefficient bounds.

This line of inquiry has been advanced by several researchers, including Lewin [20], Brannan and Taha [6], and Srivastava and collaborators [23], who laid important groundwork for the theory of bi-univalent functions. However, their contributions mostly provided non-optimal (non-sharp) estimates of the early coefficients.

Further developments included generalized bounds for subclasses of analytic bi-univalent functions. In particular, the use of integral operators involving Lucas polynomials was employed to obtain such estimates [21]. A novel subclass of bi-univalent functions involving (m,n)-Lucas polynomial was introduced by Hadi and Darus[18], who also addressed the Fekete-Szegö problem within that context.

Despite these advancements, the precise (sharp) bounds for $|\mathfrak{a}_n|$ with $n \geq 3$ remain unresolved. In response to this challenge, Hamidi and Jahangiri [15] began applying the Faber polynomial expansion technique to analyze such coefficients. Originally proposed by Faber [12], the method was later emphasized by Gong for its theoretical significance [14].

Recent contributions have expanded the scope of this approach. Bult, for instance, defined multiple new subclasses of bi-univalent functions and applied the Faber expansion to estimate general bounds for $|\mathfrak{a}_n|$ when $n \geq 3$ [7], additionally noting the irregular behavior of some initial coefficients.

Other works have adopted subordination principles combined with the Faber method to derive coefficient bounds [16]. Similarly, Altinkaya and Yalcin explored how these techniques yield intriguing patterns in the behavior of coefficients within newly defined bi-univalent classes [5]. A selection of recent contributions addressing the initial coefficient estimates and the Fekete–Szegö functional for certain special subclasses of the class Π, particularly those associated with specific families of polynomials or geometric domains, can be found in [1, 3, 9, 10, 11, 25, 26]. $\mathcal{A}$ denotes the class of analytic functions in $\mathbb{D}$ of the form above, normalized by $\vartheta(0) = 0$ and $\vartheta'(0) = 1$, and $\mathcal{S} \subset \mathcal{A}$ is the subclass of univalent functions.

Let f and g be analytic functions defined on the unit disk $\mathbb{D}$. The function f is said to be *subordinate* to g, written as

$$f(z) \prec g(z),$$

if there exists a Schwarz function ω, analytic in $\mathbb{D}$, satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for every $z \in \mathbb{D}$, such that

$$\mathfrak{f}(z) = \mathfrak{g}(\omega(z)), \quad \text{for all } z \in \mathbb{D}.$$

Moreover, when g is univalent in $\mathbb{D}$, the subordination $\mathfrak{f} \prec \mathfrak{g}$ holds if and only if

$$\mathfrak{f}(0) = \mathfrak{g}(0) \quad \text{and} \quad \mathfrak{f}(\mathbb{D}) \subseteq \mathfrak{g}(\mathbb{D}).$$

Among the important subclasses of the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{D}$, we highlight the following:

- The class $\mathcal{S}^*(\delta)$ of *starlike functions of order* δ $(0 \leq \delta < 1)$, defined by the subordination condition:

$$\frac{z\vartheta'(z)}{\vartheta(z)} \prec \frac{1+(1-2\delta)z}{1-z}. \tag{1.3}$$

- The class $\mathcal{K}(\delta) \subset \mathcal{S}$ of *convex functions of order* δ $(0 \leq \delta < 1)$, defined by:

$$1 + \frac{z\vartheta''(z)}{\vartheta'(z)} \prec \frac{1+(1-2\delta)z}{1-z}. \tag{1.4}$$

Laguerre Polynomials and Their Generating Function. Consider the second-order differential equation (see [24])

$$\kappa\, u''(\kappa) + (1+\theta-\kappa)\, u'(\kappa) + m\, u(\kappa) = 0, \quad m \in \mathbb{N}, \theta \in \mathbb{R}^+, \kappa \in \mathbb{R}.$$

whose solutions are known as the generalized Laguerre polynomials $L_m^{\theta}(\kappa)$. These polynomials satisfy the recurrence relation

$$L_{m+1}^{\theta}(\kappa) = \frac{2m+1+\theta-\kappa}{m+1} L_m^{\theta}(\kappa) - \frac{m+\theta}{m+1} L_{m-1}^{\theta}(\kappa),$$

with initial conditions

$$L_0^{\theta}(\kappa) = 1, \quad L_1^{\theta}(\kappa) = 1+\theta-\kappa,$$

and

$$L_2^{\theta}(\kappa) = \frac{\kappa^2}{2} + (\theta+2)\kappa + \frac{(\theta+1)(\theta+2)}{2}.$$

Figure 1 below illustrates several cases of the polynomial curve for different values of θ and m.

