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A SUBFAMILY OF BI-UNIVALENT FUNCTIONS INVOLVING THE RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE SUBORDINATE TO THE SINE FUNCTION

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Abstract. In this study, we provide a class of bi-univalent functions defined via the Riemann–Liouville subordination derivative with the sine function. we establish that this class is non-empty and provide best bounds for the initial coefficients, alongside the maximal estimate for the Fekete–Szegö functional.

Keywords. Fekete–Szegö inequalities, initial coefficient problems, sine function, bi-univalent functions.
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1. Introduction

In this paper, we use the standard notation by $\mathcal{A}$ the collection of functions that are analytic in the open unit disk $\mathcal{U}_d = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. A function $\mathfrak{g} \in \mathcal{A}$ admits the normalized Taylor series expansion of the form

$$\mathfrak{g}(\varsigma) = \varsigma + \sum_{l=2}^{\infty} \mathfrak{a}_l \varsigma^l, \quad \varsigma \in \mathcal{U}_d. \tag{1.1}$$

Furthermore, we indicate by $\mathcal{S}$ the subset of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}_d$. It is well-known that every function $\mathfrak{g} \in \mathcal{S}$ admits an inverse function $\mathfrak{g}^{-1}$, a fact ensured by the Koebe $(1/4)-$theorem. Specifically, one has

$$\mathfrak{g}^{-1}(\mathfrak{g}(\varsigma)) = \varsigma \quad \text{and} \quad \mathfrak{g}(\mathfrak{g}^{-1}(\nu)) = \nu, \qquad |\nu| < r_0(\mathfrak{g}),$$

where $r_0(\mathfrak{g}) \geq \frac{1}{4}$. The inverse function can be represented by the expansion

$$\mathfrak{h}(\nu) = \mathfrak{g}^{-1}(\nu) = \nu - \mathfrak{a}_2\nu^2 + \left(2\mathfrak{a}_2^2 - \mathfrak{a}_3\right)\nu^3 - \left(5\mathfrak{a}_2^3 - 5\mathfrak{a}_2\mathfrak{a}_3 + \mathfrak{a}_4\right)\nu^4 + \cdots.$$

A function $\mathfrak{g} \in \mathcal{A}$ is called *bi-univalent* in $\mathcal{U}_d$ if both $\mathfrak{g}$ and its inverse $\mathfrak{g}^{-1}$ are univalent in $\mathcal{U}_d$. The class of such functions is usually referred as Σ. One of the fundamental contributions to the study of this family was provided by Srivastava et al. [30], where several examples and coefficient-related results were established. In recent years, the investigation of bi-univalent functions has gained renewed attention, as reflected in a variety of surveys and research papers.

Specifically, several studies have investigated subclasses of univalent and bi-univalent functions that are linked to particular families of polynomials, including the Horadam

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polynomials. For these functions, estimates of initial coefficients and corresponding Fekete–Szegö type inequalities have been obtained. The Horadam polynomials are of particular significance since they encompass several well-known polynomial families as special cases, including the Lucas, Pell–Lucas, Fibonacci, Pell, and Chebyshev polynomials of the second kind. Accordingly, the present investigation establishes new results in this direction, while also emphasizing their connections with earlier findings reported in the literature.

Due to their complex structure and extensive applicability in fields such as mathematical physics, engineering, and image processing, bi-univalent functions have become a focal point of research in complex analysis and geometric function theory. A central theme in this context concerns the Bieberbach conjecture, which posits that the Taylor coefficients of a univalent function defined on the open unit disk must satisfy certain inequality constraints. The conjecture was conclusively proven by Louis de Branges in 1985, representing a landmark achievement in the development of complex function theory.

Prominent examples of bi-univalent functions comprise the Koebe function, the Bessel function, as well as several of their associated subclasses, including starlike and convex functions. In particular, certain illustrative functions belonging to the class Σ can be expressed as

$$\frac{\varsigma}{1-\varsigma}, \quad -\log(1-\varsigma), \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+\varsigma}{1-\varsigma}\right).$$

It is noteworthy that the class Σ is not empty, even though the classical Koebe function itself is not a member of Σ.

