Gulf Journal of Mathematics
Vol 21, Issue 1 (2025) 562-576

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i1.3507

MATHEMATICAL ANALYSIS OF THERMO-ELASTIC-VISCOPLASTIC CONTACT PROBLEMS WITH TIME-FRACTIONAL DERIVATIVES

HAMID ACHAB[1]*, ILYAS BOUKAROURA[2] and ABDELMOUMENE DJABI [3]

Abstract. This paper presents a mathematical model for a quasi-static frictional contact problem between a thermo-elastic-viscoplastic body and a deformable foundation, incorporating memory effects via time-fractional derivatives in both the constitutive law and the heat conduction. The model is constructed inside a variational framework, resulting in a mathematical system comprising a fractional variational inequality that dictates the displacement's evolution, and an evolutionary equation for the thermal field. We use the Galerkin approximation approach, the Banach's fixed point theorem and certain analytical instruments associated with fractional variational inequalities to demonstrate the existence of a weak solution.

Keywords. Thermo-Elastic-Viscoplastic, Time-fractional derivative, weak solutions.
2020 Mathematics Subject Classification. Primary 74C10; Secondary 49J40

1. Introduction

Contact problems are central to the mechanics of deformable bodies. They aim to characterize the behavior of interacting surfaces under mechanical loading, including the transmission of forces, deformation under stress, and the influence of friction, which may lead to either sliding or sticking. These phenomena often involve elasticity, viscosity, and plasticity, resulting in nonlinear and highly complex boundary conditions that challenge both modeling and numerical analysis. Several works, such as [12, 11], offer general theoretical foundations for frictional contact involving viscoelastic-plastic materials.

The mathematical theory of fractional calculus dates back to the works of Leibniz and l'Hôpital in the late 17th century [7]. In recent decades, this theory has seen significant developments and wide applicability, particularly in modeling complex physical systems with memory-dependent behavior. Fractional derivatives have been incorporated into constitutive laws, such as those of the fractional Kelvin–Voigt and Maxwell models [10, 5, 15, 1]. These models have shown particular promise in capturing the hereditary effects inherent to viscoelastic and plastic materials.

Date: Received: Sep 6, 2025; Accepted: Oct 4, 2025.

* Corresponding author

Recent studies [14, 16] have employed fractional calculus in the analysis of hemivariational inequalities, using techniques such as the Rothe method to establish the existence of weak solutions. In [1, 2], a mathematical model was proposed for a thermo-viscoelastic body in frictional contact with a thermally conductive foundation, incorporating fractional time derivatives in both the mechanical and thermal equations. Similarly, Xuan et al. [15] examined a related problem involving fractional time-order terms and long-memory effects.

Motivated by this body of work, we propose a novel mathematical model for frictional contact that incorporates fractional time derivatives into both the heat conduction law and the constitutive equation for viscoplasticity, together with the Clarke subdifferential to describe friction. The model is designed to capture long-memory effects characteristic of materials exhibiting coupled thermal, viscoelastic, and plastic behavior. We derive the corresponding variational formulation and establish the existence and uniqueness of a weak solution by employing tools such as the Galerkin method and the Banach fixed-point theorem.

The structure of this paper is as follows. In Section 2, we introduce the mathematical model that characterizes the frictional contact between a viscoplastic body and a deformable foundation, where the formulation integrates fractional time derivatives together with thermal effects. In Section 3, we introduce the adopted notation and provide some preliminary results, followed by a detailed presentation of the assumptions on the data and the construction of the variational formulation of the problem. Section 4 is devoted to establishing the existence of a weak solution to the proposed model.

2. The Mathematical Model

The material under investigation is assumed to exhibit thermo-elasto-visco-plastic behavior and occupies a bounded domain $\Omega \subset \mathbb{R}^m$, where $m = 2$ or 3, with a Lipschitz boundary Γ. The boundary Γ is partitioned into three disjoint, relatively open subsets: Γ_1, Γ_2, and Γ_3, such that $\operatorname{meas}(\Gamma_1) > 0$. The time interval under consideration is denoted by $\mathrm{I} \equiv [0, T]$, where $T > 0$.

We make use of the following notation:

$$\Sigma = \Omega \times \mathrm{I}, \quad \Xi = \Gamma \times \mathrm{I}, \quad \Xi_0 = (\Gamma_1 \cap \Gamma_2) \times \mathrm{I}$$
$$\Xi_1 = \Gamma_1 \times \mathrm{I}, \quad \Xi_2 = \Gamma_2 \times \mathrm{I}, \quad \Xi_3 = \Gamma_3 \times \mathrm{I}.$$

Throughout this interval, the body is subjected to a body force of density f_0 and a volumetric heat source of intensity χ. Mechanical and thermal boundary conditions are imposed on distinct parts of the boundary Γ. Mechanically, the body is clamped on Γ_1, where the displacement field is assumed to vanish. A surface traction of density f_2 is applied on Γ_2. Thermally, the temperature is assumed to vanish on Γ_0. In the reference configuration, the body is in contact with the foundation along the contact surface Γ_3.

Let u denote the displacement field, σ the stress tensor, and $\boldsymbol{\varepsilon}(\mathbf{u}(t))$ the linearized strain tensor associated with the displacement. In this study, we adopt a constitutive law that incorporates a fractional time derivative to model the thermo-elasto-viscoplastic behavior of the material. This law is given by:

$$\sigma(t) = \mathcal{L}\boldsymbol{\varepsilon}\left({}_0^C D_t^\alpha u(t)\right) + \mathcal{M}\big(\boldsymbol{\varepsilon}(u\,(t))\big)$$
$$+ \int_0^t \mathcal{S}\big(\sigma\,(r) - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u(r)\right), \boldsymbol{\varepsilon}(u\,(r)), \theta(r)\big)dr, \quad \text{in } \Sigma. \tag{2.1}$$

Here, $\mathcal{L}$ and $\mathcal{M}$ are operators that represent, respectively, the viscous and elastic properties of the material while $\mathcal{S}$ is a continuous nonlinear function that characterizes the viscoplastic behavior of the material.

For $\alpha \in]0,1[$, the notation ${}_0^C D_t^\alpha \mathbf{u}(t)$ denotes the time-fractional derivative of order α of the function u, in the sense of Caputo.

The thermo-elasto-viscoplastic constitutive law with time-fractional derivative (2.1) incorporates temperature effects, which are described by the following evolution equation:

$$ {}_0^C D_t^\alpha \theta(t) - \operatorname{div} \mathcal{K}\nabla\theta(t) = \psi\big(\sigma(t) - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u(t)\right), \boldsymbol{\varepsilon}(u(t)), \theta(t)\big) + \chi(t) \quad \text{in } \Sigma, \tag{2.2} $$

where $\mathcal{K}$ is the thermal conductivity tensor, χ is the density of volume heat sources and ψ is a nonlinear constitutive function that characterizes the heat produced by the work of internal forces.

For mechanical frictional contact conditions, we apply the multivalued normal compliance condition, which involves the fractional derivative mentioned in [15] and defined by:

$$ -\sigma_\nu \in \partial j_\nu({}_0^C D_t^\alpha u_\nu), \quad -\sigma_\tau \in \partial j_\tau({}_0^C D_t^\alpha u_\tau). \tag{2.3} $$

Here, σ_ν and σ_τ are the normal and tangential components of the stress tensor, given by: $\sigma_\nu = (\sigma\nu)\cdot\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu$.

Similarly, u_ν and u_τ denote the normal and tangential displacements, with $u_\nu = u\cdot\nu$ and $u_\tau = u - u_\nu\nu$.

Moreover, for heat conduction, we apply the following conditions on the contact surface:

$$ \mathcal{K}_0 \frac{\partial\theta}{\partial\nu} = \kappa_e(\theta - \theta_F), \quad \text{on } \Xi_3, \tag{2.4} $$

where κ_e is the heat transfer coefficient at the body–obstacle interface, and θ_F the foundation temperature.

