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SIMULTANEOUS RECONSTRUCTION OF BOUNDARY COEFFICIENTS IN A PARABOLIC SYSTEM VIA REGULARIZED LEVENBERG-MARQUARDT OPTIMIZATION

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Abstract. This work addresses the problem of simultaneously recovering two spatially varying boundary coefficients in a parabolic system. To handle the ill-posedness, we use Tikhonov regularization. The forward model is approximated using finite elements for spatial discretization together with an implicit Euler scheme for the temporal direction. The resulting nonlinear least-squares problem is solved by the Levenberg–Marquardt algorithm, improved with a surrogate functional approach that gives explicit update formulas and speeds up convergence. Numerical tests with noisy data show that our method is accurate, stable, and efficient.

Keywords. Inverse problems, Boundary coefficient reconstruction, Tikhonov regularization, Levenberg-Marquardt optimization, Surrogate functionals.
2020 Mathematics Subject Classification. Primary 35R30, 65M32; Secondary 65J20, 65K10

1. Introduction

This paper analyzes a parabolic system governed by the initial-boundary value problem below:

$$\begin{cases} \partial_t u - \Delta u = f(x,t) & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial n} + \gamma(x) u = g(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u}{\partial n} = q(x) R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

- The spatial domain for our problem is a bounded region $\Omega \subset \mathbb{R}^d$ (with $d = 2, 3$). Its boundary is decomposed into three disjoint parts: $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.
- The internal source term, denoted by $f(x,t)$, is a known function belonging to the space $L^2(0,T; L^2(\Omega))$.
- $\gamma(x) \in L^\infty(\Gamma_1)$ is a known Robin coefficient satisfying $0 < \gamma_1 \le \gamma(x) \le \gamma_2$ almost everywhere on Γ_1.

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- $h(x,t) \in L^2(0,T;L^\infty(\Gamma_1))$ and $R(x,t) \in L^2(0,T;L^\infty(\Gamma_2))$ are known time-dependent boundary weight functions.
- $g(x)$ is the unknown boundary coefficient to be determined, assumed to lie in the admissible set:

$$K_1 = \{g \in L^2(\Gamma_1) \mid g_1 \le g(x) \le g_2\}.$$

- $q(x)$ is the unknown Neumann heat flux function to be identified on Γ_2, assumed to satisfy:

$$K_2 = \{q \in L^2(\Gamma_2) \mid q_1 \le q(x) \le q_2\}.$$

This study focuses on an inverse problem governed by a time-dependent parabolic partial differential equation (PDE). The primary goal is to simultaneously identify two unknown spatially varying functions: the boundary coefficient $g(x)$, defined on a part of the boundary denoted by Γ_1, and the Neumann boundary flux $q(x)$, defined on another portion of the boundary denoted by Γ_2.

Measurements are acquired from the boundary, specifically consisting of observations of the state variable $u(x,t)$ on the segment Γ_3 throughout the time interval $(0,T]$. These measurements are denoted by $z^\delta(x,t)$ and may be contaminated by noise of level $\delta > 0$. That is,

$$z^\delta(x,t) = u(x,t) + \text{noise}, \quad \text{for } (x,t) \in \Gamma_3 \times (0,T].$$

This leads to the following formulation of the inverse problem:

For given measurements z^δ on $\Gamma_3 \times (0,T]$, find the functions $g(x)$ on Γ_1 and $q(x)$ on Γ_2 such that the resulting solution $u(x,t)$ of the parabolic system matches the observed data as accurately as possible.

Inverse problems deal with finding unknown parameters or inputs from indirect or incomplete measurements. First studied in the early 20th century [13, 14], they now appear in many fields such as medical imaging, remote sensing, fluid mechanics, and finance [2, 3, 4, 6, 10, 16, 17, 18]. They are especially useful when direct measurements cannot be made, but information can be estimated through mathematical models.

A well-known example is the inverse heat conduction problem (IHCP), where we try to determine heat fluxes or thermal coefficients from temperature data. These problems are inherently ill-posed in the sense of Hadamard [11], meaning that small errors in the data can cause large errors in the results [21]. Because of this, regularization methods are needed to obtain stable and reliable solutions [1, 5, 8, 7].

Problems with Robin-type boundary conditions are of special interest, since the coefficient may change in space or time. Recent works have used nonlinear conjugate gradient methods [12], Rothe's method [20], and uniqueness proofs in one-dimensional settings [19]. More recently, the work in [15] introduced a Levenberg-Marquardt framework enhanced by surrogate functionals, enabling the concurrent estimation of a Robin boundary coefficient and a Neumann flux within elliptic equations. While the simultaneous identification of boundary parameters using Tikhonov regularization and the Levenberg–Marquardt method has been investigated for elliptic systems [15], its extension to the parabolic case represents a significant and non-trivial advancement. The introduction of time dependence creates challenges absent in the steady-state setting: the problem becomes more severely ill-posed, since one must reconstruct spatial functions from noisy temporal data; the analysis and computation are complicated by the need to solve both forward and backward-in-time PDEs for sensitivity and adjoint calculations; and the numerical implementation must maintain stability throughout the entire time evolution. The primary contribution of this work is to comprehensively address these challenges. We rigorously establish the well-posedness of the regularized inverse problem, derive the necessary time-dependent sensitivity and adjoint formulations, and develop an efficient algorithm that combines the Levenberg-Marquardt method with a surrogate functional approach. This strategy yields explicit update formulas that reduce the high computational burden while ensuring stable and accurate reconstructions. To the best of our knowledge, this constitutes the first complete framework for the simultaneous recovery of boundary coefficients in a parabolic system, providing a robust tool for this fundamentally more complex class of inverse problems.

