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ON WEAKLY SEMIPRIME IDEALS IN NONCOMMUTATIVE RING

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Abstract. We extend the concept of weakly semiprime ideals, originally defined by A. Badawi for commutative rings, to the noncommutative setting. We define a proper ideal I of a noncommutative ring R to be weakly semiprime if for any $a \in R$, $0 \neq aRa \subseteq I$ implies $a \in I$. Fundamental properties of such ideals are investigated, particularly those that are weakly semiprime but not semiprime. Key results include showing that such ideals are contained in the prime radical of the ring. These ideals are also characterized in decomposable rings (direct products), and a method for constructing examples using the idealization of a bimodule is provided. We further introduce and study related concepts, including "ideal weakly semiprime" ideals (where $0 \neq A^2 \subseteq I$ for an ideal A implies $A \subseteq I$), weakly completely semiprime ideals, and an extension of these notions to submodules of an R-module.

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1. Introduction

In [7] Badawi introduced the notion of weakly semiprime ideals in a commutative ring with identity and called an ideal I weakly semiprime if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$. Throughout this paper let R be a noncommutative ring but not necessarily with unity unless indicated. Recall that a proper ideal I of R is called semiprime if $a \in R$ and $aRa \subseteq I$ implies $a \in I$. In this paper, we define a proper ideal I of R to be weakly semiprime if $a \in R$ and $0 \neq aRa \subseteq I$ implies $a \in I$. Recall from (Anderson and Smith [3]) that an ideal I of a commutative ring R is said to be weakly prime if $a, b \in R$ and $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. Every weakly prime ideal of R is weakly semiprime. However, the converse is not true in general. In [10] Hirano et al. introduced weakly prime ideals in a noncommutative ring. They define an ideal I of R to be weakly prime if $a, b \in R$ and $0 \neq aRb \subseteq I$ implies $a \in I$ or $b \in I$. They then also show that this is equivalent to: If A and B are ideals of R such that $0 \neq AB \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. Every weakly prime ideal of R is weakly semiprime. However the converse is not true. For example, the ideal $P = M_2(\{\overline{0}, \overline{8}\})$ of $R = M_2(\mathbb{Z}_{16})$ is a weakly semiprime ideal of R which is neither a semiprime nor a weakly prime ideal. P is weakly semiprime since

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it can be checked that if $xRx \subseteq P$, then $xRx = \left\{ \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \right\}$. P is not semiprime since $\begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} M_2(\mathbb{Z}_4) \begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} = \left\{ \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \right\} \subseteq P$ with $\begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \notin P$, and not weakly prime since $\begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} M_2(\mathbb{Z}_4) \begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \subseteq M_2(\{\overline{0}, \overline{8}\}) = P$, with $\begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \notin P$ and $\begin{pmatrix} \overline{4} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix} \notin P$.

In a recent paper [1] Abouhalaka introduced a weakly semiprime ideal in a noncommutative ring. The author defines an ideal P of a noncommutative ring R to be a weakly semiprime ideal if whenever A is an ideal of R with $0 \neq A^2 \subseteq P$, then $A \subseteq P$. We will call these ideals ideal weakly semiprime.

Lemma 1.1. *Let R be a ring with identity. If I is an ideal weakly semiprime ideal of R, then I is weakly semiprime.*

Proof. Let $a \in R$ such that $\{0\} \neq aRa \subseteq I$. Hence, there exists $r \in R$ such that $0 \neq ara \in I$. Now, $0 \neq ara = 1ar1ar \in (RaR)(RaR) \subseteq I$. Since I is ideal weakly semiprime, we have $a \in RaR \subseteq I$ and we are done. □

From [7, Example 2.7] we have the following example:

Example 1.2. Let $A = \mathbb{Z}_{16}[X]$. Then $J = (X^2 + 8X)$ and $L = (X, 8)$ are ideals of A and $J \subset L$. Let $R = A/J$ and $I = L/J$ and $x = X + J \in I$. Then I is a weakly semiprime ideal of R that is nether semiprime nor weakly prime. $H = (J + (4, 2x))/J$ is an ideal of R where $\{0\} \neq H^2 \subset I$ but $H \nsubseteq I$.

Hence in general weakly semiprime ideals are not ideal weakly semiprime.

In section 2 we prove basic properties of weakly semiprime ideals. It is shown that among others in Theorem 1.3 that a non-semiprime, weakly semiprime ideal is contained in the prime radical. In section 3 we characterize weakly semiprime ideals in decomposable rings (direct products). In section 4 we show how to construct weakly semiprime ideals using the method of idealization. In section 5 we introduce the notion of a weakly completely semiprime ideal. We show that if I is a weakly completely semiprime ideal that is not a completly semiprime ideal then I is contained in the completely prime radical (generalized nil radical) of a ring R i.e. $\mathcal{N}_g(R)$ is the intersection of all the completely prime ideals of R. In section 6 we introduce the notion of the weakly semiprime radical $\mathcal{P}_{iws}(A)$ of an ideal A of R. We prove that $\mathcal{P}_{iws}(A)$ is equal to the intersection of all the ideal weakly semiprime ideals that contain A. In the last section we introduce the notion of weakly semiprime submodules of an left R-module M. Different characterzations of weakly semiprime submodules are obtained.

2. Basic properties of weakly semiprime ideals

Following [7, Definition 2.2] we have the following definition for noncommutative rings.

Definition 2.1. Let I be a weakly semiprime ideal of a ring R and $a \in R$. We say a is an unbreakable-zero element of I if $aRa = 0$ and $a \notin I$.