Furthermore, numerous studies have introduced and examined different subclasses of the bi-univalent function class Σ, extending the pioneering contributions of Srivastava et al. [30]. Nonetheless, many of the more recent works have established only non-sharp bounds for the initial Taylor coefficients, specifically $|\mathfrak{a}_2|$ and $|\mathfrak{a}_3|$ (see, for instance, [3, 39, 11, 12, 18, 36, 37]).

Another classical problem in geometric function theory concerns the Fekete–Szegö functional

$$\mathfrak{a}_3 - \mu\mathfrak{a}_2^2 \quad \text{for} \quad \mathfrak{g} \in \mathcal{S},$$

which traces its origins to the counterexample by Fekete and Szegö [14] to the Littlewood–Paley conjecture, originally proposing that the coefficients of odd univalent functions do not exceed one in magnitude. This functional continues to be extensively studied, particularly for various subclasses of univalent functions, and remains an active area of current research [2, 6, 13, 16, 22, 34, 43].

Regarding the concept of subordination between analytic functions, let $\mathfrak{g}$ and $\mathfrak{h}$ be analytic in $\mathcal{U}_d$. we say that $\mathfrak{g}$ is subordinate to $\mathfrak{h}$, denoted $\mathfrak{g} \prec \mathfrak{h}$ (or equivalently, $\mathfrak{g}(\varsigma) \prec \mathfrak{h}(\varsigma)$ for all $\varsigma \in \mathcal{U}_d$), if there exists a Schwarz function ω, analytic in $\mathcal{U}_d$, satisfying $\omega(0) = 0$ and $|\omega(\varsigma)| < 1$ for every $\varsigma \in \mathcal{U}_d$, such that

$$\mathfrak{g}(\varsigma) = \mathfrak{h}(\omega(\varsigma)).$$

Moreover, if $\mathfrak{h}$ is univalent in $\mathcal{U}_d$, then the subordination condition $\mathfrak{g} \prec \mathfrak{h}$ holds if and only if $\mathfrak{g}(0) = \mathfrak{h}(0)$ and $\mathfrak{g}(\mathcal{U}_d) \subseteq \mathfrak{h}(\mathcal{U}_d)$.

Fractional calculus, as an extension of classical calculus, provides a generalization of the traditional operators of differentiation and integration by allowing their orders to vary from integer values to arbitrary real or complex numbers [20, 25, 26]. These generalized operators have been extensively employed in numerous scientific diskiplines,

ranging from electrical circuits [1] and viscoelastic fluid dynamics [7] to weight fractional operator [21] and the telegraph equation [5, 10, 15, 17, 27, 33, 35, 42]. Such developments indicate that fractional calculus is not merely a mathematical curiosity, but a powerful tool for modeling complex systems. Consequently, various generalizations of fractional differentiation and integration have been proposed to further expand the theoretical and practical scope of this field [4, 8, 38, 40, 41].

In the context of analytic and bi-univalent functions, fractional calculus has found significant applications. Classical formulations of fractional operators and their generalizations have been effectively applied to derive characterization results, obtain bounds on coefficients, and establish distortion inequalities [31]. In particular, Srivastava and Owa [32] introduced fractional operators within the complex domain, providing a fundamental tool for defining novel subclasses of analytic functions.