2. Regularization and Well-posedness

This section begins by establishing the well-posedness of the forward solution to system (1.1) through a preliminary lemma. Subsequently, a Tikhonov regularization framework is presented, which reformulates the ill-posed inverse problem as a stable variational minimization problem.

Lemma 2.1. [9]
Consider an open, bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3, whose boundary $\partial\Omega$ is of class C^1 and is partitioned into three segments: $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Assume:

(1) $\gamma(x) \in L^\infty(\Gamma_1)$ *with* $0 < \gamma_1 \leq \gamma(x) \leq \gamma_2$
(2) $g(x) \in K_1 = \{g \in L^2(\Gamma_1) : g_1 \leq g(x) \leq g_2\}$
(3) $q(x) \in K_2 = \{q \in L^2(\Gamma_2) : q_1 \leq q(x) \leq q_2\}$
(4) $f(x,t) \in L^2(0,T; L^2(\Omega))$
(5) $h(x,t) \in L^2(0,T; L^\infty(\Gamma_1))$
(6) $R(x,t) \in L^2(0,T; L^\infty(\Gamma_2))$

Then system (1.1) admits a unique solution u belonging to the space $L^2(0,T; H^1(\Omega))$ satisfying the estimate:

$$\|u\|_{1,\Omega} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|gh\|_{L^2(0,T;L^2(\Gamma_1))} + \|qR\|_{L^2(0,T;L^2(\Gamma_2))}\right)$$

Tikhonov's regularization. To address the ill-posedness of the inverse problem, we recast it as a stabilized minimization problem using Tikhonov regularization. The goal is to find the unknown functions $g(x)$ and $q(x)$ such that the solution of the PDE matches the observed data as closely as possible, while also ensuring that the solution remains stable and physically meaningful.

To achieve this, we define a cost functional that includes two key components:

(1) A term that measures how close the computed solution $u(g,q)$ is to the observed data z^δ on the boundary Γ_3.
(2) A regularization term that penalizes large or unstable values of g and q to enforce smoothness and stability.

The regularized cost functional is written as:

$$J(g,q) = \int_0^T \|u(g,q) - z^\delta\|^2_{\Gamma_3} dt + \beta\|g\|^2_{\Gamma_1} + \eta\|q\|^2_{\Gamma_2},$$

where:

- The function $u(g,q)$ denotes the solution to the forward problem, which is uniquely determined by the boundary parameters g and q,
- $z^\delta(x,t)$ is the (possibly noisy) measurement data on Γ_3,
- $\beta > 0$ and $\eta > 0$ are regularization parameters that control the strength of the penalization,

- $\|g\|^2$ and $\|q\|^2$ ensure that the solution remains stable and does not overfit noisy data.

The regularized inverse problem is therefore formulated as the minimization of $J(g, q)$ over the admissible set $K_1 \times K_2$. That is,

$$\min_{(g,q)\in K_1\times K_2} J(g,q) = \min_{(g,q)} \left[\int_0^T \|u(g,q) - z^\delta\|^2_{\Gamma_3} dt + \beta\|g\|^2_{\Gamma_1} + \eta\|q\|^2_{\Gamma_2}\right] \tag{2.1}$$

The subsequent analysis demonstrates the regularizing properties of the optimization problem (2.1), proving that solutions always exist and depend continuously on the noisy data z^δ. We begin by establishing existence in the following theorem.

Theorem 2.2. *Under the assumptions of Lemma 2.1, then the minimization problem (2.1) has at least one solution.*

Proof. Since $J(g, q) \geq 0$, there exists a minimizing sequence $\{(g_k, q_k)\} \subset K_1 \times K_2$ such that:

$$\lim_{k\to\infty} J(g_k, q_k) = \inf_{(g,q)\in K_1\times K_2} J(g, q).$$

The regularization terms $\beta\|g_k\|^2_{\Gamma_1} + \eta\|q_k\|^2_{\Gamma_2}$ ensure $\{(g_k, q_k)\}$ is bounded in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. Then, by Banach-Alaoglu Theorem, there exists a subsequence (still denoted $\{(g_k, q_k)\}$) and $(g^*, q^*) \in K_1 \times K_2$ such that:

$$g_k \rightharpoonup g^* \text{ in } L^2(\Gamma_1), \quad q_k \rightharpoonup q^* \text{ in } L^2(\Gamma_2).$$

Let $u_k \equiv u(g_k, q_k)$. From Lemma 2.1, $\{u_k\}$ is bounded in $L^2(0, T; H^1(\Omega))$. By Aubin-Lions Lemma, there exists a subsequence $u_k \to u^*$ strongly in $L^2(\Gamma_3 \times (0, T))$.