Theorem 2.2. *Let I be a weakly semiprime ideal of a ring R and suppose that a is an unbreakable-zero element of I. Then $(a + i)R(a + i) = (a - i)R(a - i) = \{0\}$ for any $i \in I$.*

Proof. Let $i \in I$. Then $(a+i)R(a+i) = aRa + aRi + iRa + iRi = aRi + iRa + iRi \subseteq I$. If $(a+i)R(a+i) \neq \{0\}$, then since I is weakly semiprime, we have $(a+i) \in I$ and hence $a \in I$ which is not possible. Hence $(a+i)R(a+i) = \{0\}$. Similarly, since $(a-i)R(a-i) = aRa - aRi - iRa + iRi = -aRi - iRa + iRi \in I$, we have $a - i \in I$ if $(a-i)R(a-i) \neq \{0\}$. Hence $a \in I$ which is not possible. Thus $(a-i)R(a-i) = 0$. □

Recall that the prime radical of a ring R i.e. $\mathcal{P}(R)$ is the intersection of all the prime ideals of R. Also recall that $\mathcal{P}(R)$ is a semiprime ideal of R.

Theorem 2.3. *Let I be a weakly semiprime ideal of a ring R that is not semiprime. Then $I \subseteq \mathcal{P}(R)$.*

Proof. Since I is weakly semiprime that is not semiprime, we conclude that I has an unbreakable-zero element, say a. Let $i \in I$. Then $(a+i)R(a+i) = \{0\}$ for any $i \in I$ by Proposition 2.2. Then $(a+i)R(a+i) \subseteq \mathcal{P}(R)$ and since $\mathcal{P}(R)$ is a semiprime ideal of R, we have $(a+i) \in \mathcal{P}(R)$. Furthermore, since a is an unbreakable-zero element of I, we have $aRa = \{0\} \subseteq \mathcal{P}(R)$. Hence $a \in \mathcal{P}(R)$ and consequently $i \in \mathcal{P}(R)$ and we have $I \subseteq \mathcal{P}(R)$. □

Proposition 2.4. *Let R be a semiprime ring. The following are equivalent:*

(1) *Every proper ideal of R is weakly semiprime*
(2) *Every proper ideal of R is semiprime.*

Proof. (1) ⇒ (2) Suppose I is a proper weakly semiprime ideal of R and let $a \in R$ such that $aRa \subseteq I$. If $aRa \neq \{0\}$, then $a \in I$ since I is weakly semiprime. If $aRa = \{0\}$, then $a = 0 \in I$ since R is a semiprime ring.

(2) ⇒ (1) This is clear. □

Proposition 2.5. *Let $I \subseteq P$ be proper ideals of a ring R. Then the following hold:*

(1) *If P is a weakly semiprime ideal of R, then P/I is a weakly semiprime ideal of R/I.*
(2) *If I and P/I are weakly semiprime ideals of R and R/I respectively, then P is weakly semiprime.*

Proof. **1.** Let $\{0\} \neq (a+I)R/I(a+I) \subseteq P/I$, where $(a+I) \in R/I$. Hence, $\{0\} \neq aRa \subseteq P$. Since P is a weakly semiprime ideal of R, $a \in P$ and hence $(a+I) \in P/I$.

2. Let $a \in R$ such that $\{0\} \neq aRa \subseteq P$. Now $(a+I)R/I(a+I) \subseteq P/I$. If $aRa \subseteq I$, then since I is a weakly semiprime ideal, we have $a \in I \subseteq P$. Suppose $aRa \nsubseteq I$, then $\{0\} \neq (a+I)R/I(a+I) \subseteq P/I$. Since P/I is a weakly semiprime ideal of R/I, we have $(a+I) \in P/I$ and hence $a \in P$ and we are done. □

Remark 2.6. P/I can be weakly semiprime without P being weakly semiprime. For example if $R = K[x,y]$ with K a field. Let $P = \langle x, y^2 \rangle$ and $I = \langle x, y \rangle^2$. Now $I \subsetneq P$ with P/I weakly semiprime but P not weakly semiprime since $(x+y)^2 \in P$ with $(x+y) \notin P$.

Theorem 2.7. *Let I be a weakly semiprime ideal of a ring R that is not semiprime and suppose that $iRi \neq 0$ for some $i \in I$. Then*

(1) $2iRi = 0$
(2) *If a is an unbreakable-zero element of I, then $(iRa + aRi) \neq \{0\}$ and $2(iRa + aRi) = \{0\}$.*

Proof. **1.** Let b be an unbreakable-zero element of I. Hence $bRb = 0$ and $b \notin I$.

Now from Proposition 2.2 $(b+i)R(b+i) + (b-i)R(b-i) = 0$. Hence

$bRb + bRi + iRb + iRi + bRb - bRi - iRb + iRi = 0$. Since b is an unbreakable-zero element of I, we have $2iRi = 0$

2. Let a be an unbreakable-zero element of I. Now
$(a + i)R(a + i) = aRa + aRi + iRa + iRi = 0$ and hence
$2(a + i)R(a + i) = 2(aRa + aRi + iRa + iRi) = 0$ and from (1) we have
$2aRi + 2iRa = 0$. Since $(a + i)R(a + i) = aRa + aRi + iRa + iRi = 0$ and $iRi \neq 0$, we have $iRa + aRi \neq 0$. □

Corollary 2.8. *(See[7, Theorem 2.12]) Let R be a commutative ring with identity and I a weakly semiprime ideal that is not semiprime. If $i^2 \neq 0$ for some $i \in I$, then:*

(1) $2i^2 = 0$ *and*
(2) *If a is an unbreakable-zero element of I, then $2ai \neq 0$ and $4ai = 0$.*

3. Weakly semiprime ideals in decomposable rings.

Definition 3.1. A ring R is called decomposable if $R = R_1 \times R_2$ for some non-trivial rings R_1 and R_2.

It is well known that any ideal of $R_1 \times R_2$ has the form $I \times J$, where I and J are ideals of R_1 and R_2, respectively. It follows from [7, Example 2.14] that if I is a weakly semiprime ideal of R_1 and J is a weakly semiprime ideal of R_2, then $I \times J$ needs not be a weakly semiprime ideal of $R_1 \times R_2$.