Definition 1.1. For a function $\mathfrak{g}$ that is analytic in a simply connected region, the fractional integral of order α is expressed by

$$\mathbb{I}^{\alpha}\mathfrak{g}(\varsigma)=\frac{1}{\Gamma(\alpha)}\int_{0}^{\varsigma}\mathfrak{g}(\varpi)(\varsigma-\varpi)^{\alpha-1}\,d\varpi,\quad \alpha>0.$$

Definition 1.2. For a function $\mathfrak{g}$ that is analytic in a simply connected region, the fractional derivative of order α is expressed by

$$\mathbb{D}^{\alpha}\mathfrak{g}(\varsigma)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\varsigma}\int_{0}^{\varsigma}\mathfrak{g}(\varpi)(\varsigma-\varpi)^{-\alpha}\,d\varpi,\quad 0\leq\alpha<1,$$

or equivalently,

$$\mathbb{D}^{\alpha}\mathfrak{g}(\varsigma)=\frac{d}{d\varsigma}\mathbb{I}^{1-\alpha}\mathfrak{g}(\varsigma).$$

Remark 1.3. From the above definitions, we have

(1) $\mathbb{D}^{\alpha}\varsigma^{\rho}=\frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}\varsigma^{\rho-\alpha},\quad \rho>-1,\ 0\leq\alpha<1,$

(2) $\mathbb{I}^{\alpha}\varsigma^{\rho}=\frac{\Gamma(\rho+1)}{\Gamma(\rho+\alpha+1)}\varsigma^{\rho+\alpha},\quad \rho>-1,\ 0\leq\alpha<1.$

These fundamental results provide a rigorous framework for extending classical differential and integral operators to fractional orders, which enables the investigation of novel subclasses of analytic and bi-univalent functions via fractional calculus.

The general framework for starlike functions was first introduced by Ma and Minda, who defined the classes

$$\mathcal{S}^{*}(\varphi):=\{\mathfrak{g}\in\mathcal{A}:\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}\prec\varphi(\varsigma)\},\quad \mathcal{C}(\varphi):=\{\mathfrak{g}\in\mathcal{A}:1+\frac{\varsigma\mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)}\prec\varphi(\varsigma)\},$$

where φ is an analytic function with positive real part, $\varphi'(0)>0$, and maps the unit disk onto a domain starlike with respect to 1 and symmetric about the real axis [23]. Recall that a domain D is starlike with respect to a point $\nu_0\in D$ if every line segment connecting ν_0 to any other point of D lies entirely within D. By choosing specific forms of φ, these classes reduce to several well-known subclasses, including Janowski starlike functions $\mathcal{S}^{*}[A,B]$, starlike functions of order α $\mathcal{S}^{*}(\alpha)$, and their convex counterparts $\mathcal{C}[A,B]$ and $\mathcal{C}(\alpha)$ [19].

Building on these foundations, Sokół and Stankiewicz introduced the class $\mathcal{S}_{L}^{*}:=\mathcal{S}^{*}(\sqrt{1+\varsigma})$, which geometrically represents functions $\mathfrak{g}\in A$ such that $\frac{\varsigma\mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)}$ lies in the right-half of the lemniscate of Bernoulli $|\nu^{2}-1|<1$ [28]. Motivated by this construction, Mendiratta et al. derived a family concerning to the left-half of a shifted lemniscate of Bernoulli, examining its geometric properties, coefficient estimates, and radius of

starlikeness [24]. Later, Sharma et al. introduced the class $\mathcal{S}_C^* := \mathcal{S}^*(1+4\varsigma/3+2\varsigma^2/3)$, investigating various structural properties of the functions in this class [29].

Motivated by the aforementioned studies, we recall a subclass of starlike functions where the analytic mapping is defined by the sine function, specifically,

$$\mathcal{S}_{sin}^* := \left\{ \mathfrak{g} \in \mathcal{A} : \frac{\varsigma \mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} \prec 1 + \sin \varsigma \right\},$$

as introduced by N. E. Cho et al. [9] (see Figure 1).