The weak formulation of the parabolic system shows $u^* = u(g^*, q^*)$.
Now by using lower Semi-continuity of the norm, we get:

$$\liminf_{k\to\infty} \|g_k\|^2_{\Gamma_1} \geq \|g^*\|^2_{\Gamma_1}, \quad \liminf_{k\to\infty} \|q_k\|^2_{\Gamma_2} \geq \|q^*\|^2_{\Gamma_2}.$$

Combined with the strong convergence of u_k, we obtain:

$$J(g^*, q^*) \leq \liminf_{k\to\infty} J(g_k, q_k) = \inf_{(g,q)\in K_1\times K_2} J(g, q).$$

Thus, (g^*, q^*) is a minimizer. □

Theorem 2.3. *Let $\{z^\delta_n\}$ be a sequence in $L^2(0, T; L^2(\Gamma_3))$ such that $z^\delta_n \to z^\delta$ as $n \to \infty$. Let $\{(g^{(n)}, q^{(n)})\}$ be minimizers of the regularized problems (2.1). Then there exists a subsequence of $\{(g^{(n)}, q^{(n)})\}$ converging weakly in $L^2(\Gamma_1) \times L^2(\Gamma_2)$ to a minimizer (g^*, q^*) of the limiting problem with data z^δ.*

Proof. For each n, $(g^{(n)}, q^{(n)})$ satisfies:

$$J(g^{(n)}, q^{(n)}) \leq J(g, q) \quad \forall (g, q) \in K_1 \times K_2.$$

Thus, $\{(g^{(n)}, q^{(n)})\}$ is bounded in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. So, we can extract a subsequence (still denoted $\{(g^{(n)}, q^{(n)})\}$) such that:

$$g^{(n)} \rightharpoonup g^* \text{ in } L^2(\Gamma_1), \quad q^{(n)} \rightharpoonup q^* \text{ in } L^2(\Gamma_2).$$

Since $z^\delta_n \to z^\delta$ in $L^2(\Gamma_3 \times (0, T))$ and $u(g^{(n)}, q^{(n)}) \to u(g^*, q^*)$ strongly:

$$\lim_{n\to\infty} \|u(g^{(n)}, q^{(n)}) - z^\delta_n\|^2_{L^2(\Gamma_3\times(0,T))} = \|u(g^*, q^*) - z^\delta\|^2_{L^2(\Gamma_3\times(0,T))}.$$

By lower semi-continuity:

$$J(g^*, q^*) \leq \liminf_{n\to\infty} J(g^{(n)}, q^{(n)}) \leq \liminf_{n\to\infty} J(g, q) = J(g, q) \quad \forall (g, q) \in K_1 \times K_2.$$

Hence, (g^*, q^*) is a minimizer. □

3. Sensitivity and Adjoint Method Theory

This section analyzes the sensitivity of the parabolic system's solution $u(x,t)$ to perturbations in the unknown boundary parameters $g(x)$ and $q(x)$. This is important for solving the inverse problem using optimization methods like the Levenberg–Marquardt algorithm.

To do this, we use Gateaux derivatives, which help us understand how the solution changes in specific directions.

Sensitivity Analysis.

Lemma 3.1. .

1- *Let $d \in L^2(\Gamma_1)$ be a small change (or direction) in the function $g(x)$. The resulting change in the solution, written as $u'_g(g,q)d$, satisfies the following system:*

$$\begin{cases} \partial_t u'_g - \Delta u'_g = 0 & \textit{in } \Omega \times (0,T], \\ \frac{\partial u'_g}{\partial n} + \gamma(x) u'_g = d \cdot h(x,t) & \textit{on } \Gamma_1 \times (0,T], \\ \frac{\partial u'_g}{\partial n} = 0 & \textit{on } (\Gamma_2 \cup \Gamma_3) \times (0,T], \\ u'_g(x,0) = 0 & \textit{in } \Omega. \end{cases}$$

2- *Now let $p \in L^2(\Gamma_2)$ be a small change in $q(x)$. The effect on the solution, written as $u'_q(g,q)p$, is given by:*

$$\begin{cases} \partial_t u'_q - \Delta u'_q = 0 & \textit{in } \Omega \times (0,T], \\ \frac{\partial u'_q}{\partial n} + \gamma(x) u'_q = 0 & \textit{on } \Gamma_1 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = p \cdot R(x,t) & \textit{on } \Gamma_2 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = 0 & \textit{on } \Gamma_3 \times (0,T], \\ u'_q(x,0) = 0 & \textit{in } \Omega. \end{cases}$$

Proof. We first consider the sensitivity of the solution with respect to the coefficient $g(x)$. For a perturbation $g^\varepsilon = g + \varepsilon d$ with $\varepsilon > 0$ and $d \in L^2(\Gamma_1)$, let $u^\varepsilon = u(g^\varepsilon, q)$ denote the corresponding solution of the forward problem. Defining the difference quotient $w^\varepsilon = (u^\varepsilon - u)/\varepsilon$, one verifies that w^ε solves

$$\begin{cases} \partial_t w^\varepsilon - \Delta w^\varepsilon = 0 & \text{in } \Omega \times (0,T], \\ \dfrac{\partial w^\varepsilon}{\partial n} + \gamma(x) w^\varepsilon = d(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \dfrac{\partial w^\varepsilon}{\partial n} = 0 & \text{on } (\Gamma_2 \cup \Gamma_3) \times (0,T], \\ w^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Passing to the limit as $\varepsilon \to 0$ yields the sensitivity equation for $u'_g(g,q)d$.