Proposition 3.2. *Let $R = R_1 \times R_2$, where $R_i (i = 1, 2)$ is a noncommutative ring with nonzero identity. Then the following hold:*

(1) *If $P_1 \times R_2$ is a weakly semiprime ideal of R, then P_1 is a weakly semiprime ideal of R_1.*
(2) *If $R_1 \times P_2$ is a weakly semiprime ideal of R, then P_2 is a weakly semiprime ideal of R_2.*

Proof. **1.** Let $P_1 \times R_2$ be a weakly semiprime ideal of R. Suppose $\{0\} \neq aRa \subseteq P_1$ where $a \in R_1$. Now $(a, 1) \in R_1 \times R_2$ such that $\{(0, 0)\} \neq (a, 1)R(a, 1) \subseteq P_1 \times R_2$. Since $P_1 \times R_2$ is a weakly semiprime ideal of R, we have $(a, 1) \in P_1 \times R_2$ and we have $a \in P_1$ and we are done.

2. The proof is similar to that in case 1. □

Lemma 3.3. *Let $R = R_1 \times R_2$, where $R_i (i = 1, 2)$ is a noncommutative ring with nonzero identity and let $J = I_1 \times I_2$ be a proper ideal of R. If J is a weakly semiprime ideal of R that is not semiprime, then I_1 and I_2 are weakly semiprime ideals of R_1, R_2 respectively and I_1 and or I_2 is not semiprime.*

Proof. Suppose that $0 \neq aR_1a \subseteq I_1$ for some $a \in R_1$ and $0 \neq bR_2b \subseteq I_2$ for some $b \in R_2$. Then $\{(0, 0)\} \neq (a, b)R(a, b) = (aR_1a, bR_2b) \subseteq J$. Since J is weakly semiprime, we have $(a, b) \in J$. Thus $a \in I_1$ and $b \in I_2$. Thus I_1 is a weakly semiprime ideal of R_1 and I_2 is a weakly semiprime ideal of R_2. Since J is not semiprime, J has an unbreakable-zero element, say $(c, d) \in R$. Since (c, d) is an unbreakable zero element of J, we have $(c, d)R(c, d) = \{(0, 0)\}$ and $(c, d) \notin J$. Hence $cR_1c = \{0\}$ and $dR_2d = \{0\}$ with $c \notin I_1$ and or $d \notin I_2$. Hence c is an unbreakable-zero element of I_1 and or d is an unbreakable-zero element of I_2. Thus I_1 is not a semiprime ideal of R_1 and or I_2 is a not a semiprime ideal of R_2. □

Theorem 3.4. *Let $R = R_1 \times R_2$, where $R_i (i = 1, 2)$ is a noncommutative ring with nonzero identity and let J be a proper ideal of R. The following statements are equivalent:*

(1) *J is a weakly semiprime ideal of R that is not semiprime such that $xRx = \{(0,0)\}$ for every $x \in J$.*
(2) *$J = I_1 \times I_2$ where I_1, I_2 are weakly semiprime ideals of R_1, R_2 respectively and I_1 is not semiprime and or I_2 is not semiprime and $aR_1a = bR_2b = \{0\}$ for every $a \in R_1$ and for every $b \in R_2$.*

Proof. (1) $\Rightarrow$ (2) Since $J = I_1 \times I_2$ is a weakly semiprime ideal of R, it follows from Lemma 3.3 that I_1 and I_2 are weakly semiprime ideals of R_1, R_2 respectively. Since $xRx = \{(0,0)\}$ for every $x \in J$, we have $aR_1a = \{0\}$ for every $a \in I_1$ and $bR_2b = \{0\}$ for every $b \in I_2$. Since J is not semiprime, J has an unbreakable-zero element, say $(c,d) \in R$. Since (c,d) is an unbreakable element of J, we have $(c,d)R(c,d) = \{(0,0)\}$ and $(c,d) \notin J$. Hence $cR_1c = \{0\}$ and $dR_2d = \{0\}$ with $c \notin I_1$ and or $d \notin I_2$. Hence c is an unbreakable-zero element of I_1 and or d is an unbreakable-zero element of I_2. Thus I_1 is not semiprime and or I_2 is not semiprime.

(2) $\Rightarrow$ (1) If $(a,b) \in R$ and $(a,b)R(a,b) \subseteq I_1 \times I_2$, then we have $(a,b)R(a,b) = \{(0,0)\}$ and hence $J = I_1 \times I_2$ is weakly semiprime. The rest is clear. □

Theorem 3.5. *(See [7, Theorem 2.16] for the commutative case) Let $R = R_1 \times R_2$, where $R_i (i = 1,2)$ is a noncommutative ring with nonzero identity and let J be a proper ideal of R. The following statements are equivalent:*

(1) *J is a weakly semiprime ideal of R that is not semiprime such that $xRx \neq (0,0)$ for some $x \in J$.*
(2) *$J = I_1 \times I_2$ where I_1 is a weakly semiprime ideal of R_1 that is not semiprime such that $aR_1a \neq \{0\}$ for some $a \in I_1$ and I_2 is a semiprime ideal of R_2 such that $bR_2b = \{0\}$ for every $b \in I_2$ and or I_2 is a weakly semiprime ideal of R_2 that is not semiprime such that $bR_2b \neq \{0\}$ for some $b \in I_2$ and I_1 is a semiprime ideal of R_1 such that $aR_1a = \{0\}$ for every $a \in I_1$.*

Proof. (1) $\Rightarrow$ (2) Since $J = I_1 \times I_2$ is a weakly semiprime ideal of R that is not semiprime, it follows Lemma 3.3 that I_1 is a weakly semiprime ideal of R_1 that is not semiprime and or I_2 is a weakly semiprime ideal of R_2 that is not semiprime. We consider three cases.