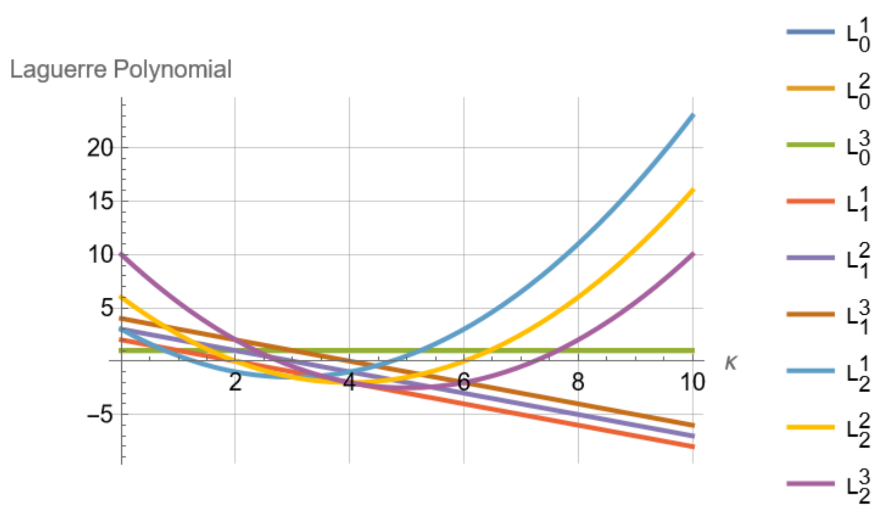

FIGURE 1. Plot of the generalized Laguerre polynomials $L_m^{\theta}(\kappa)$ for selected values of m and θ.

Laguerre polynomials play a significant role in quantum mechanics, particularly appearing in the radial component of the Schrdinger equation solutions for hydrogen-like atoms. Additionally, they characterize the stationary Wigner functions associated with oscillator models within the quantum phase space framework. Moreover, these polynomials are involved in the quantum analysis of the Morse potential as well as the isotropic harmonic oscillator in three dimensions. However, we note that, for $\theta = 0$, the generalized Laguerre polynomials reduce to the classical Laguerre polynomials:

$$L_m^0(\kappa) = L_m(\kappa).$$

In [24], the generating function of the generalized Laguerre polynomials is given by:

$$\mathcal{G}(\kappa, \theta; z) = \sum_{m=0}^{\infty} L_m^{\theta}(\kappa) z^m = \frac{\exp\left(-\frac{\kappa z}{1-z}\right)}{(1-z)^{\theta+1}}, \quad |z| < 1.$$

This study is motivated by the need to further explore the properties and applications of Laguerre polynomials and their associated fractional operators in complex analysis. Despite significant research on related classes of analytic and bi-univalent functions, the specific combination of fractional operators with Laguerre polynomials has not been thoroughly investigated. The main contribution of this work is to introduce new subclasses of bi-univalent functions defined via these operators and to derive best bounds for their initial coefficients. This work is to introduce novel classes of bi-univalent functions employing fractional operators and polynomials, provide the best possible estimates of their initial coefficients, and construct a theoretical framework that gives the way for Variety of applications in complex analysis and relevent branches of mathematics.

Definition 1.4. Let f be analytic in $\mathbb{D}$. Then $f \in \mathcal{F}_{\mathfrak{q}}^{\kappa,\theta}(\lambda, \varsigma)$ if the following subordination holds:

$$\left\{\frac{2z\left[\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right]^{\lambda}}{\vartheta(z)-\vartheta(-z)}\right\}^{\varsigma} \cdot \left\{\frac{2\left[\mathcal{D}_{\mathfrak{q}}\left(z\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right)\right]^{\lambda}}{\mathcal{D}_{\mathfrak{q}}\left(\vartheta(z)-\vartheta(-z)\right)}\right\}^{1-\varsigma} \prec \mathcal{G}(\kappa, \theta; z), \tag{1.5}$$

and

$$\left\{\frac{2\mathfrak{w}\left[\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right]^{\lambda}}{\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})}\right\}^{\varsigma} \cdot \left\{\frac{2\left[\mathcal{D}_{\mathfrak{q}}\left(\mathfrak{w}\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right)\right]^{\lambda}}{\mathcal{D}_{\mathfrak{q}}\left(\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})\right)}\right\}^{1-\varsigma} \prec \mathcal{G}(\kappa, \theta; \mathfrak{w}). \tag{1.6}$$

where $\lambda \geq 1$, $\varsigma \geq 0$, and $\mathcal{G}(\kappa, z)$ is the generating function and the inverse function $\vartheta^{-1} := \Theta$ is defined and characterized by (1.2).