Next, we study the sensitivity with respect to the Neumann flux $q(x)$. For a perturbation $q^\varepsilon = q + \varepsilon p$, let $u^\varepsilon = u(g, q^\varepsilon)$. With $w^\varepsilon = (u^\varepsilon - u)/\varepsilon$, one obtains

$$\begin{cases} \partial_t w^\varepsilon - \Delta w^\varepsilon = 0 & \text{in } \Omega \times (0,T], \\ \dfrac{\partial w^\varepsilon}{\partial n} + \gamma(x) w^\varepsilon = 0 & \text{on } \Gamma_1 \times (0,T], \\ \dfrac{\partial w^\varepsilon}{\partial n} = p(x) R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \dfrac{\partial w^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ w^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

As $\varepsilon \to 0$, the sequence w^ε converges weakly in $L^2(0,T;H^1(\Omega))$ to $u'_q(g,q)p$, which satisfies the above system.

This establishes the claimed sensitivity relations. □

Remark 3.2. ˙

1- Understanding these sensitivities helps us improve the estimates of $g(x)$ and $q(x)$ in the inverse problem. These results are later used in the Levenberg-Marquardt (LM) method to efficiently compute the directional derivatives, which makes the optimization process faster and more accurate.

2- (Regularity) Both sensitivity equations have unique solutions in $L^2(0,T;H^1(\Omega))$ by the same arguments as in Lemma 2.1.

Adjoint Formulations.

Lemma 3.3 (Adjoint Relations)**.** ˙
Let $\omega, v \in L^2(\Gamma_3)$ be given. Define the adjoint states $\varphi(u'^*_g\omega = \varphi)$ and $\psi(u'^*_q v = \psi)$ by the terminal-boundary value problems

$$\begin{aligned} -\partial_t\varphi - \Delta\varphi &= 0 && \textit{in } \Omega \times (0,T], \\ \frac{\partial\varphi}{\partial n} + \gamma\varphi &= 0 && \textit{on } \Gamma_1 \times (0,T], \\ \frac{\partial\varphi}{\partial n} &= 0 && \textit{on } \Gamma_2 \times (0,T], \\ \frac{\partial\varphi}{\partial n} &= -\omega && \textit{on } \Gamma_3 \times (0,T], \\ \varphi(x,T) &= 0 && \textit{in } \Omega, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\partial_t\psi - \Delta\psi &= 0 && \textit{in } \Omega \times (0,T], \\ \frac{\partial\psi}{\partial n} + \gamma\psi &= 0 && \textit{on } \Gamma_1 \times (0,T], \\ \frac{\partial\psi}{\partial n} &= 0 && \textit{on } \Gamma_2 \times (0,T], \\ \frac{\partial\psi}{\partial n} &= -v && \textit{on } \Gamma_3 \times (0,T], \\ \psi(x,T) &= 0 && \textit{in } \Omega. \end{aligned} \tag{3.2}$$

Then the following adjoint relations hold:

$$\left\langle \int_0^T u_g' d\,dt, \omega \right\rangle_{\Gamma_3} = \left\langle d, \int_0^T h(x,t)\varphi\,dt \right\rangle_{\Gamma_1} \tag{3.3}$$

$$\left\langle \int_0^T u_q' p\,dt, v \right\rangle_{\Gamma_3} = \left\langle p, \int_0^T R(x,t)\psi\,dt \right\rangle_{\Gamma_2} \tag{3.4}$$

for all $d \in L^2(\Gamma_1)$ and $p \in L^2(\Gamma_2)$.

Proof. .
We begin with the case of the coefficient g. Let u_g' be the solution of the sensitivity system corresponding to a perturbation $d \in L^2(\Gamma_1)$, and let φ solve the adjoint problem (3.1) with terminal condition $\varphi(x,T) = 0$. Multiplying the sensitivity equation for u_g' by φ and integrating over $\Omega \times (0,T)$, and then applying integration by parts in both time and space, we obtain

$$\int_0^T \int_{\Gamma_3} u_g'\, \omega\, ds\, dt = \int_0^T \int_{\Gamma_1} d(x)h(x,t)\,\varphi(x,t)\,ds\,dt,$$

where we used the boundary conditions for u_g' and φ on Γ_1 and Γ_3. This establishes the adjoint relation (3.3).

For the Neumann flux, let u_q' be the sensitivity solution associated with a perturbation $p \in L^2(\Gamma_2)$, and let ψ solve the adjoint problem (3.2). Proceeding in the same way, we multiply the equation for u_q' by ψ, integrate over $\Omega \times (0,T)$, and apply integration by parts. Using the boundary conditions on Γ_2 and Γ_3, we find

$$\int_0^T \int_{\Gamma_3} u_q'\, v\, ds\, dt = \int_0^T \int_{\Gamma_2} p(x)R(x,t)\,\psi(x,t)\,ds\,dt,$$

which yields the relation (3.4).

Thus, both adjoint identities are established. □

Remark 3.4. (Well-posedness) Both adjoint problems are well-posed backward parabolic equations with unique solutions $\varphi, \psi \in L^2(0,T; H^1(\Omega))$ by standard theory.