The classical mechanical fractional contact problem for thermo-elastic-viscoplastic materials is formulated as:

Problem $\mathcal{P}$. Find *a displacement field* $u : \Sigma \to \mathbb{R}^m$, *a stress field* $\sigma : \Sigma \longrightarrow \mathbb{S}^m$ and *a temperature* $\theta : \Sigma \to \mathbb{R}$ where:

$$ \begin{aligned} \sigma = {} & \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right) + \mathcal{M}\big(\boldsymbol{\varepsilon}(u)\big) \\ & + \int_0^t \mathcal{S}\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) dr \quad \text{in } \Sigma, \end{aligned} \tag{2.5} $$

$$ \operatorname{Div}\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Sigma, \tag{2.6} $$

$$ {}_0^C D_t^\alpha \theta - \operatorname{div}\mathcal{K}\nabla\theta = \psi\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) + \chi \quad \text{in } \Sigma, \tag{2.7} $$

$$ u = 0 \quad \text{on } \Xi_1, \tag{2.8} $$

$$ \sigma\nu = \mathbf{f}_2 \quad \text{on } \Xi_2, \tag{2.9} $$

$$ \theta = 0 \quad \text{on } \Xi_0, \tag{2.10} $$

$$ -\sigma_\nu \in \partial j_\nu\left({}_0^C D_t^\alpha u_\nu\right) \quad \text{on } \Xi_3, \tag{2.11} $$

$$ -\sigma_\tau \in \partial j_\tau\left({}_0^C D_t^\alpha u_\tau\right) \quad \text{on } \Xi_3, \tag{2.12} $$

$$ \mathcal{K}\frac{\partial\theta}{\partial\nu} = \kappa_e(\theta - \theta_F) \quad \text{on } \Xi_3, \tag{2.13} $$

$$ u(0) = u_0 \quad \text{in } \Omega, \tag{2.14} $$

$$ \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{2.15} $$

In the problem (2)–(2.15), Equation (2) represents the constitutive law for thermo-elastic-viscoplastic materials, involving Caputo-type time-fractional derivatives. Equation (2.6) represents the stress equilibrium condition while equation (2.7) describes the heat evolution over time using a time-fractional derivative. Equations (2.8) and (2.9) represent the boundary conditions related to displacement and traction forces, whereas equations (2.10) and (2.13) describe the thermal boundary conditions. Relation (2.11) characterizes the multivalued normal compliance condition involving the fractional derivative. Inclusion (2.12) represents the multivalued friction law, with ∂j_τ denoting the Clarke subdifferential of j_τ introduced in [9]. u_0 and θ_0 are the initial displacement and the initial temperature, respectively.

3. Variational formulation

This section includes the description of all the notations used to derive the variational formulation of the problem $\mathcal{P}$, as well as the introduction of basic concepts related to the formal framework of our study.

3.1. **Preliminaries and notations.** In this section, we introduce the used notation and review some fundamental mathematical concepts that form the theoretical framework for the subsequent analysis. For a more comprehensive treatment of these topics, the reader is referred to works described in [3, 13]. Throughout this work, we employ the standard notation for function spaces of the form $L^p(\mathrm{I};X)$ and $W^{k,p}(\mathrm{I};X)$, where $1 \leq p \leq \infty$ and $k >$ in addition to continuous function spaces $\mathcal{C}(\mathrm{I};X)$ as well as the space of continuously differentiable functions $\mathcal{C}^1(\mathrm{I};X)$.

Here, X denotes a real Hilbert space and I represents a finite time interval. When dealing with physical quantities of tensorial nature, we denote by S^m the space of second-order symmetric tensors over $\mathbb{R}^m$, where $m = 2,3$. This notation is essential in the modeling of stress and strain fields in multi-physics mechanical systems. We introduce the following spaces, considering a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$:

$$H = \left\{\vartheta = (\vartheta_i) : \vartheta_i \in L^2(\Omega)\right\} = L^2(\Omega)^m,$$
$$\mathcal{H} = \left\{\varsigma = (\varsigma_{ij}) : \varsigma_{ij} = \varsigma_{ji} \in L^2(\Omega)\right\},$$
$$H^1(\Omega)^m = \{\vartheta = (\vartheta_i) \in H : \varepsilon(\vartheta) \in \mathcal{H}\},$$
$$\mathcal{H}_1 = \{\boldsymbol{\varsigma} \in \mathcal{H} : \mathrm{Div}\boldsymbol{\varsigma} \in H\}.$$

These spaces provide the mathematical foundation for generating weak solutions and represent the theoretical framework on which the study of the problem is based.

We introduce the closed subspace of $H^1(\Omega)^m$ defined by:

$$\mathcal{V} = \left\{\vartheta \in H^1(\Omega)^m : \vartheta = 0 \text{ on } \Gamma_1\right\}, \quad \mathcal{W} = \left\{\zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\}.$$

The inner products on $\mathcal{V}$ and $\mathcal{W}$ are defined as:

$$(\upsilon,\vartheta)_{\mathcal{V}} = (\boldsymbol{\varepsilon(\upsilon)}, \varepsilon(\vartheta))_{\mathcal{H}}, \ \forall\, \upsilon,\vartheta \in \mathcal{V}, \tag{3.1}$$
$$(\xi,\zeta)_{\mathcal{W}} = (\nabla\xi, \nabla\zeta)_H, \ \ \forall\, \xi,\zeta \in \mathcal{W}. \tag{3.2}$$

Since meas $\Gamma_1 > 0$, the Korn's and Friedrichs-Poincaré inequalities holds, thus:

$$\|\varepsilon(\vartheta)\|_{\mathcal{H}} \geq c_k \|\vartheta\|_{H^1(\Omega)^m}, \quad \forall\, \vartheta \in \mathcal{V}, \tag{3.3}$$
$$\|\nabla\zeta\|_H \geq c_F \|\zeta\|_{\mathcal{W}}, \quad \forall\, \zeta \in \mathcal{W}, \tag{3.4}$$

where c_k and c_F are positive constants depending on the data of the problem; however, they are independent of the solutions.

Moreover, by the Sobolev trace theorem and the equalities (3.1)–(3.2), there exist c_s and c_d which are positive constants, such that:

$$\begin{aligned} \|\vartheta\|_{L^2(\Gamma_3)^m} &\leq c_s\|\vartheta\|_{\mathcal{V}}, \quad \forall\vartheta\in\mathcal{V}, & (3.5)\\ \|\zeta\|_{L^2(\Gamma_3)} &\leq c_d\|\zeta\|_{\mathcal{W}}, \quad \forall\zeta\in\mathcal{W}. & (3.6)\end{aligned}$$

Let $\gamma : H^1(\Gamma))^m \to H_\Gamma = (H^{1/2}(\Gamma))^m$ be the trace operator. For all $\vartheta \in H^1(\Omega)^m$, we utilize the notation ϑ to represent $\gamma\vartheta$ for ϑ over Γ, and for all $\sigma \in\mathcal{H}_1$, the following three Green's formulas holds:

$$(\sigma, \varepsilon(\vartheta))_{\mathcal{H}} + (\operatorname{Div}\sigma, \vartheta)_H = \int_\Gamma \sigma\nu.\vartheta\, da, \quad \forall\vartheta \in H^1(\Omega)^m. \tag{3.7}$$

3.2. **Assumptions on the data.** The operators $\mathcal{L}, \mathcal{M}$ and $\mathcal{S}$ satisfy the following assumptions:

$$\begin{cases} (\mathbf{a})\ \mathcal{L} = (l_{ijlk}) : \Omega\times S^m \longrightarrow S^m,\ l_{ijlk} = l_{lkij} = l_{ijlk} \in L^\infty(\Omega), \text{and} \\ \text{there exists } M_{\mathcal{L}} > 0,\ \text{ such that } \|\mathcal{L}(.,\zeta_1) - \mathcal{L}(.,\zeta_2)\| \leq M_{\mathcal{L}}\|\zeta_1-\zeta_2\| \\ \forall\zeta_1,\zeta_2\in\mathbb{S}^m, \text{a.e.in } \Omega. \\ (\mathbf{b}) \text{ there exists } m_{\mathcal{L}} > 0,\ \text{ such that } \mathcal{L}\zeta\cdot\zeta \geq m_{\mathcal{L}}\|\zeta\|^2,\ \forall\zeta\in\mathbb{S}^d, \text{a.e.in } \Omega. \end{cases} \tag{3.8}$$

$$\begin{cases} \mathcal{M} = (m_{ijlk}) : \Omega\times S^m \longrightarrow S^m,\ m_{ijlk} = m_{lkij} = m_{ijlk} \in L^\infty(\Omega), \text{and} \\ \text{there exists } M_{\mathcal{M}} > 0,\ \text{ such that } \ \|\mathcal{M}(.,\tau_1) - \mathcal{M}(.,\tau_2)\| \leq M_{\mathcal{M}}\|\tau_1-\tau_2\| \\ \forall\tau_1,\tau_2\in\mathbb{S}^m,\ \text{ a.e.in } \Omega. \end{cases} \tag{3.9}$$

$$\begin{cases} (\mathbf{a})\ \mathcal{S} : \Omega\times\mathbb{S}^m\times\mathbb{S}^m\times\mathbb{R}\to\mathbb{S}^m,\ \text{ and there exists } M_{\mathcal{S}} > 0,\ \text{ such that} \\ \|\mathcal{S}(.,\tau_1,\varsigma_1,\omega_1) - \mathcal{S}(.,\tau_2,\varsigma_2,\omega_2)\| \leq M_{\mathcal{S}}(\|\tau_1-\tau_2\| + \|\varsigma_1-\varsigma_2\| + \|\omega_1-\omega_2\|) \\ \forall\tau_1,\tau_2,\varsigma_1,\varsigma_2\in\mathbb{S}^m, \text{ and } \omega_1,\omega_2\in\mathbb{R} \text{ a.e.in } \Omega. \\ (\mathbf{b}) \text{ the function } \mathcal{S}(.,\tau,\varsigma,\omega) \text{ is measurable on } \Omega, \forall\tau,\varepsilon\in\mathbb{S}^m, \text{ and } \omega\in\mathbb{R}. \\ \ \text{and } \mathcal{S}(.,0,0,0)\in\mathcal{H}. \end{cases} \tag{3.10}$$