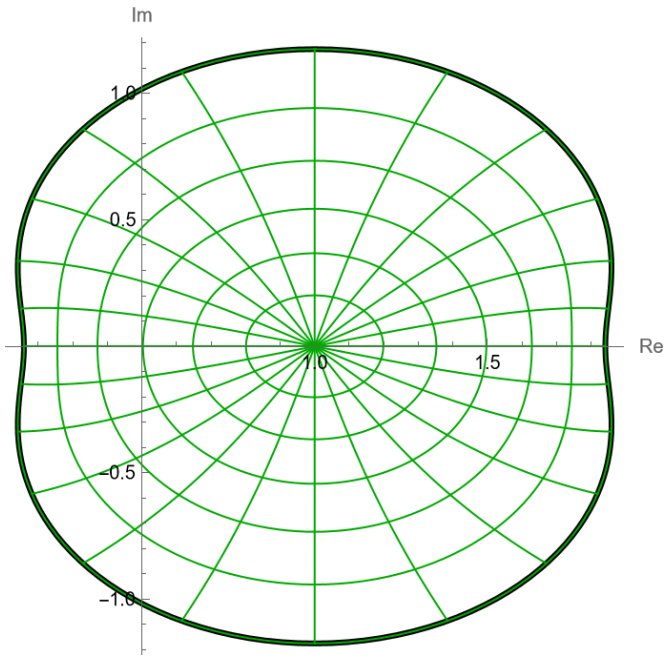


FIGURE 1. Image of the unit disk under the mapping $1 + \sin \varsigma$

2. Definition and Examples

In this section, we present our main definition along with illustrative examples concerning a subclass of bi-univalent functions.

Definition 2.1. Let $\alpha \in [0, 1)$. We say that $\mathfrak{g} \in \Sigma$ is in the class $\mathbb{B}_{\sin}^{\alpha}$, if the subsequent relations are fulfilled:

$$\frac{\mathbb{D}^{\alpha}(\varsigma \mathfrak{g}'(\varsigma))}{\mathbb{D}^{\alpha}\mathfrak{g}(\varsigma)} \prec \Phi_{\sin}(\varsigma), \quad \varsigma \in \mathcal{U}_d \tag{2.1}$$

and

$$\frac{\mathbb{D}^{\alpha}(\nu \mathfrak{h}'(\nu))}{\mathbb{D}^{\alpha}\mathfrak{h}(\nu)} \prec \Phi_{\sin}(\nu), \quad \nu \in \mathcal{U}_d \tag{2.2}$$

where $\mathfrak{h} := \mathfrak{g}^{-1}$ denotes the inverse of f. The analytic function $\Phi_{\sin}$ is given by

$$\Phi_{\sin}(\varsigma) = 1 + \sin \varsigma, \quad \varsigma \in \mathcal{U}_d. \tag{2.3}$$

The following two examples represent special cases of Definition 2.1, illustrating the starlike and convex forms, respectively.

Example 2.2. *If $\alpha = 0$, then $\mathbb{B}_{\sin}^{\alpha} := \mathcal{S}_{\sin}^{*}$. The function $\mathfrak{g} \in \Sigma$ is said to belong to the class $\mathcal{S}_{\sin}^{*}$, if the subsequent relations are fulfilled:*

$$\frac{\varsigma \mathfrak{g}'(\varsigma)}{\mathfrak{g}(\varsigma)} \prec 1 + \sin \varsigma, \quad \varsigma \in \mathcal{U}_d,$$

and

$$\frac{\nu \mathfrak{h}'(\nu)}{\mathfrak{h}(\nu)} \prec 1 + \sin \nu, \quad \nu \in \mathcal{U}_d,$$

where $\mathfrak{h} := \mathfrak{g}^{-1}$ denotes the inverse of $\mathfrak{g}$.

Example 2.3. *If $\alpha \to 1^{-}$, then $\mathbb{B}_{\sin}^{\alpha} := \mathcal{C}_{\sin}$. The function $\mathfrak{g} \in \Sigma$ is said to belong to the class $\mathcal{C}_{\sin}$, if the subsequent relations are fulfilled:*

$$1 + \frac{\varsigma \mathfrak{g}''(\varsigma)}{\mathfrak{g}'(\varsigma)} \prec 1 + \sin \varsigma, \quad \varsigma \in \mathcal{U}_d,$$

and

$$1 + \frac{\nu \mathfrak{h}''(\nu)}{\mathfrak{h}'(\nu)} \prec 1 + \sin \nu, \quad \nu \in \mathcal{U}_d,$$

where $\mathfrak{h} := \mathfrak{g}^{-1}$ denotes the inverse of $\mathfrak{g}$.