4. A Levenberg-Marquardt and Surrogate Functional Strategy

This section presents an efficient computational framework for addressing the nonlinear minimization problem (2.1). Given that the forward solution $u(g,q)$ exhibits nonlinear dependence on both boundary functions $g(x)$ and $q(x)$, the corresponding minimization problem becomes non-convex and computationally challenging. Our approach involves a two-stage methodology: first, we transform the non-convex problem into a sequence of convex subproblems using the Levenberg-Marquardt iterative scheme; second, we apply a surrogate functional technique to these convex problems to obtain explicit update formulas.

The non-convex minimization problem is addressed by first linearizing the forward operator. For an iterate $(\bar{g}, \bar{q}) \in K_1 \times K_2$, the first-order Taylor expansion

$$u(g,q) \approx u(\bar{g},\bar{q}) + u_g'(\bar{g},\bar{q})(g-\bar{g}) + u_q'(\bar{g},\bar{q})(q-\bar{q}),$$

where u'_g and u'_q denote the Fréchet derivatives, provides a local convex approximation. This leads to the Levenberg-Marquardt iterative scheme for solving (2.1):

$$\begin{aligned} J(g^{k+1}, q^{k+1}) &= \min_{(g,q)\in K_1\times K_2} J(g,q) \\ &\equiv \int_0^T \big\| u'_g(g^k, q^k)(g - g^k) + u'_q(g^k, q^k)(q - q^k) \\ &\qquad - [z^\delta - u(g^k, q^k)]\big\|^2_{L^2(\Gamma_3)} dt + \beta \|g\|^2_{L^2(\Gamma_1)} + \eta \|q\|^2_{L^2(\Gamma_2)}. \end{aligned} \tag{4.1}$$

As equation (4.1) is a simple convex quadratic minimization problem, we can write its necessary optimality conditions as follows:

$$\frac{\partial J}{\partial g}(g^{k+1}, q^{k+1})d = 0 \quad \text{for any } d \in L^2(\Gamma_1),$$

$$\frac{\partial J}{\partial q}(g^{k+1}, q^{k+1})p = 0 \quad \text{for any } p \in L^2(\Gamma_2).$$

The Gâteaux derivatives of J at (g^k, q^k) in the directions $d \in L^2(\Gamma_1)$ and $p \in L^2(\Gamma_2)$ are given by:

$$\begin{aligned} \frac{\partial J}{\partial g}(g^k, q^k)[d] = 2 \int_0^T &\langle u'_g(g^k, q^k)(g - g^k) + u'_q(g^k, q^k)(q - q^k) - (z^\delta - u(g^k, q^k)), \\ &u'_g(g^k, q^k)d\rangle_{\Gamma_3} dt + 2\beta \langle g, d\rangle_{\Gamma_1}, \end{aligned}$$

$$\begin{aligned} \frac{\partial J}{\partial q}(g^k, q^k)[p] = 2 \int_0^T &\langle u'_g(g^k, q^k)(g - g^k) + u'_q(g^k, q^k)(q - q^k) - (z^\delta - u(g^k, q^k)), \\ &u'_q(g^k, q^k)p\rangle_{\Gamma_3} dt + 2\eta \langle q, p\rangle_{\Gamma_2}. \end{aligned}$$

Although the minimization problem (4.1) is quadratic and convex, its solution is not straightforward. The first-order optimality conditions lead to a large, coupled system of equations for the updates (g^{k+1}, q^{k+1}), where the cross-terms involving the operators u'_g and u'_q produce a dense and non-diagonal structure. Solving this system implicitly at every iteration would require inverting a discretized Hessian operator, which is computationally expensive, especially in the parabolic setting where each evaluation of the forward and adjoint operators already involves time-dependent PDE simulations. To circumvent this bottleneck and obtain practical efficiency, we adopt a surrogate functional approach. The key idea is to majorize the original functional $J(g, q)$ with a strictly convex surrogate $J^s(g, q)$ that is carefully designed to remain equivalent in terms of minimizers but, crucially, becomes separable in g and q. By adding a positive-definite quadratic form that dominates the coupling terms, the surrogate functional effectively decouples the variables, diagonalizes the problem, and yields explicit update formulas. This strategy eliminates the need to solve large implicit systems, accelerates convergence, and enhances robustness against noise. Thus, the surrogate functional not only preserves the theoretical soundness of the Levenberg-Marquardt framework but also makes the simultaneous recovery of boundary coefficients in parabolic systems computationally feasible.

Now, we introduce the surrogate functional $J^s(g,q)$ of $J(g,q)$:

$$J^s(g,q) = J(g,q) + A\|g-g^k\|^2_{\Gamma_1} + B\|q-q^k\|^2_{\Gamma_2} - \int_0^T \|u'_g(g^k,q^k)(g-g^k) + u'_q(g^k,q^k)(q-q^k)\|^2_{\Gamma_3}\,dt, \tag{4.2}$$

The positive constants A and B are selected to ensure the quadratic form

$$A\|d\|^2_{\Gamma_1} + B\|p\|^2_{\Gamma_2} - \int_0^T \|u'_g(g^k,q^k)d + u'_q(g^k,q^k)p\|^2_{\Gamma_3}dt$$

remains non-negative for all $d \in L^2(\Gamma_1)$ and $p \in L^2(\Gamma_2)$, thereby guaranteeing $J^s(g,q) \geq J(g,q)$. A sufficient condition for this is given by the operator norm bounds:

$$A \geq 2\|u'_g(g^k,q^k)\|^2_{\mathcal{L}(L^2(\Gamma_1),L^2(\Gamma_3))}, \quad B \geq 2\|u'_q(g^k,q^k)\|^2_{\mathcal{L}(L^2(\Gamma_2),L^2(\Gamma_3))}.$$

In this expression, $\|\cdot\|_{\mathcal{L}(U,V)}$ represents the operator norm between the spaces U and V. The boundedness of the derivatives u'_g and u'_q, required for these norms to be finite, is established by Lemma 3.1.