The function $\psi : \Omega\times\mathbb{S}^m\times\mathbb{S}^m\times\mathbb{R}\to\mathbb{S}^m$ satisfies:

$$\begin{cases} (\mathbf{a}) \text{ there exists } M_\psi > 0 \text{ such that:} \\ \quad \|\psi(.,\tau_1,\varsigma_1,\omega_1) - \psi(.,\tau_2,\varsigma_2,\omega_2)\| \leq M_\psi(\tau_1-\tau_2\| + \|\varsigma_1-\varsigma_2\| + \|\omega_1-\omega_2\|) \\ \forall\tau_1,\tau_2,\varsigma_1,\varsigma_2\in\mathbb{S}^M, \text{ and } \omega_1,\omega_2\in\mathbb{R} \text{ a.e. in } \Omega. \\ (\mathbf{b}) \text{ the function } \psi(.,\tau,\varsigma,\chi) \text{ is measurable on } \Omega,\ \forall\tau,\varsigma\in\mathbb{S}^d, \text{ and } \chi\in\mathbb{R}^m. \\ \text{and } \psi(.,0,0,0)\in L^2(\Omega). \end{cases} \tag{3.11}$$

The *normal potential* $j_\nu : \Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies:

$$\begin{cases} (\mathbf{a}) \text{ one can find a positive constant } M_\nu > 0 \text{ satisfying:} \\ \ j^0_\nu(.,\vartheta_1,\vartheta_2-\vartheta_1) + j^0_\nu(.,\vartheta_2,\vartheta_2-\vartheta_1) \leq M_\nu\|\vartheta_1-\vartheta_2\|^2_{\mathbb{R}}, \quad \forall\upsilon_1,\upsilon_2\in\mathbb{R}, \text{a.e.in } \Gamma_3 \\ (\mathbf{b}) \text{ for a.e. } x\in\Gamma_3,\ \text{ the function } j_\nu(x,.) \text{ satisfies Lipschitz continuity on } \mathbb{R} \\ (\mathbf{c}) \text{ a bound } m_\nu > 0 \text{ can be found such that } |\partial j_\nu(.,\vartheta)| \leq m_\nu, \quad \forall\vartheta\in\mathbb{R} \text{ a.e. in } \Gamma_3 \\ (\mathbf{d}) \ \text{ for a.e. } x\in\Gamma_3,\ \text{ the function } j_\nu(x,.) \text{ exhibits regularity} \\ (\mathbf{e}) \ \text{ the function } j_\nu(.,\vartheta) \text{ is measurable on } \Gamma_3,\ \forall\vartheta\in\mathbb{R} \text{ and there exists} \\ \ r\in L^2(\Gamma_3;\mathbb{R}) \text{ such that } j_\nu(.,r(.))\in L^1(\Gamma_3) \end{cases} \tag{3.12}$$

The *potential function* $j_\tau : \Gamma_3 \times \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfies:

$$\begin{cases} (\mathbf{a}) \text{ one can find a positive constant } M_\tau > 0 \text{ satisfying:} \\ \quad j_\tau^0(.,\vartheta_1,\vartheta_2-\vartheta_1) + j_\tau^0(.,\vartheta_2,\vartheta_2-\vartheta_1) \le M_\tau \|\vartheta_1-\vartheta_2\|^2_{\mathbb{R}^m}, \quad \forall \upsilon_1,\upsilon_2 \in \mathbb{R}^m, \text{a.e.in } \Gamma_3 \\ (\mathbf{b}) \text{ for a.e. } x \in \Gamma_3, \text{ the function } j_\tau(x,.) \text{ satisfies Lipschitz continuity on } \mathbb{R}^m \\ (\mathbf{c}) \text{ a bound } m_\tau > 0 \text{ can be found such that } |\partial j_\tau(.,\vartheta)| \le m_\tau, \quad \forall \vartheta \in \mathbb{R}^m \text{ a.e. in } \Gamma_3 \\ (\mathbf{d}) \text{ for a.e. } x \in \Gamma_3, \text{ the function } j_\tau(x,.) \text{ exhibits regularity} \\ (\mathbf{e}) \text{ the function } j_\tau(.,\vartheta) \text{ is measurable on } \Gamma_3, \quad \forall \vartheta \in \mathbb{R}^m \text{ and there exists} \\ \quad r \in L^2(\Gamma_3;\mathbb{R}^m) \text{ such that } j_\tau(.,r(.)) \in L^1(\Gamma_3) \end{cases} \tag{3.13}$$

The *thermal tensors* satisfy the following conditions :

$$\begin{cases} \mathcal{K} = k_{ij}, \quad k_{ij} = k_{ji} \in L^\infty(\Omega), \\ k_{ij}\zeta_i\zeta_i \ge m_{\mathcal{K}}\,\zeta_i\zeta_i \quad \text{ where, } m_{\mathcal{K}} > 0 \quad \forall \zeta_i \in \mathbb{R}^m. \end{cases} \tag{3.14}$$

The forces, the traction and the heat source satisfy the following regularities:

$$\mathbf{f}_0 \in \mathcal{C}(\mathrm{I}; L^2(\Omega)), \quad \mathbf{f}_2 \in \mathcal{C}(\mathrm{I}; L^2(\Gamma_2)^d), \quad \chi \in \mathcal{C}(\mathrm{I}; L^2(\Omega)). \tag{3.15}$$

The initial conditions, the thermal potential and the coefficient of heat exchange satisfy:

$$\mathbf{u}_0 \in \mathcal{V}, \quad \theta_0 \in \mathcal{W}, \quad \theta_F \in L^2(\mathrm{I}; L^2(\Gamma_3)), \text{ and } \kappa_e \in L^\infty(\Gamma_3, \mathbb{R}^+). \tag{3.16}$$

Using the Riesz representation theorem, we define the mappings $\mathbf{f} : [0,T] \to \mathcal{V}$, $h : [0,T] \longrightarrow \mathcal{W}^*$ as follows:

$$(\mathbf{f}(t), \vartheta)_{\mathcal{V}} = \int_\Omega \mathbf{f}_0(t)\vartheta\, dx + \int_{\Gamma_2} \mathbf{f}_2(t)\vartheta\, da, \tag{3.17}$$

$$(h(t), \zeta)_{\mathcal{W}^*\times\mathcal{W}} = \int_\Omega \chi(t)\zeta dx + \int_{\Gamma_3} \kappa_e\theta_F(t)\zeta da. \tag{3.18}$$

Next, we define the mappings $K : \mathcal{W} \longrightarrow \mathcal{W}^*$ by:

$$(K\theta, \xi)_{\mathcal{W}^*\times\mathcal{W}} = \int_\Omega \mathcal{K}\nabla\theta.\nabla\xi\, dx + \int_{\Gamma_3} \kappa_e\theta\xi da. \tag{3.19}$$

The operator K satisfies that :

$$\|K\xi_1 - K\xi_2\|_{\mathcal{W}'} \le M_k\|\xi_1-\xi_2\|_{\mathcal{W}}, \quad \forall \xi_1,\xi_2 \in \mathcal{W}, \text{ With } M_k > 0. \tag{3.20}$$

For all $\vartheta \in \mathcal{V}$ and $\zeta \in \mathcal{W}$, by applying Green's formulas and relying on the assumptions stated above, we derive the variational form of the mechanical Problem given in (2.5)–(2.15).