Case one. Suppose that I_1 is a weakly semiprime ideal of R_1 that is not semiprime, hence I_1 has unbreakable-zero element $c \in R_1$. We show that I_2 is a semiprime ideal of R_2 such that $bR_2b = 0$ for every $b \in I_2$. Let $b \in I_2$. Since $(c,b)R(c,b) = (cR_1c, bR_2b) = (0, bR_2b) \subseteq J$ and $c \notin I_1$, we conclude that $bR_2b = \{0\}$. If $bR_2b \neq \{0\}$, then $\{(0,0)\} \neq (c,b)R(c,b) \subseteq J$ and since J is a weakly semiprime ideal of R, $(c,b) \in I_1 \times I_2$ and hence $c \in I_1$ a contradiction. Since $xRx \neq \{(0,0)\}$ for some $x \in J$ and $bR_2b = \{0\}$ for every $b \in I_2$, we conclude that there is an $h \in I_1$ such that $hR_1h \neq \{0\}$. Let $f \in R_2$ and suppose that $fR_2f \subseteq I_2$. Since $\{(0,0)\} \neq (h,f)R(h,f) = (hR_1h, fR_2f) \subseteq J$ and J is a weakly semiprime ideal of R, we conclude that $(h,f) \in I_1 \times I_2$ and hence $f \in I_2$. Thus I_2 is a semiprime ideal of R_2 such that $bR_2b = \{0\}$ for every $b \in I_2$.

Case two : Suppose that I_2 is a weakly semiprime ideal of R_2 that is not semiprime. Then by similar argument as in case one, we conclude that I_1 is a semiprime ideal of R_1 such that $aR_1a = 0$ for every $a \in I_1$.

Case three: Suppose that I_2 is a weakly semiprime ideal of R_2 that is not semiprime and suppose that I_1 is a weakly semiprime ideal of R_1 that is not semiprime. Then by similar argument as in case one and two, we conclude that I_1 is a semiprime ideal of R_1 such that $aR_1a = 0$ for every $a \in I_1$ and I_2 is a semiprime ideal of R_2 such that $bR_1b = \{0\}$ for every $b \in I_2$.

(2) $\Rightarrow$ (1). It can be easily verified and it is left to the reader. □

4. Idealization

We now show how to construct weakly semiprime ideals using the Method of Idealization. In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r, m)(s, n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq \{0\} \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus \{0\}$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ \{0\} & R \end{bmatrix}$ via the mapping $(r, m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$. It follows from [12] that the prime ideals of $R \boxplus M$ are exactly the ideals of the form $I \boxplus M$ where I is a prime ideal of R.

Theorem 4.1. *Let R be a ring, M an $R-R$-bimodule and I a proper ideal of R. If $I \boxplus M$ is a weakly semiprime ideal of $R \boxplus M$, then I is a weakly semiprime ideal of R and for any unbreakable zero a of I we have $aMa = \{0\}$.*

Proof. Suppose $I \boxplus M$ is a weakly semiprime ideal of $R \boxplus M$. Let $\{0\} \neq aRa \subseteq I$ where $a \in R$. Now $(0, 0) \neq (a, 0)R \boxplus M(a, 0) \subseteq I \boxplus M$ and $I \boxplus M$ a weakly semiprime ideal gives $(a, 0) \in I \boxplus M$. Hence $a \in I$. So I is weakly semiprime. Now suppose a is an unbreakable zero of I. We claim that $aMa = \{0\}$. Assume say $aMa \neq \{0\}$, so there exists $m \in M$ such that $ama \neq 0$. Now we have $(0, 0) \neq (a, 0)(1, m)(a, 0) \in (a, 0)R \boxplus M(a, 0) \subseteq aRa \boxplus M = \{0\} \boxplus M \subseteq I \boxplus M$. But $(a, 0) \notin I \boxplus M$ contradicting the fact that $I \boxplus M$ is a weakly semiprime ideal. □

Theorem 4.2. *Let R be a ring, M an $R-R$-bimodule and I a proper ideal of R. If I is a weakly semiprime ideal of R and for any unbreakable zero a of I we have $aM = Ma = aMa = \{0\}$, then $I \boxplus M$ is a weakly semiprime ideal of $R \boxplus M$.*

Proof. Assume $\{(0, 0)\} \neq (a, m)R \boxplus M(a, m) \subseteq I \boxplus M$ for $a \in R$ and $m \in M$. We have $aRa \subseteq I$. Two cases are possible:

Case 1: $\{0\} \neq aRa \subseteq I$. Now I a weakly semiprime ideal of R gives $a \in I$. Hence $(a, m) \in I \boxplus M$ as desired.

Case 2: $\{0\} = aRa \subseteq I$. We may assume $a \notin I$. Hence a is a unbreakable zero of I and from assumption $aM = Ma = aMa = \{0\}$. Now $(a, m)R \boxplus M(a, m) = (aRa, aM + aMa + Ma) = (\{0\}, \{0\})$ a contradiction. Hence $a \in I$ and we have $(a, m) \in I \boxplus M$ as desired. □

Theorem 4.3. *Let R be a ring such that $aRa = 0$ implies $a = 0$ for every $a \in R$, M be a $R-R$-bimodule, and $R \boxplus M$ be the idealization of M by R. Then, an ideal J of $R \boxplus M$ is weakly semiprime if and only if $J = \{0\} \boxplus F$, where F is an $R-R$-bi submodule of M or $J = I \boxplus M$ where, I is a weakly semiprime ideal of R.*

Proof. Let R be a ring such that $aRa = 0$ implies $a = 0$ for every $a \in R$, M be a $R-R$-bimodule, and $R \boxplus M$ be the idealization of M by R, and let J be an ideal of $R \boxplus M$. We show that if J is a weakly semiprime ideal of $R \boxplus M$, then $J = \{0\} \boxplus F$ or $J = I \boxplus M$. If $J \subseteq \{0\} \boxplus M$, the result is clear. Now, assume that $J \not\subseteq \{0\} \boxplus M$. Then there exists $(a, e') \in J$ such that $a \neq 0$. Let $I = \{a \in A : (a, e) \in J$ for some $e \in M\}$. It is clear that $J \subseteq I \boxplus M$. Now we claim that $J = I \boxplus M$. Let $(a, e) \in I \boxplus M$. Two cases are then possible:

Case 1: $a = 0$. We claim that $(0, e) \in J$. If $(0, e) \notin J$, then $(0, e)R \boxplus M(0, e) = \{(0, 0)\}$ and $(a, e') \in J$. From Proposition 2.2 we have $[(a, e') + (0, e)]R \boxplus M[(a, e') + (0, e)] = [(a, e' + e)]R \boxplus M[(a, e' + e) = (aRa, T) = \{(0, 0)\}$ where $T \subseteq M$. Therefore, $aRa = \{0\}$ and so $a = 0$ by hypothesis, a desired contradiction.