Remark 4.1 (Choice of surrogate constants). The positivity of the surrogate functional $J^s(g,q)$ requires the constants A and B to satisfy

$$A \geq 2\|u'_g(g^k,q^k)\|^2_{\mathcal{L}(L^2(\Gamma_1),L^2(\Gamma_3))}, \quad B \geq 2\|u'_q(g^k,q^k)\|^2_{\mathcal{L}(L^2(\Gamma_2),L^2(\Gamma_3))}.$$

In practice, these operator norms are difficult to estimate directly. To ensure convergence, we initialize with moderate values (e.g., $A_0 = B_0 = 1$) and adaptively increase them if needed during the iteration. This practical strategy guarantees the positivity of $J^s(g,q)$ while keeping the updates efficient.

By applying the adjoint relations established in (3.5) and (3.6), the surrogate functional $J^s(g,q)$ can be reformulated in the following manner:

$$\begin{aligned} J^s(g,q) &= -2\int_0^T \langle u'_g(g^k,q^k)(g-g^k) + u'_q(g^k,q^k)(q-q^k), z^\delta - u(g^k,q^k)\rangle_{\Gamma_3}\,dt \\ &\quad + \beta\|g\|^2_{\Gamma_1} + \eta\|q\|^2_{\Gamma_2} + A\|g-g^k\|^2_{\Gamma_1} + B\|q-q^k\|^2_{\Gamma_2} \\ &\quad + \int_0^T \|z^\delta - u(g^k,q^k)\|^2_{\Gamma_3}\,dt \\ &= -2\langle g-g^k, \int_0^T h(x,t)u'_g(g^k,q^k)^*(z^\delta - u(g^k,q^k))\,dt\rangle_{\Gamma_1} \\ &\quad - 2\langle q-q^k, \int_0^T R(x,t)u'_q(g^k,q^k)^*(z^\delta - u(g^k,q^k))\,dt\rangle_{\Gamma_2} \\ &\quad + (\beta+A)\|g-g^k\|^2_{\Gamma_1} + (\eta+B)\|q-q^k\|^2_{\Gamma_2} \\ &\quad + \int_0^T \|z^\delta - u(g^k,q^k)\|^2_{\Gamma_3}\,dt. \end{aligned} \tag{4.3}$$

Since $J^s(g,q)$ is quadratic in both g and q, the minimizer (g^*,q^*) is characterized by the first-order optimality conditions:

$$\frac{\partial J^s}{\partial g}(g^*,q^*) = 0 \quad \text{and} \quad \frac{\partial J^s}{\partial q}(g^*,q^*) = 0.$$

Solving these equations yields the explicit update formulas:

$$g^{k+1} = g^k + \frac{1}{\beta + A} \int_0^T h(x,t) \cdot u_g'(g^k, q^k)^* (z^\delta - u(g^k, q^k))\, dt, \quad x \in \Gamma_1,$$
$$q^{k+1} = q^k + \frac{1}{\eta + B} \int_0^T R(x,t) \cdot u_q'(g^k, q^k)^* (z^\delta - u(g^k, q^k))\, dt, \quad x \in \Gamma_2,$$

Based on the preceding analysis, we now present an efficient algorithm for the simultaneous reconstruction of g and q.

Algorithm 1

Step 1. Initialize g^0, q^0, set tolerance parameters $\varepsilon_1, \varepsilon_2$ and initialize surrogate constants $A = B = 1$. (These may be adaptively increased to ensure positivity of J^s.)
Step 2. At iteration k, solve the forward problem to get $u(g^k, q^k)$.
Step 3. Solve the adjoint problems to get $u_g'(g^k, q^k)^*(z^\delta - u(g^k, q^k))$ and $u_q'(g^k, q^k)^*(z^\delta - u(g^k, q^k))$.
Step 4. Update g^{k+1} and q^{k+1} using the explicit formulas.
Step 5. If the relative error satisfies

$$\frac{\|g^{k+1} - g^k\|_{L^2(\Gamma_1)}}{\|g^k\|} < \varepsilon_1 \quad \text{and} \quad \frac{\|q^{k+1} - q^k\|_{L^2(\Gamma_2)}}{\|q^k\|} < \varepsilon_2,$$

then stop; otherwise, set $k \leftarrow k + 1$ and repeat.