Special Cases.

- For $\varsigma = 1$ and $\lambda = 1$:

$$\frac{2z\,\mathcal{D}_{\mathfrak{q}}\vartheta(z)}{\vartheta(z)-\vartheta(-z)} \prec \mathcal{G}(\kappa, \theta; z),$$

$$\frac{2\mathfrak{w}\,\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})}{\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})} \prec \mathcal{G}(\kappa, \theta; \mathfrak{w}).$$

- For $\varsigma = 0$ and $\lambda = 1$:

$$\frac{2\,\mathcal{D}_{\mathfrak{q}}(z\,\mathcal{D}_{\mathfrak{q}}\vartheta(z))}{\mathcal{D}_{\mathfrak{q}}(\vartheta(z)-\vartheta(-z))} \prec \mathcal{G}(\kappa, \theta; z),$$

$$\frac{2\,\mathcal{D}_{\mathfrak{q}}(\mathfrak{w}\,\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w}))}{\mathcal{D}_{\mathfrak{q}}(\Theta(\mathfrak{w})-\Theta(-\mathfrak{w}))} \prec \mathcal{G}(\kappa, \theta; \mathfrak{w}).$$

- For $\varsigma = \frac{1}{2}$ and $\lambda = 2$:

$$\sqrt{\left(\frac{2z[\mathcal{D}_{\mathfrak{q}}\vartheta(z)]^2}{\vartheta(z)-\vartheta(-z)}\right)\cdot\left(\frac{2[\mathcal{D}_{\mathfrak{q}}(z\mathcal{D}_{\mathfrak{q}}\vartheta(z))]^2}{\mathcal{D}_{\mathfrak{q}}(\vartheta(z)-\vartheta(-z))}\right)} \prec \mathcal{G}(\kappa,\theta;z),$$

$$\sqrt{\left(\frac{2\mathfrak{w}[\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})]^2}{\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})}\right)\cdot\left(\frac{2[\mathcal{D}_{\mathfrak{q}}(\mathfrak{w}\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w}))]^2}{\mathcal{D}_{\mathfrak{q}}(\Theta(\mathfrak{w})-\Theta(-\mathfrak{w}))}\right)} \prec \mathcal{G}(\kappa,\theta;\mathfrak{w}).$$

2. Initial Coefficients of the Class $\mathcal{F}_q^{\kappa,\theta}(\lambda,\varsigma)$

This section is devoted to deriving sharp upper estimates of the first Taylor coefficients for functions belonging to the class $\mathcal{F}_q^{\kappa,\theta}(\lambda,\varsigma)$, as defined earlier in (1.1). These results enable a clearer and more accurate understanding of the properties of the geometric behavior of functions that fall within this category.

Theorem 2.1. *The function ϑ in (1.1) is in the class $\mathcal{F}_q^{\kappa,\theta}(\lambda,\varsigma)$. Then the following upper coefficient bounds hold:*

$$\mathfrak{a}_2 \leq \frac{2|L_1^\theta(\kappa)|\sqrt{L_1^\theta(\kappa)}}{\sqrt{\left|\begin{array}{l}\Big[\lambda\left(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2\right]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\\ +2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right]\Big][L_1^\theta(\kappa)]^2-2\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2\lceil 2\rfloor_{\mathfrak{q}}^2 L_2^\theta(\kappa)\end{array}\right|}} \tag{2.1}$$

and

$$\mathfrak{a}_3 \leq \frac{|L_1^\theta(\kappa)|}{|(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right]|}+\frac{|L_1^\theta(\kappa)|^2}{\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2\lceil 2\rfloor_{\mathfrak{q}}^2} \tag{2.2}$$