Case 2: $a \neq 0$. Hence, there exists $f \in M$ such that $(a, f) \in J$. We claim that $(0, f - e) \in J$. If $(0, f - e) \notin J$, then $(0, f - e)R \boxplus M(0, f - e) = \{(0, 0)\}$. Hence, by the same argument as above, we have $[(a, f) - (0, f - e)]R \boxplus M[(a, f) - (0, f - e)] = (a, e)R \boxplus M(a, e) = (aRa, L) = \{(0, 0)\}$ for some $L \subseteq M$. Hence $aRa = 0$ and so $a = 0$, by hypothesis, a desired contradiction. Hence, $(0, f - e) \in J$ and $(a, f) \in J$, then $[(a, f) - (0, f - e)] = (a, e) \in J$. Therefore, $(a, e) \in J$ in all cases. Hence, $J = I \boxplus M$. It remains to show that I is a weakly semiprime ideal of R. Let $\{0\} \neq aRa \subseteq I$ where $a \in R$. Then $\{(0, 0)\} \neq (a, e)R \boxplus M(a, e) \subseteq I \boxplus M$ implies $(a, e) \in I \boxplus M$ since $I \boxplus M$ is a weakly semiprime ideal of $R \boxplus M$. Therefore, $a \in I$ and so I is a weakly semiprime ideal of R which completes the proof. Conversely, assume that $J = \{0\} \boxplus F$. Suppose that $y \in R \boxplus M$ and $y(R \boxplus M)y \subseteq J$, say $y = (r, m)$ with $r \in R$ and $m \in M$. Now $(r, m)R \boxplus M(r, m) = (rRr, mRr + rMr + rRm) \subseteq J = \{0\} \boxplus F$ implies $rRr = \{0\}$ and so $r = 0$, by hypothesis. Hence $y(R \boxplus M)y = \{(0, 0)\}$. Hence J is a weakly semiprime ideal by definition. Now, assume that $J = I \boxplus M$, where I is a weakly semiprime proper ideal of R. Let $\{(0, 0)\} \neq (a, e)R \boxplus M(a, e) \subseteq J$, then $\{(0, 0)\} \neq (aRa, eRa + aMa + aRe) \subseteq J$. We claim that $aRa \neq \{0\}$. Suppose not, then, $a = 0$ by hypothesis and hence $(a, e)R \boxplus M(a, e) = \{(0, 0)\}$ a desired contradiction. Hence, $\{0\} \neq aRa \subseteq I$ and so $a \in I$ since I is a weakly semiprime ideal of R. Therefore, $(a, e) \in J$ and so J is a weakly semiprime ideal of $R \boxplus M$ which completes the proof of the Theorem. $\square$

Remark 4.4. The condition "$aRa = \{0\}$ implies $a = 0$ for every $a \in R$" in Theorem 4.3 is necessary. Indeed, let $R \boxplus M$ be the trivial ring extension of $\mathbb{Z}/8\mathbb{Z}$ by the $\mathbb{Z}/8\mathbb{Z}$-module $4\mathbb{Z}/8\mathbb{Z}$, that is $R \boxplus M = \mathbb{Z}/8\mathbb{Z} \boxplus 4\mathbb{Z}/8\mathbb{Z}$. Let $I = 4\mathbb{Z}/8\mathbb{Z} \boxplus 4\mathbb{Z}/8\mathbb{Z}$ be an ideal of $R \boxplus M$. Then I is not a weakly semiprime ideal of $R \boxplus M$ since $(0, 0) \neq (\overline{2}, \overline{0})(\overline{2}, \overline{0}) \in I$, but $(\overline{2}, \overline{0}) \notin I$.

5. Weakly completely semiprime ideals

Definition 5.1. A proper ideal I of R is called weakly completely semiprime if $a \in R$ and $0 \neq a^2 \in I$ implies $a \in I$.

Definition 5.2. Let I be a weakly completely semiprime ideal of a ring R and $a \in R$. We say a is an unbreakable-zero element of I if $a^2 = 0$ and $a \notin I$.

Proposition 5.3. *Let I be a weakly completely semiprime ideal of a ring R and suppose that a is an unbreakable-zero element of I. Then$(a + i)^2 = (a - i)^2 = 0$ for any $i \in I$..*

Proof. Let $i \in I$. Then $(a + i)^2 = a^2 + ai + ia + i^2 = ai + ia + i^2 \in I$. If $(a + i)^2 \neq 0$, then since I is weakly completely semiprime, we have $(a + i) \in I$ and hence $a \in I$ which is not possible. Hence $(a + i)^2 = 0$. Similarly, since $(a - i)^2 = a^2 - ai - ia + i^2 = -ai - ia + i^2 \in I$. If $(a - i)^2 \neq 0$, we have $a - i \in I$ hence $a \in I$ which is not possible Thus $(a - i)^2 = 0$. $\square$

Recall that the completely prime radical (generalized nil radical) of a ring R i.e. $\mathcal{N}_g(R)$ is the intersection of all the completely prime ideals of R. Also recall that $\mathcal{N}_g(R)$ is a completely semiprime ideal of R.