5. Numerical Experiments

This section presents numerical experiments implementing Algorithm 1 for the simultaneous recovery of the Robin coefficient $g(x)$ and heat flux $q(x)$ in the parabolic system (1.1). The simulations are conducted using MATLAB with the computational domain defined as $\Omega = (0,1) \times (0,2)$. The boundary partitions are specified as follows:

$$\Gamma_1 = \{(x,y) \in \partial\Omega \,|\, x = 1,\, 0 \le y \le 2\}, \quad \Gamma_2 = \{(x,y) \in \partial\Omega \,|\, y = 0,\, 0 \le x \le 1\},$$

$$\Gamma_3 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2).$$

A structured triangular mesh is generated by subdividing each rectangular cell of the computational grid along its diagonal. The forward and adjoint problems are solved using linear finite element methods.

We take the source term, time-dependent heat flux factor, and boundary temperature function as:

$$f(x,t) = (xy+1)t + t, \qquad R(x,t) = x^2 + t, \qquad h(x,t) = 1 - y - 3t.$$

Noisy measurement data z^δ on Γ_3 is generated by solving the forward problem with exact parameters and adding 2% uniform random noise:

$$z^\delta = u + \delta R u, \quad \text{where } \delta = 0.01.$$

The following parameter values are employed in all experiments: regularization parameters $\beta = \eta = 10^{-4}$, surrogate weights $A = B = 1$, and stopping tolerances $\varepsilon_1 = \varepsilon_2 = 2 \times 10^{-4}$. Two numerical tests are conducted to evaluate the simultaneous reconstruction of $g(x)$ and $q(x)$ in system (1.1), with $\gamma(x) = 1$ specified throughout.

Example 1: We define the exact functions to be recovered as:

$$g_{\text{exact}}(x) = 1 + \frac{1}{4}y^2, \quad q_{\text{exact}}(x) = 1 + \frac{1}{2}\sin\left(\frac{\pi}{2}x\right).$$

The initial guesses are chosen as constants:

$$g^0(x) = 1, \quad q^0(x) = 1.5 \quad \text{for all mesh points.}$$

Figure 5.1 displays a comparison between the exact coefficient profiles and their numerical reconstructions obtained using Algorithm 1.

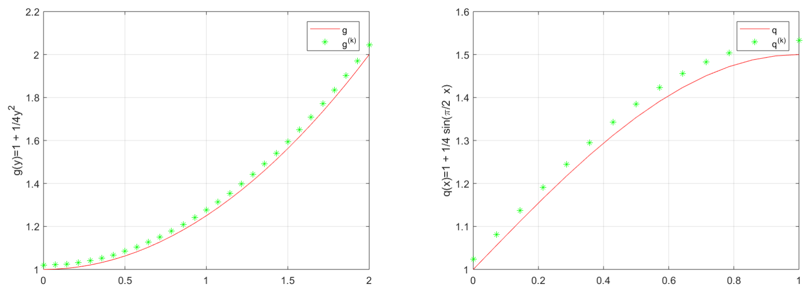

FIGURE 1. Reconstruction results for boundary coefficients $g(x)$ (left) and $q(x)$ (right), showing exact versus computed profiles.

After 17 iterations of Algorithm 1, the relative errors between the exact and reconstructed coefficients are:

$$\frac{\|g^{17} - g_{\text{exact}}\|_{L^2(\Gamma_1)}}{\|g_{\text{exact}}\|_{L^2(\Gamma_1)}} = 0.0208, \qquad \frac{\|q^{17} - q_{\text{exact}}\|_{L^2(\Gamma_2)}}{\|q_{\text{exact}}\|_{L^2(\Gamma_2)}} = 0.0225.$$

Example 2: In this example, we choose more nonlinear target functions:

$$g_{\text{exact}}(x) = 2 - \frac{1}{4}y^2, \quad q_{\text{exact}}(x) = 2 + (x-1)^2.$$

We use the initial guess:

$$g^0(x) = 2, \quad q^0(x) = 2.$$

Figure 5.2 displays a comparison between the exact coefficient profiles and their numerical reconstructions obtained using Algorithm 1.

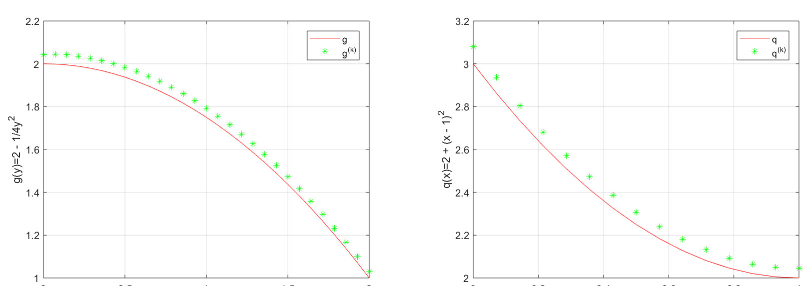

FIGURE 2. Reconstruction results for boundary coefficients $g(x)$ (left) and $q(x)$ (right), showing exact versus computed profiles.