Problem $\mathcal{PV}$. Find *a displacement field* $u : \mathrm{I} \to \mathcal{V}$, *a stress field* $\boldsymbol{\sigma} : \mathrm{I} \to \mathcal{H}_1$ and *a temperature* $\theta : \mathrm{I} \to \mathcal{W}$, such that:

$$\sigma = \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right) + \mathcal{M}\big(\boldsymbol{\varepsilon}(u)\big) + \int_0^t \mathcal{S}\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big)dr, \tag{3.21}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\vartheta))_{\mathcal{H}} + \int_{\Gamma_3} j_\nu^0\left({}_0^C D_t^\alpha u_\nu(t); \vartheta_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left({}_0^C D_t^\alpha u_\tau(t); \vartheta_\tau\right) da \ge (\mathbf{f}(t), \vartheta)_{\mathcal{V}}, \tag{3.22}$$

$$\begin{aligned}\left({}_0^C D_t^\alpha \theta(t) + K\theta(t), \zeta\right)_{\mathcal{W}^*\times\mathcal{W}} = {} & \big(\psi\big(\boldsymbol{\sigma} - \mathcal{L}\varepsilon({}_0^C D_t^\alpha u(t)), \boldsymbol{\varepsilon}(u(t)), \theta(t)\big), \zeta\big) \\ & + (h(t), \zeta)_{\mathcal{W}^*\times\mathcal{W}}, \end{aligned} \tag{3.23}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0. \tag{3.24}$$

4. Existence and uniqueness result

The existence and uniqueness of the solution to the Problem $\mathcal{PV}$ is proven through the following.

Theorem 4.1. *Assume that the equations (3.8)–(3.16) and the conditions $m_{\mathcal{L}} > c_s^2(M_\nu + M_\tau)$ hold. Then, there exists a unique weak solution to the $\mathcal{PV}$ problem which also exhibits regularity.*

$$u \in W^{1,2}(\mathrm{I}; \mathcal{V}), \quad \boldsymbol{\sigma} \in W^{1,2}(\mathrm{I}; \mathcal{H}_1), \quad \theta \in W^{1,2}(\mathrm{I}; \mathcal{W}). \tag{4.1}$$

Theorem 4.1 is proven through a sequence of steps based on results from variational inequalities, the Caputo derivative, the Galerkin method and the Banach's fixed-point theorem.

In this section, we adopt the assumptions of Theorem (4.1) and assume that C denotes a generic positive constant that depends on the Problem data and may vary from line to line.

To begin, let $\pi \in \mathcal{C}(\mathbf{I}; \mathcal{H})$ be given. The Riesz representation theorem allows us to define the functional:

$$(f_\pi(t), \vartheta)_V = (f(t), \vartheta) - (\pi(t), \vartheta), \qquad \forall \vartheta \in \mathcal{V}. \tag{4.2}$$

In addition, we introduce the following auxiliary mechanical problem:

Problem $\mathcal{PV}_{1\pi}$. Find *a displacement field* $u_\pi : \mathrm{I} \to \mathcal{V}$ such that:

$$\begin{aligned} &\left(\mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u_\pi(t)\right), \boldsymbol{\varepsilon}(\vartheta)\right)_{\mathcal{H}} + \left(\mathcal{M}\big(\boldsymbol{\varepsilon}(u_\pi(t))\big), \boldsymbol{\varepsilon}(\vartheta)\right)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0\left({}_0^C D_t^\alpha u_{\nu\pi}(t); \vartheta_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left({}_0^C D_t^\alpha u_{\tau\pi}(t); \vartheta_\tau\right) da \geq (\mathbf{f}_\pi(t), \vartheta)_{\mathcal{V}}, \quad \forall \vartheta \in \mathcal{V}, \end{aligned} \tag{4.3}$$

$$u_\pi(0) = u_0. \tag{4.4}$$

We establish the following result in relation to $\mathcal{PV}_{1\pi}$:

Lemma 4.2. *The Problem $\mathcal{PV}_{1\pi}$ has a unique weak solution. Moreover. If u_{π_1} and u_{π_2} are two solutions of $\mathcal{PV}_{1\pi}$ corresponding to the data $\pi_1, \pi_2 \in \mathcal{C}(\mathrm{I}; \mathcal{H})$, then there exists $C > 0$ such that:*

$$\|u_{\pi_1}(t) - u_{\pi_2}(t)\|_{\mathcal{V}} \leq C\left(\|\pi_1(t) - \pi_2(t)\|_{\mathcal{H}}, \quad \forall\, t \in \mathrm{I}. \tag{4.5}$$

Proof. We set ${}_0^C D_t^\alpha u_\pi = w_\pi$. According to Proposition 3 in [16], we then have $u_\pi(t) = {}_0I_t^\alpha w_\pi(t) + u_0$, for a.e. $t \in \mathrm{I}$. Consequently, $\mathcal{PV}_{1\pi}$ may be reformulated in the following form:

Problem $\mathcal{PV}_{2\pi}$. Find *a displacement field* $w_\pi : \mathrm{I} \to \mathcal{V}$ such that:

$$\begin{aligned} &(\mathcal{L}\varepsilon(w_\pi(t)), \boldsymbol{\varepsilon}(\vartheta))_{\mathcal{H}} + \left(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_\pi(t) + u_0)\big), \boldsymbol{\varepsilon}(\vartheta)\right)_{\mathcal{H}} \\ &\int_{\Gamma_3} j_\nu^0\left(w_{\nu\pi}(t); \vartheta_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left(w_{\tau\pi}(t); \vartheta_\tau\right) da \geq (f_\pi(t), \vartheta)_{\mathcal{V}}, \quad \forall \vartheta \in \mathcal{V}, \\ &u_\pi(0) = u_0. \end{aligned} \tag{4.6}$$

This means that $u_\pi \in W^{1,2}(\mathrm{I}; \mathcal{V})$ is a solution to the Problem $\mathcal{PV}_{1\pi}$ if and only if $w_\pi \in L^1(0, T; \mathcal{V})$ solves the Problem $\mathcal{PV}_{2\pi}$.

We consider the mapping $J : L^2(\Gamma_3, \mathbb{R}^m) \longrightarrow \mathbb{R}$ defined by:

$$J(w) = \int_{\Gamma_3} (j_\nu(., w_\nu) + j_\tau(., w_\tau)) da \tag{4.7}$$

Based on Assumptions (3.12) and (3.13), together with Corollary 4.15 in [9], we deduce that the function J satisfies a local Lipschitz condition and that the following inequality holds:

$$J^0(w,\vartheta) \leq \int_{\Gamma_3} (j_\nu^0(.,w_\nu;\vartheta_\nu) + j_\tau^0(.,w_\tau;\vartheta_\tau))da, \quad \forall u,\vartheta \in L^2(\Gamma_3,\mathbb{R}^m). \tag{4.8}$$

Now, consider the following auxiliary Problem, for all $\vartheta \in \mathcal{V}$, find $w_\pi : \mathrm{I} \to \mathcal{V}$ such that:

$$\begin{aligned}(\mathcal{L}\varepsilon(w_\pi(t)), \boldsymbol{\varepsilon}(\vartheta))_{\mathcal{H}} + \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_\pi(t) + u_0)\big), \boldsymbol{\varepsilon}(\vartheta)\big)_{\mathcal{H}}& \\ +J^0(\gamma w(t);\gamma\vartheta) \geq (f_\pi(t),\vartheta)_{\mathcal{V}}\,,&\end{aligned} \tag{4.9}$$

where the operator $\gamma : \mathcal{V} \longrightarrow L^2(\Gamma_3,\mathbb{R}^m)$ is compact and represents the trace operator. It follows from Assumption (3.8) that the operator $\mathcal{L}$ belongs to $\mathcal{L}(\mathcal{V},\mathcal{V}^*)$ and is coercive. According to Assumption (3.9), the operator $\mathcal{M}$ belongs to $\mathcal{L}(\mathcal{V},\mathcal{V}^*)$. Moreover, based on hypotheses (3.15), (3.17), and the regularity of π, we have $f_\pi \in \mathcal{C}(\mathrm{I};\mathcal{V}^*)$. Using Assumptions (3.12) and (3.13), together with Corollary 4.15 in [9], we conclude that:

$$\|\partial J(\vartheta)\|_{\mathcal{V}^*} \leq C(1 + \|\vartheta\|_{\mathcal{V}}), \quad \forall \vartheta \in \mathcal{V}. \tag{4.10}$$

At this point, all the assumptions of Theorem 16 in [8] have been verified. Therefore, Problem (4.9) admits a unique solution $w_\pi \in L^1(\mathrm{I};\mathcal{V})$. Consequently, w_π is the unique solution to Problem $\mathcal{PV}_{2\pi}$. In conclusion, we deduce that $u_\pi \in W^{1,2}(\mathrm{I};\mathcal{V})$ constitutes the unique weak solution to Problem $\mathcal{PV}_{1\pi}$.