The following example demonstrates that the class $\mathbb{B}_{\sin}^{\alpha}$ is non-empty, as it contains infinitely many bi-univalent functions.

Example 2.4. *Consider the cluster of mappings given by*

$$\mathfrak{g}_{\rho}(\varsigma) = \frac{\varsigma}{1 - \rho\varsigma}, \qquad |\rho| < 1.$$

Through a straightforward symbolic manipulation one observes that

$$\frac{\varsigma\, \mathfrak{g}_{\rho}'(\varsigma)}{\mathfrak{g}_{\rho}(\varsigma)} = \frac{1}{1 - \rho\varsigma}, \qquad 1 + \frac{\varsigma\, \mathfrak{g}_{\rho}''(\varsigma)}{\mathfrak{g}_{\rho}'(\varsigma)} = \frac{1 + \rho\varsigma}{1 - \rho\varsigma}.$$

Consequently, the entire family is nested within the class $\mathcal{S}_{\sin}^{*}$ whenever

$$|\rho| \leq 0.455,$$

while its confinement to the class $\mathcal{C}_{\sin}$ emerges precisely under the restriction

$$|\rho| \leq 0.295.$$

For a geometric intuition of these inclusions, consult Figure 2.

3. Coefficient Bounds of $\mathbb{B}_{\sin}^{\alpha}$

We start by reviewing the class $\mathcal{P}$, which will be frequently used throughout this section. The function r is considered a member of the class $\mathcal{P}$ whenever it is analytic in $\mathcal{U}_d$ and meets

$$r(\varsigma) = 1 + \sum_{m=1}^{\infty} c_m \varsigma^m, \quad \varsigma \in \mathcal{U}_d, \tag{3.1}$$

with

$$\Re\{r(\varsigma)\} > 0.$$

At the beginning of this section, we present the following lemmas, as they are essential for the theoretical development in the remainder of this work.

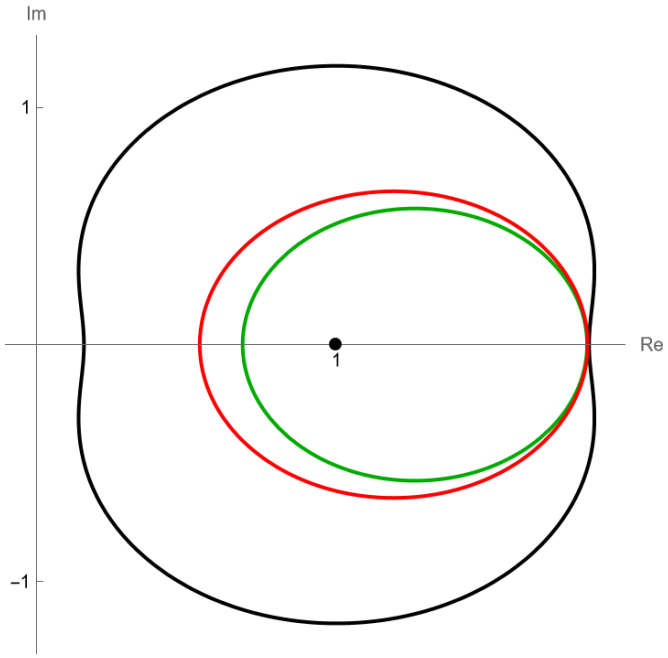


FIGURE 2. In the illustration, the trajectory of $\frac{1}{1-\rho\varsigma}$ is rendered in green, whereas the path of $\frac{1+\rho\varsigma}{1-\rho\varsigma}$ is traced in red.