After 31 iterations of Algorithm 1, the relative errors between the exact and reconstructed coefficients are:

$$\frac{\|g^{31} - g_{\text{exact}}\|_{L^2(\Gamma_1)}}{\|g_{\text{exact}}\|_{L^2(\Gamma_1)}} = 0.0243, \qquad \frac{\|q^{31} - q_{\text{exact}}\|_{L^2(\Gamma_2)}}{\|q_{\text{exact}}\|_{L^2(\Gamma_2)}} = 0.0249.$$

6. Conclusion

In this work, we have investigated the numerical identification of boundary coefficients in a parabolic system, focusing on the simultaneous recovery of the coefficient $g(x)$ and the heat flux $q(x)$ from noisy boundary measurements. The study addressed the inherent ill-posedness of the inverse problem by employing Tikhonov regularization and the Levenberg-Marquardt (LM) method combined with a surrogate functional strategy.

First, we established the well-posedness of the forward problem and analyzed the stability and convergence of the regularized formulation. Second, we developed a computational algorithm based on the finite element method for spatial discretization and the backward Euler scheme for time integration. The proposed method demonstrated robustness in handling noisy data and achieved accurate reconstructions of the unknown boundary functions.

Numerical experiments validated the effectiveness of the approach, showing successful recovery of $g(x)$ and $q(x)$ with relative errors below 2.5% in both test cases. These results demonstrate that regularization and optimization methods can effectively solve challenging inverse problems in heat transfer.

References

1. Agbodjan, K. S., Ahmed, O. S., Messaoudi, D., Cheng, T., & Jiang, D. (2019). Quadratic convergence of Levenberg-Marquardt method for general nonlinear inverse problems with two parameters. International Journal of Computer Mathematics, 96(11), 2269-2284. https://doi.org/10.1080/00207160.2019.1668559
2. Alquier, P., Gautier, E., & Stoltz, G. (Eds.). (2011). Inverse problems and high-dimensional estimation: Stats in the Château Summer School, August 31, September 4, 2009 (Vol. 203). Springer. https://doi.org/10.1007/978-3-642-19989-9
3. Aster, R. C., Borchers, B., & Thurber, C. H. (2018). Parameter estimation and inverse problems (3rd ed.). Elsevier. https://doi.org/10.1016/C2015-0-02458-3
4. Banks, H. T., & Kunisch, K. (2012). Estimation techniques for distributed parameter systems. Springer. https://doi.org/10.1007/978-1-4612-3700-6
5. Beck, J. V., Blackwell, B., & St. Clair, C. R. (1985). Inverse heat conduction: Ill-posed problems. Wiley.
6. Beilina, L., & Smirnov, Y. G. (Eds.). (2018). Nonlinear and inverse problems in electromagnetics: PIERS 2017, St. Petersburg, Russia, May 22-25 (Vol. 243). Springer.
7. Daubechies, I., Defrise, M., & De Mol, C. (2004). An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. Communications on Pure and Applied Mathematics, 57(11), 1413-1457. https://doi.org/10.48550/arXiv.math/0307152
8. Danilaev, P. G. (2001). Coefficient inverse problems for parabolic type equations and their applications. VSP.
9. Evans, L. C. (1997). Partial differential equations and Monge-Kantorovich mass transfer. Current Developments in Mathematics, 1997(1), 65-126. https://dx.doi.org/10.4310/CDM.1997.v1997.n1.a2
10. Groetsch, C. W. (1993). Inverse problems in the mathematical sciences (Vol. 52). Springer. https://doi.org/10.1007/978-3-322-99202-4

11. Hadamard, J. (1902). Sur les problèmes aux dérivées partielles et leur signification physique. Bulletin de lUniversité de Princeton, 13.
12. Hao, D. N., Thanh, P. X., & Lesnic, D. (2013). Determination of the heat transfer coefficients in transient heat conduction. Inverse Problems, 29(9), 095020. https://doi.org/10.1088/0266-5611/29/9/095020
13. Hawkes, W. P. (1998). Advances in imaging and electron physics. Academic Press.
14. Isakov, V. (1998). Inverse problems for partial differential equations. Springer. https://doi.org/10.1007/978-3-319-51658-5
15. Jiang, D., & Talaat, T. (2017). Simultaneous identification of Robin coefficient and heat flux in an elliptic system. International Journal of Computer Mathematics, 94(1), 185-196. https://doi.org/10.1080/00207160.2015.1099634
16. Kabanikhin, S. I. (2012). Inverse and ill-posed problems: Theory and applications. De Gruyter. https://doi.org/10.1515/9783110224016
17. Kirsch, A. (2011). An introduction to the mathematical theory of inverse problems (2nd ed.). Springer. https://doi.org/10.1007/978-1-4419-8474-6
18. Neto, F. D. M., & da Silva Neto, A. J. (2012). An introduction to inverse problems with applications. Springer. https://doi.org/10.1007/978-3-642-32557-1
19. Slodicka, M., Lesnic, D., & Onyango, T. (2010). Determination of a time-dependent heat transfer coefficient in a nonlinear inverse heat conduction problem. Inverse Problems in Science and Engineering, 18(1), 65-81. https://doi.org/10.1080/17415970903234273
20. Slodicka, M., & Van Keer, R. (2002). Determination of a Robin coefficient in semilinear parabolic problems by means of boundary measurements. Inverse Problems, 18(1), 139-147.
21. Wikstrom, P., Blasiak, W., & Berntsson, F. (2007). Estimation of the transient surface temperature and heat flux of a steel slab using an inverse method. Applied Thermal Engineering, 27(14-15), 2463-2472. https://doi.org/10.1016/j.applthermaleng.2007.02.005

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