On the other hand, suppose that $w_{\pi_1}, w_{\pi_2} \in \mathcal{V}$ are two solutions to Problem $\mathcal{PV}_{2\pi}$. Then, for all $\vartheta \in \mathcal{V}$, it follows that:

$$\begin{aligned}(\mathcal{L}\varepsilon(w_{\pi_1}(t)), \boldsymbol{\varepsilon}(\vartheta))_{\mathcal{H}} + \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_{\pi_1}(t) + u_0)\big), \boldsymbol{\varepsilon}(\vartheta)\big)_{\mathcal{H}}& \\ +\int_{\Gamma_3} j_\nu^0\left(w_{\nu\pi_1}(t);\vartheta_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left(w_{\tau\pi_1}(t);\vartheta_\tau\right) da \geq (f_{\pi_1}(t),\vartheta)_{\mathcal{V}}\,,&\end{aligned} \tag{4.11}$$

$$\begin{aligned}(\mathcal{L}\varepsilon(w_{\pi_1}(t)), \boldsymbol{\varepsilon}(\vartheta))_{\mathcal{H}} + \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_{\pi_1}(t) + u_0)\big), \boldsymbol{\varepsilon}(\vartheta)\big)_{\mathcal{H}}& \\ +\int_{\Gamma_3} j_\nu^0\left(w_{\nu\pi_1}(t);\vartheta_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left(w_{\tau\pi_1}(t);\vartheta_\tau\right) da \geq (f_{\pi_1}(t),\vartheta)_{\mathcal{V}}\,.&\end{aligned} \tag{4.12}$$

Take $\vartheta = w_{\pi_2} - w_{\pi_1}$ in (4.11) and $\vartheta = w_{\pi_1} - w_{\pi_2}$ in (4.12) to obtain:

$$\begin{aligned}(\mathcal{L}\varepsilon(w_{\pi_1}(t)), \boldsymbol{\varepsilon}(w_{\pi_2} - w_{\pi_1}))_{\mathcal{H}} + \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_{\pi_1}(t) + u_0)\big), \boldsymbol{\varepsilon}(w_{\pi_2} - w_{\pi_1})\big)_{\mathcal{H}}& \\ \int_{\Gamma_3} j_\nu^0\left(w_{\nu\pi_1}(t);(w_{\pi_2} - w_{\pi_1})_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left(w_{\tau\pi_1}(t);(w_{\pi_2} - w_{\pi_1})_\tau\right) da& \\ \geq (f_\pi(t), w_{\pi_2} - w_{\pi_1})_{\mathcal{V}}\,,&\end{aligned} \tag{4.13}$$

$$\begin{aligned}(\mathcal{L}\varepsilon(w_{\pi_2}(t)), \boldsymbol{\varepsilon}(w_{\pi_1} - w_{\pi_2}))_{\mathcal{H}} + \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^\alpha w_{\pi_2}(t) + u_0)\big), \boldsymbol{\varepsilon}(w_{\pi_1} - w_{\pi_2})\big)_{\mathcal{H}}& \\ \int_{\Gamma_3} j_\nu^0\left(w_{\nu\pi_2}(t);(w_{\pi_1} - w_{\pi_2})_\nu\right) da + \int_{\Gamma_3} j_\tau^0\left(w_{\tau\pi_2}(t);(w_{\pi_1} - w_{\pi_2})_\tau\right) da& \\ \geq (f_\pi(t), w_{\pi_1} - w_{\pi_2})_{\mathcal{V}}\,.&\end{aligned} \tag{4.14}$$

By combining the two inequalities mentioned above, we get:

$$\begin{aligned}
&(\mathcal{L}\varepsilon(w_{\eta_1}(t)) - \mathcal{L}\varepsilon(w_{\eta_2}(t)), \boldsymbol{\varepsilon}(w_{\eta_2}(t) - w_{\eta_1}(t)))_{\mathcal{H}} \\
&\leq \big(\mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^{\alpha} w_{\eta_1}(t) + u_0)\big) - \mathcal{M}\big(\boldsymbol{\varepsilon}({}_0I_t^{\alpha} w_{\eta_2}(t) + u_0)\big), \boldsymbol{\varepsilon}(w_{\eta_2}(t) - w_{\eta_1}(t))\big)_{\mathcal{H}} \\
&\quad + (\pi_1(t) - \pi_2(t), w_{\pi_2}(t) - w_{\pi_1}(t)) + R(w_{\pi_1}, w_{\pi_2}; w_{\pi_1}, w_{\pi_2}),
\end{aligned} \tag{4.15}$$

where:

$$\begin{aligned}
R(w_{\pi_1}, w_{\pi_2}; w_{\pi_1}, w_{\pi_2}) &= \int_{\Gamma_3} j_\tau^0\,(w_{\tau\pi_2}(t); (w_{\pi_1} - w_{\pi_2})_\tau)\, da \\
&+ \int_{\Gamma_3} j_\tau^0\,(w_{\tau\pi_1}(t); (w_{\pi_2} - w_{\pi_1})_\tau)\, da + \int_{\Gamma_3} j_\nu^0\,(w_{\nu\pi_2}(t); (w_{\pi_1} - w_{\pi_2})_\nu)\, da \\
&+ \int_{\Gamma_3} j_\nu^0\,(w_{\nu\pi_1}(t); (w_{\pi_2} - w_{\pi_1})_\nu)\, da.
\end{aligned} \tag{4.16}$$

Using assumptions (3.12), (3.13) and (3.5), we find:

$$|R(w_{\pi_1}, w_{\pi_2}; w_{\pi_1}, w_{\pi_2})| \leq c_s^2(M_\nu + M_\tau)\|w_{\eta_1} - w_{\eta_2}\|_{\mathcal{V}}^2 \tag{4.17}$$

From hypothesis (3.8), (3.9) and (4.17), we find:

$$\begin{aligned}
m_{\mathcal{L}}\|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}}^2 &\leq c_s^2(M_\nu + M_\tau)\|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}}^2 \\
&+ M_{\mathcal{M}}\|{}_0I_t^{\alpha} w_{\pi_1}(t) - {}_0I_t^{\alpha} w_{\pi_2}(t)\|_V \|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}} \\
&+ \|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}}\|\pi_1(t) - \pi_2(t)\|_{\mathcal{H}}.
\end{aligned} \tag{4.18}$$

Hence, applying a fractional generalized Gronwall argument under the condition $m_{\mathcal{L}} > c_s^2(M_\nu + M_\tau)$ yields:

$$\|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}} \leq C\|\pi_1(t) - \pi_2(t)\|_{\mathcal{H}}. \tag{4.19}$$

On the other hand, taking into account the relation $u_\pi(t) = {}_0I_t^{\alpha} w_\pi(t) + u_0$, we derive the following estimate:

$$\begin{aligned}
\|u_{\pi_1}(t) - u_{\pi_2}(t)\|_{\mathcal{V}} =& \|{}_0I_t^{\alpha} w_{\pi_1}(t) - {}_0I_t^{\alpha} w_{\pi_2}(t)\|_{\mathcal{V}} \\
\leq& \frac{t^\alpha}{\alpha\Gamma(\alpha)}\|w_{\pi_1}(t) - w_{\pi_2}(t)\|_{\mathcal{V}}.
\end{aligned} \tag{4.20}$$

From (4.19) and (4.20), we arrive at (4.5), thus completing the proof of Lemma 4.2. □

In the next step, let $\mu \in C(\mathrm{I}; \mathcal{W})$ be given, we consider the following variational Problem:

Problem $\mathcal{PV}_\mu$. Find *a temperature* $\theta_\mu : \mathrm{I} \longrightarrow \mathcal{W}$ such that:

$$\big({}_0^C D_t^{\alpha}\theta_\mu(t) + K\theta_\mu(t), \zeta\big)_{\mathcal{W}'\times\mathcal{W}} = (\mu(t) + h(t), \zeta)_{\mathcal{W}'\times\mathcal{W}}, \quad \forall\zeta \in \mathcal{W}, \tag{4.21}$$

$$\theta_\mu(0) = \theta_0. \tag{4.22}$$

Lemma 4.3. *There exists a unique solution $\theta_\mu \in W^{1,2}(\mathrm{I}; \mathcal{W})$ that satisfies (4.21) and (4.22). Moreover, if θ_{μ_1} and θ_{μ_2} are two solutions of $\mathcal{PV}_\mu$, then there exists a constant $C > 0$ such that*

$$\|\theta_{\mu_1}(t) - \theta_{\mu_2}(t)\|_{\mathcal{W}} \leq C\,\|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}}, \quad \forall t \in \mathrm{I}. \tag{4.23}$$

Proof. We proceed by employing the Galerkin method. Let $\{\lambda_j\}_{j=1}^{\infty}$ denote an orthogonal basis of $\mathcal{W}$, which simultaneously serves as an orthonormal basis of $L^2(\Omega)$ (see [4]). Fix a fixed positive integer n, we introduce the function $\theta\mu_n : \mathrm{I} \to \mathcal{W}$ defined by:

$$\theta_{\mu_n}(t) = \sum_{i=1}^{n} x_n^i(t)\lambda_i. \tag{4.24}$$

We denote by W_n the finite-dimensional subspace spanned by $\lambda_1, \ldots, \lambda_n$. Consequently, $\theta_{\mu_n} \in W_n$ and $\theta_{\mu_n} \longrightarrow \theta_\mu$ in $\mathcal{W}$.