Theorem 5.4. *Let I be a weakly completely semiprime ideal of a ring R that is not completely semiprime. Then $I \subseteq \mathcal{N}_g(R)$.*

Proof. Since I is weakly completely semiprime that is not completely semiprime, we conclude that I has an unbreakable-zero element, say a. Let $i \in I$. Then $(a+i)^2 = 0$ for any $i \in I$ by Proposition 5.3. Then $(a+i)^2 \in \mathcal{N}_g(R)$ and since $\mathcal{N}_g(R)$ is a completely semiprime ideal of R,we have $(a+i) \in \mathcal{N}_g(R)$. Furthermore, since a is an unbreakable-zero element of I, we have $a^2 = 0 \in \mathcal{N}_g(R)$. Hence $a \in \mathcal{N}_g(R)$ and consequently $i \in \mathcal{N}_g(R)$ and we have $I \subseteq \mathcal{N}_g(R)$. □

Proposition 5.5. *If R is a commutative ring with identity element and I is a weakly semiprime ideal, then I is a weakly completely semiprime ideal.*

Proof. Suppose I is weakly semiprime and let $b \in R$ such that $0 \neq b^2 \in I$. Now, $0 \neq b^2 \in b^2R = bRb \subseteq I$ and since I is weakly semiprime, $b \in I$ and we are done. □

Example 5.6. Not every weakly semiprime ideal is a weakly completely semiprime ideal: Let $M_2(\mathbb{Z})$ be the full matrix ring with entries from the ring of integers $\mathbb{Z}$. $M_2(2\mathbb{Z})$ is a prime ideal and hence also a weakly semiprime ideal of $M_2(\mathbb{Z})$. To show that $M_2(2\mathbb{Z})$ is not a weakly completely semiprime ideal, consider $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2(\mathbb{Z})$. Now $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \in M_2(2\mathbb{Z})$ but $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin M_2(2\mathbb{Z})$.

6. Rings in which every ideal is weakly completely semiprime

We are interested in the structure of rings in which every ideal is weakly completely semiprime . Note that by definition, a weakly completely semiprime ideal is a proper ideal of a ring. It is therefore not possible that every ideal of a ring is a weakly completely semiprime ideal. However, a ring whose zero ideal is completely semiprime is called a completely semiprime ring. In this sense, every ring is a weakly completely semiprime ring since the zero ideal is always weakly completely semiprime. We may therefore say that every ideal of a ring is weakly completely semiprime when every proper ideal of the ring is a weakly completely semiprime ideal. If $R^2 = \{0\}$, then it is evident that every ideal of R is weakly completely semiprime.

Proposition 6.1. *Every ideal of a ring R is weakly completely semiprime if and only if for every $x \in R$ we have $\langle x^2 \rangle = \langle x \rangle$, or $\langle x^2 \rangle = \{0\}$.*

Proof. Suppose every ideal of R is a weakly completely semiprime ideal and let $x \in R$. If $\langle x^2 \rangle \neq R$ and if $\langle x^2 \rangle = \{0\}$, then we done. If $\langle x^2 \rangle \neq \{0\}$, then $\langle x^2 \rangle$ is a weakly completely semiprime ideal. Now, since $0 \neq x^2 \in \langle x^2 \rangle$, we have $x \in \langle x^2 \rangle$. Hence $\langle x^2 \rangle = \langle x \rangle$. If $\langle x^2 \rangle = R$, then $\langle x \rangle = R$.

Conversely, let K be any proper ideal of R and suppose $0 \neq x^2 \in K$ for $x \in R$. Now we have $\langle x \rangle = \langle x^2 \rangle \subseteq K$. Hence $x \in K$ and we are done. □

Example 6.2. Let F be a field and $R = F \oplus F \oplus F$. Then for every element $a \in R$, we have $\langle a \rangle = \langle a^2 \rangle$. $I = F \oplus \{0\} \oplus \{0\}$ is weakly completely semiprime but evidently not weakly completely prime.

7. Ideal weakly semiprime ideals

We now have the following:

Definition 7.1. Let I be a proper ideal of R. I is an ideal weakly semiprime ideal of R if for any ideal A of R such that $\{0\} \neq A^2 \subseteq I$, we have $A \subseteq I$..

Theorem 7.2. *Let R be a ring, M an $R-R$-bimodule and I a proper ideal of R. If $I \boxplus M$ is an ideal weakly semiprime ideal of $R \boxplus M$, then I is an ideal weakly semiprime ideal of R.*

Proof. Suppose $I \boxplus M$ is an ideal weakly semiprime ideal of $R \boxplus M$. Let $\{0\} \neq A^2 \subseteq I$ where $A \lhd R$. Now $\{(0,0)\} \neq (A \boxplus M)^2 = A^2 \boxplus M \subseteq I \boxplus M$ and $I \boxplus M$ an ideal weakly semiprime ideal gives $A \boxplus M \subseteq I \boxplus M$. Hence $A \subseteq I$. So I is ideal weakly semiprime. . $\square$

7.1. **The ideal weakly semiprime radical.** From [8] we have the notion of a weakly prime radical $\mathcal{P}_w(I)$ of an ideal I of a ring R as the intersection of all the weakly prime ideals of R containing I.

Lemma 7.3. *Let R be any ring. Then $\mathcal{P}_w(R)$ is an ideal weakly semiprime ideal.*

Proof. Suppose I is an ideal of R such that $\{0\} \neq I^2 \subseteq \mathcal{P}_w(R) = \{P_i : P_i$ is a weakly prime ideal of $R\}$. Hence $\{0\} \neq I^2 \subseteq P_i$ for every i. Since every P_i is weakly prime, we have $I \subseteq P_i$ for every i. Hence $I \subseteq \mathcal{P}_w(R)$ and hence $\mathcal{P}_w(R)$ is ideal weakly semiprime. $\square$

Motivated by the work in [8] we introduce the notion of an ideal weakly n-system. See also [1, Definition 2]

Definition 7.4. Let R be a ring. A nonempty set $S \subseteq R\backslash\{0\}$ is called a ideal weakly n-system if, for an ideal A of R, if $A \cap S \neq \emptyset$ and $A^2 \neq \{0\}$, then $A^2 \cap S \neq \emptyset$.