Proof. For $\vartheta \in \mathcal{F}_q^{\kappa,\theta}(\lambda,\varsigma)$ and the fact $\Theta = \vartheta^{-1}$ is the inverse of Taylor function. Utilizing Definition (1.4) in conjunction with the relations given in (1.5) and (1.6), the subordination conditions can be reformulated in the following manner:

$$\left\{\frac{2z\left[\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right]^\lambda}{\vartheta(z)-\vartheta(-z)}\right\}^{\varsigma}\left\{\frac{2\left\{\mathcal{D}_{\mathfrak{q}}\left[z\,\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right]\right\}^\lambda}{\mathcal{D}_{\mathfrak{q}}\left[\vartheta(z)-\vartheta(-z)\right]}\right\}^{1-\varsigma} = \mathcal{G}(\kappa,\theta;c(z)), \tag{2.3}$$

and

$$\left\{\frac{2\mathfrak{w}\left[\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right]^\lambda}{\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})}\right\}^{\varsigma}\left\{\frac{2\left\{\mathcal{D}_{\mathfrak{q}}\left[\mathfrak{w}\,\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right]\right\}^\lambda}{\mathcal{D}_{\mathfrak{q}}\left[\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})\right]}\right\}^{1-\varsigma} = \mathcal{G}(\kappa,\theta;d(\mathfrak{w})), \tag{2.4}$$

where

$$c(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad d(\mathfrak{w}) = d_1 w + d_2\mathfrak{w}^2 + d_3\mathfrak{w}^3 + \cdots$$

are analytic functions of Schwarz type, satisfying $c(0) = d(0) = 0$ and $|c(z)| \leq 1$, $|d(\mathfrak{w})| \leq 1$ for every $z, \mathfrak{w} \in \Omega$. Consequently, the coefficient bounds $|c_j| \leq 1$ and $|d_j| \leq 1$ hold for all $j \in \mathbb{N}$.

Expanding the left-hand side of equations (2.3) and (2.4), we get

$$\begin{aligned}&\left\{\frac{2z\left[\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right]^{\lambda}}{\vartheta(z)-\vartheta(-z)}\right\}^{\varsigma}\left\{\frac{2\left[\mathcal{D}_{\mathfrak{q}}\left(z\mathcal{D}_{\mathfrak{q}}\vartheta(z)\right)\right]^{\lambda}}{\mathcal{D}_{\mathfrak{q}}\left(\vartheta(z)-\vartheta(-z)\right)}\right\}^{1-\varsigma}\\&=1+\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]\lceil 2\rfloor_{\mathfrak{q}}\mathfrak{a}_2 z+\Big\{\left(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1\right)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right]\mathfrak{a}_3\\&+\frac{\lambda}{2}\left(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2\right]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\mathfrak{a}_2^2\Big\}z^2+\cdots\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\left\{\frac{2\mathfrak{w}\left[\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right]^{\lambda}}{\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})}\right\}^{\varsigma}\left\{\frac{2\left[\mathcal{D}_{\mathfrak{q}}\left(\mathfrak{w}\mathcal{D}_{\mathfrak{q}}\Theta(\mathfrak{w})\right)\right]^{\lambda}}{\mathcal{D}_{\mathfrak{q}}\left(\Theta(\mathfrak{w})-\Theta(-\mathfrak{w})\right)}\right\}^{1-\varsigma}\\&=1-\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]\lceil 2\rfloor_{\mathfrak{q}}\mathfrak{a}_2\mathfrak{w}+\Big\{\left(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1\right)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right](2\mathfrak{a}_2^2-\mathfrak{a}_3)\\&+\frac{\lambda}{2}\left(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2\right]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\mathfrak{a}_2^2\Big\}\mathfrak{w}^2+\cdots.\end{aligned}\tag{2.6}$$

On the other hand, expanding the right-hand side of equations (2.3) and (2.4), we have

$$\mathcal{B}(k,c(z))=1+L_1^{\theta}(k)c_1z+\left[L_1^{\theta}(k)c_2+L_2^{\theta}(k)c_1^2\right]z^2+\cdots,\tag{2.7}$$

$$\mathcal{B}(k,d(\mathfrak{w}))=1+L_1^{\theta}(k)d_1\mathfrak{w}+\left[L_1^{\theta}(k)d_2+L_2^{\theta}(k)d_1^2\right]\mathfrak{w}^2+\cdots.\tag{2.8}$$