For each positive integer n, we consider the following approximate problem:

Find $\theta_{\mu_n} : \mathrm{I} \longrightarrow W_n$ such that:

$$\left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \lambda_j\right) + (K\theta_{\mu_n}(t), \lambda_j) = (\mu(t) + h(t), \lambda_j), \quad \forall \lambda_j \in W_n, \tag{4.25}$$

$$\theta_{\mu_n}(0) = \theta_0. \tag{4.26}$$

From (4.24), we have:

$$\begin{aligned}
({}_0^C D_t^\alpha \theta_{\mu_n}(t), \lambda_j) &= {}_0^C D_t^\alpha x_n^i(t),\\
(K\theta_{\mu_n}, \lambda_j) &= K x_n^i(t),\\
(\mu(t) + h(t), \lambda_j) &= \mu_i(t) + h_i(t).
\end{aligned}$$

Problems (4.25) and (4.26) can be written as follows:

$${}_0^C D_t^\alpha x_n^i(t) = g(t, x_n^i(t)), \tag{4.27}$$

$$x_n^i(0) = (\theta_0, \lambda_j), \tag{4.28}$$

where $g(t, x_n^i(t) = \mu_i(t) + h_i(t) - K x_n^i(t)$.

It follows from assumption (3.20) that there exists a constant $C > 0$ satisfying:

$$|g(t, x_{i_1}(t) - g(t, x_{i_2}(t)| \leq C|x_{i_1} - x_{i_2}|. \tag{4.29}$$

Then, by applying a standard result from fractional calculus, we conclude the existence of a unique absolutely continuous function $x_n(t) = (x_n^1(t), \ldots, x_n^n(t))$ that satisfies the equations (4.27) and (4.28) for a.e. $t \in [0, T]$.

Estimates: Multiply (4.25) by $x_n^i(t)$, for $i = 1, ..., n$. We find:

$$\left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \theta_{\mu_n}(t)\right) + (K\theta_{\mu_n}(t), \theta_{\mu_n}(t)) = (\mu(t) + h(t), \theta_{\mu_n}(t)). \tag{4.30}$$

Since $\frac{1}{2}\|u\|^2$ is a convex function, it follows from Proposition 2.2 in [8] that

$${}_0^C D_t^\alpha \frac{1}{2}\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2 \leq \left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \theta_{\mu_n}(t)\right). \tag{4.31}$$

Applying the inequality:

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \qquad \text{for } \epsilon > 0, \tag{4.32}$$

we obtain:

$$(\mu(t) + h(t), \theta_{\mu_n}(t)) \leq \frac{1}{4\epsilon}\|\mu(t) + h(t)\|_{\mathcal{W}}^2 + \epsilon\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2. \tag{4.33}$$

From (4.31), (4.33) and using assumption (3.14), we conclude that:

$${}_0^C D_t^\alpha \frac{1}{2}\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2 + C_1\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2 \leq C_2\|\mu(t) + h(t)\|_{\mathcal{W}}^2. \tag{4.34}$$

As well, according to proposition 3 in [16], we obtain:

$$\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2 + \frac{2C_1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}\|\theta_{\mu_n}(s)\|_{\mathcal{W}}^2 dr \leq C_3\left(\|\mu(t)+h(t)\|_{\mathcal{W}}^2 + \|\theta_0\|_{\mathcal{W}}^2\right). \quad (4.35)$$

Let $\zeta \in \mathcal{W}$, with $\|\zeta\|_{\mathcal{W}} \leq 1$, and write $\zeta = \zeta_1 + \zeta_2$ where $\zeta_1 \in \operatorname{span}\{\lambda_i\}_{i=1}^n$ and $(\zeta_2, \lambda_i) = 0$ for $i = 1, ..., n$. Using (4.25), we find:

$$\left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \zeta_1\right) + (K\theta_{\mu_n}(t), \zeta_1) = (\mu(t) + h(t), \zeta_1). \quad (4.36)$$

Similarly to (4.34), we conclude that:

$$\|{}_0^C D_t^\alpha \theta_{\mu_n}(t)\|_{\mathcal{W}^*} \leq C\left(\|\theta_{\mu_n}(t)\|_{\mathcal{W}} + \|\mu(t)+h(t)\|_{\mathcal{W}}\right). \quad (4.37)$$

From (4.35), we deduce that there exists a constant $C > 0$ such that:

$$\|{}_0^C D_t^\alpha \theta_{\mu_n}(t)\|_{L^2(\mathrm{I};\mathcal{W}^*)} \leq C. \quad (4.38)$$

Passage to the limit: According to estimates (4.35) and (4.38), we observe that the sequence $\{\theta_{\mu_n}\}_{n=1}^\infty$ is bounded in $L^2(\mathrm{I};\mathcal{W})$, and the sequence $\left\{{}_0^C D_t^\alpha \theta_{\mu_n}\right\}_{n=1}^\infty$ is bounded in $L^2(\mathrm{I};\mathcal{W}^*)$.

Consequently (see Theorem 4.2 in [8]), there exist a subsequence $\{\theta_{\mu_{n_l}}\}_{l=1}^\infty \subset \{\theta_{\mu_n}\}_{n=1}^\infty$ and a function $\theta_\mu \in L^2(\mathrm{I};\mathcal{W})$, with ${}_0^C D_t^\alpha \theta_\mu \in L^2(\mathrm{I};\mathcal{W}^*)$, such that:

$$\begin{aligned} \theta_{\mu_{\chi n}} &\longrightarrow \theta_\mu \quad \text{strongly in } L^2(\mathrm{I};\mathcal{W}), \\ {}_0^C D_t^\alpha \theta_{\mu_{\chi n}} &\longrightarrow {}_0^C D_t^\alpha \theta_\mu \quad \text{weakly in } L^2(\mathrm{I};\mathcal{W}^*). \end{aligned} \quad (4.39)$$

The frist part of the lemma is proved. For the second part, let $\theta_{\mu_1}, \theta_{\mu_2} \in \mathcal{W}$ be two solutions of the Problem $\mathcal{PV}_\mu$. From (4.21), we can write:

$$\begin{aligned} &\left({}_0^C D_t^\alpha \theta_{\mu_1}(t) - {}_0^C D_t^\alpha \theta_{\mu_2}(t), \zeta\right)_{\mathcal{W}^*\times\mathcal{W}} + (K\theta_{\mu 1}(t) - K\theta_{\mu_2}(t), \zeta)_{\mathcal{W}^*\times\mathcal{W}} \\ &\qquad = (\mu_1(t) - \mu_2(t), \zeta)_{\mathcal{W}^*\times\mathcal{W}}. \end{aligned} \quad (4.40)$$

Let ${}_0^C D_t^\alpha \theta_\mu = \varsigma_\mu$. Then, for a.e. $t \in \mathrm{I}$, we have $\theta_\mu(t) = {}_0I_t^\alpha \varsigma_\mu(t) + \theta_0$.
By choosing $\zeta = \varsigma_{\mu_1} - \varsigma_{\mu_2}$ as a test function, we deduce that:

$$\begin{aligned} &(\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t))_{\mathcal{W}^*\times\mathcal{W}} = (\mu_1(t) - \mu_2(t), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t))_{\mathcal{W}^*\times\mathcal{W}} \\ &- \left(K\left({}_0I_t^\alpha \varsigma_{\mu_1}(t) + \theta_0\right) - K\left({}_0I_t^\alpha \varsigma_\mu(t)_{\mu_2} + \theta_0\right), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\right)_{\mathcal{W}^*\times\mathcal{W}}. \end{aligned} \quad (4.41)$$