Lemma 3.1. *If the function $r \in \mathcal{P}$ is given by the series (3.1), then the following sharp estimates hold:*

$$|c_m| \leq 2, \quad m = 1, 2, \dots \tag{3.2}$$

Theorem 3.2. *Let $\mathfrak{g} \in \mathbb{B}^{\alpha}_{\sin}$. Then*

$$|\mathfrak{a}_2| \leq \min\left\{\frac{\aleph_1(\alpha)}{\aleph_2(\alpha)}, \sqrt{\frac{\aleph_1(\alpha)}{2\aleph_3(\alpha)}}\right\},$$

and

$$|\mathfrak{a}_3| \leq \min\left\{\frac{\aleph_1^2(\alpha)}{\aleph_2^2(\alpha)} + \frac{\aleph_1(\alpha)}{2\aleph_3(\alpha)}, \frac{\aleph_1(\alpha)}{\aleph_3(\alpha)}\right\},$$

where

$$\aleph_n(\alpha) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)},$$

Proof. Let $\mathfrak{g} \in \mathbb{B}^{\alpha}_{\sin}$. Then

$$\frac{\mathbb{D}^{\alpha}\big(\varsigma\mathfrak{g}'(\varsigma)\big)}{\mathbb{D}^{\alpha}\mathfrak{g}(\varsigma)} \prec \Phi_{\sin}(u(\varsigma)), \tag{3.3}$$

and

$$\frac{\mathbb{D}^{\alpha}\big(\varsigma\mathfrak{h}'(\nu)\big)}{\mathbb{D}^{\alpha}\mathfrak{h}(\nu)} \prec \Phi_{\sin}(v(\nu)), \tag{3.4}$$

where $r, s \in \mathcal{P}$ and are defined by

$$r(\varsigma) = \frac{1+u(\varsigma)}{1-u(\varsigma)} = 1 + c_1\varsigma + c_2\varsigma^2 + c_3\varsigma^3 + \cdots, \tag{3.5}$$

and

$$s(\nu)=\frac{1+v(\nu)}{1-v(\nu)}=1+d_1\nu+d_2\nu^2+d_3\nu^3+\cdots. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$u(\varsigma)=\frac{r(\varsigma)-1}{r(\varsigma)+1}=\frac{1}{2}\Big[c_1\varsigma+\big(c_2-\frac{c_1^2}{2}\big)\varsigma^2+\big(c_3-c_1c_2+\frac{c_1^3}{4}\big)\varsigma^3+\cdots\Big], \tag{3.7}$$

and

$$v(\nu)=\frac{s(\nu)-1}{s(\nu)+1}=\frac{1}{2}\Big[d_1\nu+\big(d_2-\frac{d_1^2}{2}\big)\nu^2+\big(d_3-d_1d_2+\frac{d_1^3}{4}\big)\nu^3+\cdots\Big]. \tag{3.8}$$

Substituting (3.7) and (3.8) into (2.3), we obtain

$$\Phi_{\sin}(u(\varsigma))=1+\frac{c_1}{2}\varsigma+\frac{1}{2}\Big(c_2-\frac{c_1^2}{2}\Big)\varsigma^2+\cdots, \tag{3.9}$$

and

$$\Phi_{\sin}(v(\nu))=1+\frac{d_1}{2}\nu+\frac{1}{2}\Big(d_2-\frac{d_1^2}{2}\Big)\nu^2+\cdots. \tag{3.10}$$

Also, the left-hand sides of (3.3) and (3.4) can be expanded as

$$\frac{\mathbb{D}^\alpha\big(\varsigma\mathfrak{g}'(\varsigma)\big)}{\mathbb{D}^\alpha\mathfrak{g}(\varsigma)}=1+\frac{\mathfrak{a}_2\,\aleph_2(\alpha)}{\aleph_1(\alpha)}\varsigma+\frac{-\mathfrak{a}_2^2\,\aleph_2^2(\alpha)+2\mathfrak{a}_3\,\aleph_1(\alpha)\aleph_3(\alpha)}{\aleph_1^2(\alpha)}\varsigma^2+\cdots, \tag{3.11}$$

and

$$\frac{\mathbb{D}^\alpha\big(\varsigma\mathfrak{h}'(\nu)\big)}{\mathbb{D}^\alpha\mathfrak{h}(\nu)}=1-\frac{\mathfrak{a}_2\,\aleph_2(\alpha)}{\aleph_1(\alpha)}\nu+\frac{-\mathfrak{a}_2^2\,\aleph_2^2(\alpha)+2(2\mathfrak{a}_2^2-\mathfrak{a}_3)\,\aleph_1(\alpha)\aleph_3(\alpha)}{\aleph_1^2(\alpha)}\nu^2+\cdots. \tag{3.12}$$