Lemma 7.5. [1, Proposition 3]*For a proper ideal P of R let $S = R\backslash P$. Then P is an ideal weakly semiprime ideal of R if and only if S is an ideal weakly n-system.*

Definition 7.6. Let R be a ring . For an ideal A of R, if there is a ideal weakly semiprime ideal containing A, then we define $\mathcal{P}_{iws}(A) := \{a \in R :$ every ideal weakly n-system containing a meets $A\}$. If there is no ideal weakly semiprime ideal containing A, then we put $\mathcal{P}_{iws}(A) = R$.

For an ideal A of R, observe that A and $\mathcal{P}_{iws}(A)$ are contained in precisely the same ideal weakly semiprime ideals of R.

Theorem 7.7. *Let A be an ideal of the ring R. Then either $\mathcal{P}_{iws}(A) = R$ or $\mathcal{P}_{iws}(A)$ equals the intersection of all the ideal weakly semiprime ideals of R containing A.*

Proof. Suppose that $\mathcal{P}_{iws}(A) \neq R$. This means that $\{P|P$ is an ideal weakly semiprime ideal of R and $A \subseteq P\} \neq \emptyset$. We first prove that $\mathcal{P}_{iws}(A) \subseteq \{P|P$ is an ideal weakly semiprime ideal of R and $A \subseteq P\}$. Let $m \in \mathcal{P}_{iws}(A)$ and P be any ideal weakly semiprime ideal of R containing A. Consider the ideal weakly n-system $R\backslash P$. This ideal weakly n-system cannot contain m, for otherwise it meets A and hence also P. Therefore, we have $m \in P$. Conversely, assume $m \notin \mathcal{P}_{iws}(A)$. Then, by Definition 7.6, there exists a ideal weakly n-system S containing m which is disjoint from A. By Zorn's Lemma, there exists an ideal $P \supseteq A$ which is maximal with respect to being disjoint from S. By[1, Theorem 6], P is an ideal weakly semiprime ideal of R and we have $m \notin P$, as desired.

Remark 7.8. It is well known that for the prime radical $\beta(I)$ of an ideal I of a ring R we have that I is a semiprime ideal if and only if $\beta(I) = I$. We have that if $I = \mathcal{P}_w(I)$, then I is an ideal weakly semiprime ideal. This follows from the fact that $\mathcal{P}_w(I)$ equal to the intersection of all the weakly prime ideals and that $\mathcal{P}_w(I)$ is an ideal weakly semiprime ideal. To prove that if I is an ideal semiprime ideal, then $\mathcal{P}_w(I) = I$ as is the relation between the prime radical of an ideal and semiprime ideals we need the following:

□

Question: If N is an ideal weakly n-system in the ring R and for $a \in R$, can we find a weakly m-system M in R such that $a \in M$ and $M \subseteq N$?

8. Weakly semiprime submodules

From [4] we have

Definition 8.1. [4, Definition 3.3]. Let M be a left R-module. A proper submodule N of M is called a weakly prime submodule of M if whenever $r \in R$ and $m \in M$ with $0 \neq rRm \subseteq N$, then either $m \in N$ or $r \in (N :_R M)$.

Definition 8.2. A proper submodule P of the R module M is a semiprime submodule of M if for any ideal A of R and any submodule N of M, if $A^2N \subseteq P$, then $AN \subseteq P$.

Definition 8.3. Let M be a left R-module. A proper submodule N of M is called a weakly semiprime submodule of M if whenever $r \in R$ and $m \in M$ with $0 \neq rRrRm \subseteq N$, then $rRm \subseteq N$.

Lemma 8.4. *Let M be a left R-module. Every weakly prime submodule N of M is weakly semiprime.*

Proof. Let N be a weakly prime submodule of M and let $a \in R$ and $m \in M$ such that $0 \neq aRaRm \subseteq N$. Now $0 \neq (RaR)(RaR)m \subseteq N$ and from [9, Theorem 4] we have $RaRm \subseteq N$ or $aM \subseteq N$. Hence $aRm \subseteq N$ and we are done. □

Proposition 8.5. *Let M be an R module. For a submodule P of M the following statements are equivalent.*

(1) *P is semiprime*
(2) *for every $m \in M$ and every $a \in R$, if $\langle a \rangle^2 m \subseteq P$, then $\langle a \rangle m \subseteq P$.*
(3) *for every $m \in M$ and every $a \in R$, if $aRaRm \subseteq P$, then $aRm \subseteq P$.*
(4) *for every $a \in R$ and every submodule N of M, if $aRaN \subseteq P$, then $aN \subseteq P$.*

Remark 8.6. Let M be an R module, then every semiprime submodule of M is a weakly semiprime submodule.

Proposition 8.7. *Let R be a semiprime ring and M a torsion free R-module, then every weakly semiprime submodule of M is a semiprime submodule.*

Proof. Let P be a weakly semiprime submodule of M and $a \in R$ and $m \in M$ such that $aRaRm \subseteq P$. If $aRaRm \neq \{0\}$, then $aRm \subseteq P$ since P is a weakly semiprime submodule. So suppose $aRaRm = \{0\}$. If $aRa \neq \{0\}$, then $Rm = \{0\}$ since M is torsion free. Hence $aRm \subseteq P$ and we are done. If $aRa = \{0\}$, then $a = 0$ since R is a semiprime ring. Hence $aRm \subseteq P$ and we are done. □

Proposition 8.8. *Let M be a prime R module, then every semiprime submodule of M is a weakly semiprime submodule.*

Proof. Let P be a weakly semiprime submodule of M and $a \in R$ and $m \in M$ such that $aRaRm \subseteq P$. If $aRaRm \neq \{0\}$, then $aRm \subseteq P$ since P is weakly semiprime. If $aRaRm = \{0\}$, then since M is a prime module, we have $aRm = \{0\}$ or $Rm = \{0\}$. Hence $aRm \subseteq P$ and we are done. $\square$

Theorem 8.9. *Let P be a proper submodule of the R-module M. The following are equivalent:*

(1) *P is a weakly semiprime submodule,*
(2) *for every $a \in R$, $(P : \langle aRa \rangle) = (P : \langle a \rangle) \cup (\{0\} : \langle aRa \rangle)$,*
(3) *for every $a \in R$, $(P : \langle aRa \rangle) = (P : \langle a \rangle)$ or $(P : \langle aRa \rangle) = (\{0\} : \langle aRa \rangle)$.*