By comparing the corresponding coefficients from equations (2.5) and (2.7), as well as from equations (2.6) and (2.8), we obtain the following system of equations:

$$\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]\lceil 2\rfloor_{\mathfrak{q}}\,\mathfrak{a}_2=L_1^{\theta}(\kappa)\,c_1,\tag{2.9}$$

$$-\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]\lceil 2\rfloor_{\mathfrak{q}}\,\mathfrak{a}_2=L_1^{\theta}(k)\,d_1,\tag{2.10}$$

$$\begin{aligned}&\left[\left(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1\right)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right]\mathfrak{a}_3+\frac{\lambda}{2}\left(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2\right]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\,\mathfrak{a}_2^2\right]\\&=L_1^{\theta}(k)\,c_2+L_2^{\theta}(k)\,c_1^2,\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\left[\left(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1\right)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}\right](2\mathfrak{a}_2^2-\mathfrak{a}_3)+\frac{\lambda}{2}\left(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2\right]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\,\mathfrak{a}_2^2\right]\\&=L_1^{\theta}(k)\,d_2+L_2^{\theta}(k)\,d_1^2,\end{aligned}\tag{2.12}$$

By referring to equations (2.9) and (2.10), we obtain

$$c_1=-d_1,\tag{2.13}$$

and

$$2\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}\right]^2\lceil 2\rfloor_{\mathfrak{q}}^2\,\mathfrak{a}_2^2=\left[L_1^{\theta}(\kappa)\right]^2(c_1^2+d_1^2).\tag{2.14}$$

By adding equations (2.11) and (2.12), we obtain

$$\mathfrak{a}_2^2\Big[\lambda\left(\lambda\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2-[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2]\right)\lceil 2\rfloor_{\mathfrak{q}}^2+2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\Big]$$
$$=L_1^{\theta}(k)(c_2+d_2)+L_2^{\theta}(k)(c_1^2+d_1^2). \tag{2.15}$$

Substituting equation (2.14) into equation (2.15), we obtain

$$\mathfrak{a}_2^2\Big[\lambda\left(\lambda\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2-[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\,[L_1^{\theta}(\kappa)]^2+2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)$$
$$\times[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\;[L_1^{\theta}(\kappa)]^2\Big]-2\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\,\lceil 2\rfloor_{\mathfrak{q}}^2\,L_2^{\theta}(k)\,\mathfrak{a}_2^2=[L_1^{\theta}(\kappa)]^3(c_2+d_2). \tag{2.16}$$

After simplification, we have

$$\mathfrak{a}_2^2\Big\{\Big[\lambda\left(\lambda\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2-[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2]\right)\lceil 2\rfloor_{\mathfrak{q}}^2+2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\Big][L_1^{\theta}(\kappa)]^2$$
$$-2\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\;\lceil 2\rfloor_{\mathfrak{q}}^2\,L_2^{\theta}(k)\Big\}=[L_1^{\theta}(\kappa)]^3(c_2+d_2). \tag{2.17}$$

The previous technique leads to the initial coefficient $\mathfrak{a}_2^2$ as follows

$$\mathfrak{a}_2^2=\frac{[L_1^{\theta}(\kappa)]^3(c_2+d_2)}{\begin{aligned}&\Big(\lambda\left(\lambda\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2-[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\\&\quad+2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\Big)[L_1^{\theta}(\kappa)]^2-2\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\,\lceil 2\rfloor_{\mathfrak{q}}^2L_2^{\theta}(k)\end{aligned}}. \tag{2.18}$$

Therefore, the upper bound for the second coefficient $\mathfrak{a}_2$ satisfies

$$\mathfrak{a}_2\le\frac{2|L_1^{\theta}(\kappa)|\sqrt{L_1^{\theta}(\kappa)}}{\sqrt{\left|\begin{aligned}&\Big[\lambda\left(\lambda\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2-[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}^2]\right)\lceil 2\rfloor_{\mathfrak{q}}^2\\&+2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\Big][L_1^{\theta}(\kappa)]^2-2\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\,\lceil 2\rfloor_{\mathfrak{q}}^2\,L_2^{\theta}(\kappa)\end{aligned}\right|}}$$

By subtracting equations (2.12) from (2.11), we obtain

$$2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\,(\mathfrak{a}_3-\mathfrak{a}_2^2)=L_1^{\theta}(k)(c_2-d_2). \tag{2.19}$$

Simplifying, we get

$$\mathfrak{a}_3-\mathfrak{a}_2^2=\frac{L_1^{\theta}(k)(c_2-d_2)}{2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]}. \tag{2.20}$$