Using the hypothesis (3.20), we find:

$$\|\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\|_{\mathcal{W}} \leq M_k\|{}_0I_t^\alpha \varsigma_{\mu_1}(t) - {}_0I_t^\alpha \varsigma_\mu(t)_{\mu_2}(t)\|_{\mathcal{W}} + \|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}} \quad (4.42)$$

Then, using a fractional generalized Gronwall argument, we obtain the following estimate:

$$\|\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\|_{\mathcal{W}} \leq C\|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}}. \quad (4.43)$$

Using arguments similar to those in (4.20), we can deduce the inequality (4.23). □

We now employ u_π, as obtained in Lemma 4.2, together with θ_μ, from Lemma 4.3, to formulate the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\pi\mu}$. Find *a stress field* $\boldsymbol{\sigma}_{\pi\mu} : \mathrm{I} \to \mathcal{H}$ that satisfies the following Problem:

$$\boldsymbol{\sigma}_{\pi\mu}(t) = \mathcal{M}(\varepsilon(u_\pi(t))) + \int_0^t \mathcal{S}\Big(\boldsymbol{\sigma}_{\pi\mu}(r), \varepsilon(u_\pi(r)), \theta_\mu(r)\Big) dr, \quad \text{a.e. } t \in \mathrm{I}. \quad (4.44)$$

Lemma 4.4. *There exists a unique solution to the problem $\mathcal{P}_{\pi\mu}$ that satisfies $\boldsymbol{\sigma}_{\pi\mu} \in \mathcal{C}(\mathrm{I};\mathcal{H})$.*

Moreover, if $u_{\pi_i}, \theta_{\mu_i}$ and $\boldsymbol{\sigma}_{\pi_i\mu_i}$ represent the solutions to the problems $\mathcal{PV}_{1\pi_i}$,$\mathcal{PV}_{\mu_i}$ and $\mathcal{P}_{\pi_i\mu_i}$, respectively, for $(\pi_i, \mu_i) \in \mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$, $i = 1, 2$, then there exists $C > 0$ such that:

$$\|\boldsymbol{\sigma}_{\pi_1\mu_1}(t) - \boldsymbol{\sigma}_{\pi_2\mu_2}(t)\|^2_{\mathcal{H}} \leq C\Big(\|u_{\pi_1}(t) - u_{\pi_2}(t)\|^2_{\mathcal{V}} + \int_0^t \|u_{\pi_1}(r) - u_{\pi_2}(r)\|^2_{\mathcal{V}} dr + \int_0^t \|\theta_{\pi_1}(r) - \theta_{\pi_2}(r)\|^2_{L^2(\Omega)} dr\Big). \tag{4.45}$$

Proof. For all $\boldsymbol{\sigma} \in \mathcal{C}(\mathrm{I};\mathcal{H})$, let $\Im_{\pi\mu} : \mathcal{C}(\mathrm{I};\mathcal{H}) \to \mathcal{C}(\mathrm{I};\mathcal{H})$ be the mapping defined by:

$$\Im_{\pi\mu}\boldsymbol{\sigma}(t) = \mathcal{M}(\varepsilon(u_\pi(t)) + \int_0^t \mathcal{S}(\boldsymbol{\sigma}(r), \boldsymbol{\varepsilon}(u_\pi(r)), \theta_\mu(r))dr, \tag{4.46}$$

For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in \mathcal{C}(\mathrm{I};\mathcal{H})$, we use (3.10) and (4.46) to obtain:

$$\|\Im_{\pi\mu}\boldsymbol{\sigma}_1(t) - \Im_{\pi\mu}\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq M_{\mathcal{S}} \int_0^t \|\boldsymbol{\sigma}_1(r) - \boldsymbol{\sigma}_2(r)\|_{\mathcal{H}} dr. \tag{4.47}$$

Thus, for sufficiently large n, the operator $\Im^n_{\pi\mu}$ is a contraction on. Accordingly, by the Banach fixed-point theorem, there exists a unique $\sigma \in \mathcal{C}(\mathrm{I};\mathcal{H})$ such that $\Im_{\pi\mu}\sigma = \sigma$, which confirms that σ is precisely the unique solution of the problem $\mathcal{PV}_{\pi\mu}$.

Consider now (π_1, μ_1), $(\pi_2, \mu_2) \in \mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$. Using the properties (3.9) and (3.10) of $\mathcal{M}$ and $\mathcal{S}$, we find:

$$\|\boldsymbol{\sigma}_{\pi_1\mu_1}(t) - \boldsymbol{\sigma}_{\pi_2\mu_2}(t)\|^2_{\mathcal{H}} \leq C\Big(\|u_{\pi_1}(t) - u_{\pi_2}(t)\|^2_{\mathcal{V}} + \int_0^t \|\boldsymbol{\sigma}_{\pi_1\mu_1}(r) - \boldsymbol{\sigma}_{\pi_2\mu_2}(r)\|^2_{\mathcal{H}} dr + \int_0^t \|u_{\pi_1}(r) - u_{\pi_2}(r)\|^2_{\mathcal{V}} dr + \int_0^t \|\theta_{\mu_1}(r) - \theta_{\mu_2}(r)\|^2_{\mathcal{W}} dr\Big). \tag{4.48}$$

Applying Gronwall's inequality to the obtained estimate yields (4.45), thereby completing the proof of Lemma 4.4.

□

In the final step, we introduce the operator $\mathbf{\Pi} : \mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W}) \longrightarrow \mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$, defined by:

$$\mathbf{\Pi}(\pi, \mu)(t) = (\mathbf{\Pi}^1(\pi, \mu)(t), \mathbf{\Pi}^2(\pi, \mu)(t)), \tag{4.49}$$

with:

$$(\mathbf{\Pi}^1(\pi, \mu)(t), \vartheta)_{\mathcal{V}} = \Big(\int_0^t \mathcal{S}\Big(\boldsymbol{\sigma}_{\eta\mu}(r), \boldsymbol{\varepsilon}(u_\pi(r)), \theta_\mu(r)\Big)dr, \boldsymbol{\varepsilon}(\vartheta)\Big)_{\mathcal{H}} \quad \forall\, \vartheta \in \mathcal{V}, \tag{4.50}$$

$$\mathbf{\Pi}^2(\pi, \mu)(t) = \psi\big(\boldsymbol{\sigma}_{\pi\mu}(t), \boldsymbol{\varepsilon}(u_\pi(t)), \theta_\mu(t)\big). \tag{4.51}$$

Here, u_π, θ_μ, and $\boldsymbol{\sigma}_{\pi\mu}$ represent the displacement, temperature, and stress fields, respectively, obtained in Lemmas 4.2, 4.3, and 4.4.

Lemma 4.5. *The operator $\mathbf{\Pi}$ has a unique fixed point $(\pi^*, \mu^*) \in \mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$ such that: $\mathbf{\Pi}(\pi^*, \mu^*) = (\pi^*, \mu^*)$.*

Proof. Let $(\pi_1,\mu_1),(\pi_2,\mu_2)\in\mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$. For simplicity, we denote $u_{\pi_i}=u_i$, $\boldsymbol{\sigma}_{\pi_i\mu_i}=\boldsymbol{\sigma}_i$, and $\theta_{\mu_i}=\theta_i$, for $i=1,2$. We begin by applying hypothesis (3.10), which yields:

$$\|\boldsymbol{\Pi}^1(\pi_1,\mu_1)(t)-\boldsymbol{\Pi}^1(\pi_2,\mu_2)(t)\|^2_{\mathcal{H}}\le C\Big(\int_0^t\|\boldsymbol{\sigma}_1(r)-\boldsymbol{\sigma}_2(r)\|^2_{\mathcal{H}}ds \tag{4.52}$$

$$+\int_0^t\|u_1(r)-u_2(r)\|^2_{\mathcal{V}}dr+\int_0^t\|\theta_1(r)-\theta_2(r)\|^2_{\mathcal{W}}dr\Big). \tag{4.53}$$

From estimates (4.45), we find:

$$\begin{aligned}\|\boldsymbol{\Pi}^1(\pi_1,\mu_1)(t)-\boldsymbol{\Pi}^1(\pi_2,\mu_2)(t)\|^2_{\mathcal{H}}&\le C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}}\\&+\int_0^t\|u_1(r)-u_2(r)\|^2_{\mathcal{V}}dr+\int_0^t\|\theta_1(r)-\theta_2(r)\|^2_{\mathcal{W}}dr\Big).\end{aligned} \tag{4.54}$$