Proof. (1) $\Rightarrow$ (2) Let P be a weakly semiprime submodule of M and let a be any element of R. Let $x \in (P : \langle aRa \rangle)$. Hence $RaRaRx \subseteq P$. Thus $aRaRx \subseteq P$. If $aRaRx \neq \{0\}$, then $aRx \subseteq P$. hence $RaRx \subseteq P$ and we have $x \in (P : \langle a \rangle)$. If $aRaRx = \{0\}$, then $RaRRaRx = \{0\}$ and hence $x \in (\{0\} : \langle aRa \rangle)$. Thus $(P : \langle aRa \rangle) \subseteq (P : \langle a \rangle) \cup (\{0\} : \langle aRa \rangle)$. Clearly $(P : \langle a \rangle) \cup (\{0\} : \langle aRa \rangle) \subseteq (P : \langle aRa \rangle)$. Hence $(P : \langle aRa \rangle) = (P : \langle a \rangle) \cup (\{0\} : \langle aRa \rangle)$.

(2) $\Rightarrow$ (3) this is straightforward.

(3) $\Rightarrow$ (1) Let $aRaRm \neq \{0\}$. Hence $RaRaRm \neq \{0\}$ and therefore $m \in (P : \langle aRa \rangle) = (P : \langle a \rangle)$. That is $aRm \subseteq RaRm \subseteq P$ and hence P is weakly semiprime. $\square$

Proposition 8.10. *Let R be a ring with identity and M a faithful R-module. If N is a weakly semiprime submodule of M, then $(N : M)$ is a weakly semiprime ideal of R.*

Proof. Suppose N is a weakly semiprime submodule of M and $a \in R$ such that $\{0\} \neq aRa \subseteq (N : M)$. Hence $\{0\} \neq 1aR1a1M \subseteq RaRRaRM \subseteq N$. Now, Theorem 8.9 implies that $RaRM \subseteq N$. Hence $aM \subseteq N$ and $a \in (N : M)$. Thus $(N : M)$ is a weakly semiprime ideal of R. $\square$

8.1. **Weakly semiprime submodules in terms of weakly n-systems.** We begin this section with the definition of weakly n-systems.

Definition 8.11. Let R be a ring and M be an R-module. A nonempty set $S \subseteq M \backslash \{0\}$ is called a weakly n-system if, for each $a \in R$, and for each $m \in M$, if $aRm \cap S \neq \emptyset$ and $aRaRm \neq \{0\}$, then $aRaRm \cap S \neq \emptyset$.

Proposition 8.12. *Let M be an R-module. A submodule P of M is weakly semiprime if and only if $M \backslash P$ is a weakly n-system.*

Proof. Suppose $S = M \backslash P$. Let $a \in R$ and $m \in M$ such that $aRm \cap S \neq \emptyset$ and $aRaRm \neq \{0\}$. If $aRaRm \cap S = \emptyset$, then $\{0\} \neq aRaRm \subseteq P$. Hence $aRm \subseteq P$ and since P is weakly semiprime, and $\{0\} \neq aRaRm \subseteq P$. It follows that $aRm \cap S = \emptyset$, a contradiction. Therefore, S is a weakly n-system in M. Conversely, let $S = M \backslash P$ be a weakly n-system in M. Suppose $a \in R$ and $m \in M$ such that $\{0\} \neq aRaRm \subseteq P$. If $aRm \nsubseteq P$, then $aRm \cap S \neq \emptyset$. Since $aRaRm \neq \{0\}$ and S a weakly n-system, we have $aRaRm \cap S \neq \emptyset$. Thus, $aRaRm \nsubseteq P$, a contradiction. Therefore, P is a weakly semiprime submodule of M. $\square$

From [9] we have that a submodule P of an R-module M is weakly prime if and only if $S = M \backslash P$ is a weakly m-system of M i.e. for each $a \in R$, and for each $m \in M$, if $Rm \cap S \neq \emptyset$, $aM \cap S \neq \emptyset$ and $aRm \neq \{0\}$, then $aRm \cap S \neq \emptyset$.

Proposition 8.13. *Let M be an R-module. Every weakly m-system is a weekly n-system.*

Proof. Let $S \subseteq M\backslash\{0\}$be a weakly m-system of M and let $a \in R$ and $m \in M$ such that $aRm \cap S \neq \emptyset$ and $aRaRm \neq \{0\}$. Since $aRm \subseteq Rm$ and $aRm \subseteq aM$, we have $Rm \cap S \neq \emptyset$ and $aM \cap S \neq \emptyset$. Also since $aRaRm \subseteq aRm$, we have that $aRm \neq \{0\}$. Now, since S is a weakly m-system we have $aRm \cap S \neq \emptyset$. Hence S is a weakly n-system. □

We have that every weakly m-system is a weakly n-system but not conversely. A fundamental question to ask is:

Question: If T is a weakly n-system in an $R-$module M such that $x \in T$, does there exist a weakly m-system $S \subseteq T$ such that $x \in S$?

From [9] we have a generalization of the notion $\sqrt{N}$, weakly prime radical of the submodule N of M. We adopt the following:

Definition 8.14. [9, Definition 20]Let R be a ring and M be an R-module. For a submodule N of M, if there is a weakly prime submodule containing N, then we define $\sqrt{N} := \{m \in M :$ every weakly m-system containing m meets $N\}$. If there is no weakly prime submodule containing N, then we put $\sqrt{N} = M$.

Theorem 8.15. [9, Theorem 21]*Let M be an R-module and $N \leq M$. Then either $\sqrt{N} = M$ or $\sqrt{N}$ equals the intersection of all the weakly prime submodules of M containing N.*

If the answer to the above question is yes, then it can be shown that every weakly semiprime submodule is an intersection of weakly prime submodules.

Conflict of interest. The authors declare no conflicts of interest

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