By substituting equation (2.14) into equation (2.20), we obtain

$$\mathfrak{a}_3=\frac{L_1^{\theta}(k)(c_2-d_2)}{2(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]}+\frac{[L_1^{\theta}(\kappa)]^2(c_1^2+d_1^2)}{2\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\,\lceil 2\rfloor_{\mathfrak{q}}^2}. \tag{2.21}$$

Thus, the upper bound for $\mathfrak{a}_3$ is given by

$$\mathfrak{a}_3\le\frac{|L_1^{\theta}(k)|}{|(\lambda\lceil 3\rfloor_{\mathfrak{q}}-1)\,[\varsigma-(\varsigma-1)\lceil 3\rfloor_{\mathfrak{q}}]\,|}+\frac{|L_1^{\theta}(\kappa)|^2}{\lambda^2\,[\varsigma-(\varsigma-1)\lceil 2\rfloor_{\mathfrak{q}}]^2\,\lceil 2\rfloor_{\mathfrak{q}}^2}. \tag{2.22}$$

□

Corollary 2.2. *Let f be a function in the class $\mathcal{F}_{\mathfrak{q}}^{\kappa,\theta}(1,1)$, i.e., $\varsigma = 1$, $\lambda = 1$. Then the coefficient bounds reduce to:*

$$\mathfrak{a}_2 \leq \frac{2|L_1^\theta(\kappa)|\sqrt{L_1^\theta(\kappa)}}{\sqrt{\left|2(L_1^\theta(\kappa))^2\right|}} = \sqrt{2}|L_1^\theta(\kappa)|,$$

and

$$\mathfrak{a}_3 \leq \frac{|L_1^\theta(\kappa)|}{|\lceil 3 \rfloor_{\mathfrak{q}} - 1|} + \frac{|L_1^\theta(\kappa)|^2}{\lceil 2 \rfloor_{\mathfrak{q}}^2}.$$

Corollary 2.3. *Let f be a function in the class $\mathcal{F}_{\mathfrak{q}}^{\kappa,\theta}(1,0)$, i.e., $\varsigma = 0$, $\lambda = 1$. Then*

$$\mathfrak{a}_2 \leq \frac{2|L_1^\theta(\kappa)|\sqrt{L_1^\theta(\kappa)}}{\sqrt{\left|\lceil 2 \rfloor_{\mathfrak{q}}^2 (L_1^\theta(\kappa))^2 - 2L_2^\theta(\kappa)\lceil 2 \rfloor_{\mathfrak{q}}^2\right|}},$$

and

$$\mathfrak{a}_3 \leq \frac{|L_1^\theta(\kappa)|}{|\lceil 3 \rfloor_{\mathfrak{q}} - 1|} + \frac{|L_1^\theta(\kappa)|^2}{\lceil 2 \rfloor_{\mathfrak{q}}^2}.$$

3. Fekete–Szegö Problem

The Fekete-Szegö inequality is a fundamental pillar in geometric function theory, serving as a key tool for analyzing the coefficients of univalent analytic functions. First proposed by Fekete and Szegö in 1933, this inequality provides a quantitative relationship between the second and third Taylor coefficients. It is of great importance because it is directly related to the Bieberbach conjecture, which deals with upper bounds on the coefficients of this type of function. The fundamental problem, known as the Fekete-Szegö problem, seeks to generalize these bounds to broader classes of analytic functions, thus enhancing the understanding of their geometric and functional properties.

More precisely, suppose that

$$\mathfrak{f}(z) = z + \mathfrak{a}_2 z^2 + \mathfrak{a}_3 z^3 + \cdots$$

is a univalent function normalized on the unit disk, and let the parameter $0 \leq \lambda < 1$. Then the Fekete–Szegö inequality gives an upper bound for the nonlinear functional $|\mathfrak{a}_3 - \lambda \mathfrak{a}_2^2|$, namely,

$$|\mathfrak{a}_3 - \lambda \mathfrak{a}_2^2| \leq 1 + 2\exp\left(-\frac{2\lambda}{1-\lambda}\right).$$

This result is consider a constraint of the coefficients. Many authors have worked to examine some inequalities that apply to various families of these functions, giving rise to a wide range of problems collectively familiar as the Fekete-Szegö inequality.

In this context, this section focuses on a formulation of the Fekete-Szegö inequality specific to the new class of functions we proposed earlier, thus expanding the scope of its application within the framework of our current research.