In addition, using similar arguments to (4.54), we find:

$$\begin{aligned}&\|\boldsymbol{\Pi}^2(\pi_1,\mu_1)(t)-\boldsymbol{\Pi}^2(\pi_2,\mu_2)(t)\|^2_{\mathcal{H}}\le C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}}\\&+\int_0^t\|u_1(r)-u_2(r)\|^2_{\mathcal{V}}dr+\|\theta_1(t)-\theta_2(t)\|^2_{\mathcal{W}}+\int_0^t\|\theta_1(r)-\theta_2(r)\|^2_{\mathcal{W}}dr\Big).\end{aligned} \tag{4.55}$$

Consequently:

$$\begin{aligned}&\|\boldsymbol{\Pi}(\pi_1,\mu_1)(t)-\boldsymbol{\Pi}(\pi_2,\mu_2)(t)\|^2_{\mathcal{H}\times\mathcal{W}}\le C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}}\\&+\int_0^t\|u_1(r)-u_2(r)\|^2_{\mathcal{V}}dr+\|\theta_1(t)-\theta_2(t)\|^2_{\mathcal{W}}+\int_0^t\|\theta_1(r)-\theta_2(r)\|^2_{\mathcal{W}}dr\Big).\end{aligned} \tag{4.56}$$

By substituting (4.5) and (4.23) into (4.56), we obtain:

$$\|\boldsymbol{\Pi}(\pi_1,\mu_1)(t)-\boldsymbol{\Pi}(\pi_2,\mu_2)(t)\|^2_{\mathcal{H}\times\mathcal{W}}\le C\int_0^t\|(\pi_1,\mu_1)(r)-(\pi_2,\mu_2)(r)\|^2_{\mathcal{H}\times\mathcal{W}}dr.$$

By applying this inequality n times, we obtain:

$$\|\boldsymbol{\Pi}^n(\pi_1,\mu_1)-\boldsymbol{\Pi}^n(\pi_2,\mu_2)\|^2_{\mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})}\le\frac{C^nT^n}{n!}\|(\pi_1,\mu_1)-(\pi_2,\mu_2)\|^2_{\mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W}}.$$

Once n becomes sufficiently large, the operator $\boldsymbol{\Pi}^n$ is shown to be a contraction on the Banach space $\mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$. The Banach fixed-point theorem then ensures that $\boldsymbol{\Pi}$ admits a unique fixed point in this space. □

We now proceed to prove Theorem 4.1, having established all the necessary preliminary results.
Let $(\pi^*,\mu^*)\in\mathcal{C}(\mathrm{I};\mathcal{H}\times\mathcal{W})$ be the fixed point of $\boldsymbol{\Pi}$ defined by (4.50),(4.51) and denote:

$$u=u_{\pi^*},\quad \theta=\theta_{\mu^*} \tag{4.57}$$

$$\boldsymbol{\sigma}=\mathcal{L}\varepsilon\left({}_0^CD_t^\alpha u(t)\right)+\boldsymbol{\sigma}_{\pi^*\mu^*}, \tag{4.58}$$

Let us rewrite (4.44) by specifying $\pi=\pi^*,\mu=\mu^*$ and let us use (4.57), (4.58) to deduce (3.21).

Next, we consider (4.3) and (4.30) with $\pi=\pi^*,\mu=\mu^*$ and employ the identities $\boldsymbol{\Pi}^1(\pi,\mu)(t)=\pi$ and $\boldsymbol{\Pi}^2(\pi,\mu)(t)=\mu$, together with (4.50), (4.51) and (4.57), to conclude that (3.22) and (3.23) are satisfied. Furthermore, (3.24) as well as the regularities stated in (4.1) are deduced from Lemmas 4.2, 4.3 and 4.4 in addition to the relations (4.57) and (4.58) and the assumptions on $\mathcal{L}$.

The uniqueness of the solution is deduced from the uniqueness of the fixed point of the operator $\mathbf{\Pi}$, along with the unique solvability of the auxiliary Problems $\mathcal{PV}_{1\pi}, \mathcal{PV}_{\mu}$ and $\mathcal{PV}_{\pi\mu}$, leading to the desired proof.

5. Conclusion

In this work, we studied a mathematical model describing frictional contact between a viscoplastic body and a deformable foundation, incorporating thermal effects and fractional time derivatives to account for long-memory behavior. We formulated the associated variational Problem and proved the existence and uniqueness of a weak solution using tools such as the Galerkin method and the Banach fixed-point theorem.

The proposed analysis provides a rigorous theoretical foundation for understanding complex thermo-viscoplastic contact phenomena. Future work will focus on numerical simulations to gain further practical insights into the behavior of the system. Moreover, the model may be extended to include more realistic features such as adhesion and wear, thus broadening its applicability to engineering and material science problems.

References

[1] Bouallala, M., & Essoufi, E. H. (2021). A thermo-viscoelastic fractional contact problem with normal compliance and Coulomb's friction. *Journal of Mathematical Physics, Analysis, Geometry, 17*(3), 280–294. https://doi.org/10.15407/mag17.03.280

[2] Bouallala, M., Essoufi, E. H., & Ouafik, Y. (2025). Analysis and numerical simulation of time-fractional derivative contact problem with friction in thermo-viscoelasticity. *Computational Methods in Applied Mathematics, 25*(1), 61–76.https://doi.org/10.1515/cmam-2023-0192

[3] Brezis, H. (1968). Equations et inéquations non linéaires dans les espaces vectoriels en dualité. *Annales de l'Institut Fourier, 18*(1), 115–175.

[4] Evans, L. C. (2022). *Partial differential equations* (Vol. 19). American Mathematical Society.

[5] Herrmann, R. (2011). *Fractional calculus: An introduction for physicists*. World Scientific.

[6] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (Vol. 204). Elsevier.

[7] L'Hôpital, G. (1715). *Analyse des infiniment petits*. François Montalant.

[8] Li, L., & Liu, J.-G. (2018). Some compactness criteria for weak solutions of time fractional PDEs. *SIAM Journal on Mathematical Analysis, 50*(4), 3963–3995. https://doi.org/10.1137/17M1145549

[9] Migórski, S., Ochal, A., & Sofonea, M. (2012). *Nonlinear inclusions and hemivariational inequalities: Models and analysis of contact problems* (Vol. 26). Springer.

[10] Podlubny, I. (1998). *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Vol. 198). Elsevier.

[11] Selmani, M., & Selmani, L. (2012). Analysis of a frictionless contact problem for elastic-viscoplastic material. *Nonlinear Analysis: Modelling and Control, 17*(1), 99–117. https://doi.org/10.15388/NA.17.1.14081

[12] Selmani, M., & Selmani, L. (2013). Frictional contact problem for elastic-viscoplastic materials with thermal effect. *Applied Mathematics and Mechanics, 34*(6), 761–776. https://doi.org/10.1007/s10483-013-1705-7

[13] Sofonea, M., & Matei, A. (2012). *Mathematical models in contact mechanics*. Cambridge University Press.

[14] Weng, Y., Chen, T., Li, X., & Huang, N. (2021). Rothe method and numerical analysis for a new class of fractional differential hemivariational inequality with an application.

Computers & Mathematics with Applications, 98, 118–138.https://doi.org/10.1016/j.camwa.2021.07.003

[15] Xuan, H., Cheng, X., & Yuan, L. (2025). Numerical studies of a class of thermo-viscoelastic frictional contact problem described by fractional differential hemivariational inequalities. *Journal of Scientific Computing, 103*(4). https://doi.org/10.1007/s10915-025-02815-2

[16] Zeng, S., & Migórski, S. (2018). A class of time-fractional hemivariational inequalities with application to frictional contact problem. *Communications in Nonlinear Science and Numerical Simulation, 56*, 34–48. https://doi.org/10.1016/j.cnsns.2017.07.016

[1] Ferhat Abbas, Setif 1 University, Applied Mathematics Laboratory. Department of Mathematics, Setif, 19000, Algeria.
Email address: achab.hamid@univ-setif.dz

[2] Ferhat Abbas, Setif 1 University, Applied Mathematics Laboratory. Department of Mathematics, Setif, 19000, Algeria.
Email address: ilyes.boukaroura@univ-setif.dz

[3] Ferhat Abbas, Setif 1 University, Applied Mathematics Laboratory. Department of Mathematics, Setif, 19000, Algeria.
Email address: Abdelmoumene.djabi@univ-setif.dz