Theorem 3.1. *Let $\vartheta(z)$ given by (2.20) be in the class $\mathcal{F}_q^{\kappa,\theta}(\lambda,\varsigma)$, with $k,\eta\in\mathbb{R}$, $1+\theta>0$. Then,*

$$|\mathfrak{a}_3-\eta\mathfrak{a}_2^2|\leq\begin{cases}\frac{2|L_1^\theta(k)|}{|2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}]|} & \textit{for } |h(\eta)|\leq\frac{1}{|2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}]|}\\ \\ 2|L_1^\theta(k)||h(\eta)| & \textit{for } |h(\eta)|\geq\frac{1}{|2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}]|}\end{cases}$$

where

$$h(\eta)=\frac{[L_1^\theta(\kappa)]^2(1-\eta)}{\left\{\begin{array}{l}\Big[\lambda\Big(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}^2\right]\Big)\lceil 2\rfloor_\mathfrak{q}^2+2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\\ \times\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]\Big][L_1^\theta(\kappa)]^2-2\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2\lceil 2\rfloor_\mathfrak{q}^2L_2^\theta(k)\end{array}\right\}}.$$

Proof. From Equation (2.20) , it is derived that

$$\mathfrak{a}_3-\eta\mathfrak{a}_2^2=\frac{L_1^\theta(k)\,(c_2-d_2)}{2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]}+(1-\eta)\mathfrak{a}_2^2 \tag{3.1}$$

Also, from (2.18)

$$\mathfrak{a}_3-\eta\mathfrak{a}_2^2=\frac{L_1^\theta(k)(c_2-d_2)}{2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]}$$
$$+\frac{[L_1^\theta(\kappa)]^3(1-\eta)(c_2+d_2)}{\left\{\begin{array}{l}\Big[\lambda\Big(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}^2\right]\Big)\lceil 2\rfloor_\mathfrak{q}^2\\ +2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]\Big][L_1^\theta(\kappa)]^2-2\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2\lceil 2\rfloor_\mathfrak{q}^2L_2^\theta(k)\end{array}\right\}} \tag{3.2}$$

Simplify to

$$\begin{aligned}\mathfrak{a}_3-\eta\mathfrak{a}_2^2=L_1^\theta(k)\Bigg[&\left(h(\eta)+\frac{1}{2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]}\right)c_2\\ &+\left(h(\eta)-\frac{1}{2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]}\right)d_2\Bigg],\end{aligned}$$

where

$$h(\eta)=\frac{[L_1^\theta(\kappa)]^2(1-\eta)}{\left\{\begin{array}{l}\Big[\lambda\Big(\lambda\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2-\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}^2\right]\Big)\lceil 2\rfloor_\mathfrak{q}^2+2(\lambda\lceil 3\rfloor_\mathfrak{q}-1)\\ \times\left[\varsigma-(\varsigma-1)\lceil 3\rfloor_\mathfrak{q}\right]\Big][L_1^\theta(\kappa)]^2-2\lambda^2\left[\varsigma-(\varsigma-1)\lceil 2\rfloor_\mathfrak{q}\right]^2\lceil 2\rfloor_\mathfrak{q}^2L_2^\theta(k)\end{array}\right\}}.$$

□

Corollary 3.2. *If $\vartheta(z)$ has the formula (2.20) is in the class $\mathcal{F}_q^{\kappa,\theta}(1,1)$, then*

$$|\mathfrak{a}_3-\eta\mathfrak{a}_2^2|\leq\begin{cases}\dfrac{2|L_1^\theta(k)|}{|2(\lceil 3\rfloor_\mathfrak{q}-1)|} & \textit{if } |h_1(\eta)|\leq\frac{1}{|2(\lceil 3\rfloor_\mathfrak{q}-1)|}\\ 2|L_1^\theta(k)|\,|h_1(\eta)| & \textit{if } |h_1(\eta)|\geq\frac{1}{|2(\lceil 3\rfloor_\mathfrak{q}-1)|}\end{cases}$$

where

$$h_1(\eta) = \frac{1-\eta}{\left(\lceil 2\rfloor_{\mathfrak{q}}^2 + 2(\lceil 3\rfloor_{\mathfrak{q}} - 1) - 2\lceil 2\rfloor_{\mathfrak{q}}^2 \cdot \dfrac{L_2^{\theta}(k)}{[L_1^{\theta}(\kappa)]^2}\right)}.$$

Corollary 3.3. *If $\vartheta(z)$ has the formula (2.20) is in the class $\mathcal{F}_q^{\kappa,\theta}(1,0)$, then*

$$|\mathfrak{a}_3 - \eta\mathfrak{a}_2^2| \leq \begin{cases} \dfrac{2|L_1^{\theta}(k)|}{|2(\lceil 3\rfloor_{\mathfrak{q}} - 1)(-\lceil 3\rfloor_{\mathfrak{q}})|} & \text{if } |h_2(\eta)| \leq \frac{1}{|2(\lceil 3\rfloor_{\mathfrak{q}}-1)(-\lceil 3\rfloor_{\mathfrak{q}})|} \\ 2|L_1^{\theta}(k)|\,|h_2(\eta)| & \text{if } |h_2(\eta)| \geq \frac{1}{|2(\lceil 3\rfloor_{\mathfrak{q}}-1)(-\lceil 3\rfloor_{\mathfrak{q}})|} \end{cases}$$

where

$$h_2(\eta) = \frac{[L_1^{\theta}(\kappa)]^2(1-\eta)}{\left\{\lceil 2\rfloor_{\mathfrak{q}}^2\left(\lceil 2\rfloor_{\mathfrak{q}}^2 - 2(\lceil 3\rfloor_{\mathfrak{q}} - 1)\lceil 3\rfloor_{\mathfrak{q}}\right)[L_1^{\theta}(\kappa)]^2 - 2\lceil 2\rfloor_{\mathfrak{q}}^4 L_2^{\theta}(k)\right\}}.$$

Corollary 3.4. *Let $\vartheta(z)$ be given by (2.20) and belong to the class $\mathcal{F}_q^{\kappa,\theta}(2,\frac{1}{2})$. Then,*

$$|\mathfrak{a}_3 - \eta\mathfrak{a}_2^2| \leq \begin{cases} \dfrac{2|L_1^{\theta}(k)|}{|2(2\lceil 3\rfloor_{\mathfrak{q}} - 1)(\frac{1}{2}(1+\lceil 3\rfloor_{\mathfrak{q}}))|} & \text{if } |h_3(\eta)| \leq \frac{1}{|2(2\lceil 3\rfloor_{\mathfrak{q}}-1)(\frac{1+\lceil 3\rfloor_{\mathfrak{q}}}{2})|} \\ 2|L_1^{\theta}(k)|\,|h_3(\eta)| & \text{if } |h_3(\eta)| \geq \frac{1}{|2(2\lceil 3\rfloor_{\mathfrak{q}}-1)(\frac{1+\lceil 3\rfloor_{\mathfrak{q}}}{2})|} \end{cases}$$

where

$$\begin{aligned} h_3(\eta) &= \frac{L_1^{\theta}(\kappa)^2(1-\eta)}{A\,L_1^{\theta}(\kappa)^2 - B\,L_2^{\theta}(\kappa)}, \\ A &:= 2\Big(2\Big(\frac{1+\lfloor 2\rfloor_{\mathfrak{q}}}{2}\Big)^2 - \Big(\frac{1+\lfloor 2^2\rfloor_{\mathfrak{q}}}{2}\Big)\Big)\lfloor 2\rfloor_{\mathfrak{q}}^2 + 2\big(2\lfloor 3\rfloor_{\mathfrak{q}} - 1\big)\Big(\frac{1+\lfloor 3\rfloor_{\mathfrak{q}}}{2}\Big), \\ B &:= 8\Big(\frac{1+\lfloor 2\rfloor_{\mathfrak{q}}}{2}\Big)^2\lfloor 2\rfloor_{\mathfrak{q}}^2. \end{aligned}$$

Conclusion

In this paper, we propose new families of bivalent functions associated with Laguerre polynomials and obtain the best possible estimates of the Taylor series coefficients for these functions. The bounds we obtain are $|\mathfrak{a}_2|$, $|\mathfrak{a}_3|$, and the Fekete-Szegö inequality not only extend previously known results but also provide new insights into the geometric properties of bivalent functions governed by Laguerre structures.

These results pave the way for future research into broader classes of bi univalent functions, particularly those based on other orthogonal polynomials or special functions. Subsequent studies could also focus on deriving precise limits for higher-order coefficients, or on applications of these results in the contexts of fractional analysis and the q-analysis.

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