It follows from (3.3), (3.9), and (3.11) that

$$\frac{\mathfrak{a}_2\,\aleph_2(\alpha)}{\aleph_1(\alpha)}=\frac{c_1}{2}, \tag{3.13}$$

and

$$\frac{-\mathfrak{a}_2^2\,\aleph_2^2(\alpha)+2\mathfrak{a}_3\,\aleph_1(\alpha)\aleph_3(\alpha)}{\aleph_1^2(\alpha)}=\frac{1}{2}\Big(c_2-\frac{c_1^2}{2}\Big). \tag{3.14}$$

Similarly, from (3.4), (3.10), and (3.12), we obtain

$$-\frac{\mathfrak{a}_2\,\aleph_2(\alpha)}{\aleph_1(\alpha)}=\frac{d_1}{2}, \tag{3.15}$$

and

$$\frac{-\mathfrak{a}_2^2\,\aleph_2^2(\alpha)+2(2\mathfrak{a}_2^2-\mathfrak{a}_3)\,\aleph_1(\alpha)\aleph_3(\alpha)}{\aleph_1^2(\alpha)}=\frac{1}{2}\Big(d_2-\frac{d_1^2}{2}\Big). \tag{3.16}$$

From (3.13) and (3.15), we have

$$\mathfrak{a}_2=\frac{\aleph_1(\alpha)\,c_1}{2\,\aleph_2(\alpha)}=-\frac{\aleph_1(\alpha)\,d_1}{2\,\aleph_2(\alpha)}. \tag{3.17}$$

This indicates that

$$c_1=-\,d_1. \tag{3.18}$$

Applying Lemma 3.1 to equation (3.17), we obtain

$$|\mathfrak{a}_2|\le\frac{\aleph_1(\alpha)}{\aleph_2(\alpha)}. \tag{3.19}$$

The squaring and addition of (3.13) and (3.15) lead to

$$\mathfrak{a}_2^2 = \frac{(c_1^2 + d_1^2)\,\aleph_1^2(\alpha)}{8\,\aleph_2^2(\alpha)}. \tag{3.20}$$

In addition, the sum of (3.14) and (3.16) gives

$$\frac{4\,\aleph_1(\alpha)\,\aleph_3(\alpha) - 2\,\aleph_2^2(\alpha)}{\aleph_1^2(\alpha)}\mathfrak{a}_2^2 = \frac{c_2 + d_2}{2} - \frac{c_1^2 + d_1^2}{4}.$$

From (3.20), we get

$$\frac{4\,\aleph_1(\alpha)\,\aleph_3(\alpha) - 2\,\aleph_2^2(\alpha)}{\aleph_1^2(\alpha)}\mathfrak{a}_2^2 = \frac{c_2 + d_2}{2} - \frac{2\,\aleph_2^2(\alpha)}{\aleph_1^2(\alpha)}\mathfrak{a}_2^2,$$

which implies

$$\mathfrak{a}_2^2 = \frac{(c_2 + d_2)\,\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}. \tag{3.21}$$

Thus, Lemma 3.1 implies that

$$|\mathfrak{a}_2| \leq \sqrt{\frac{\aleph_1(\alpha)}{2\,\aleph_3(\alpha)}}.$$

Subsequently, by subtracting (3.16) from (3.14), we obtain

$$\mathfrak{a}_3 = \mathfrak{a}_2^2 + \frac{(c_2 - d_2)\,\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}, \tag{3.22}$$

and hence,

$$|\mathfrak{a}_3| \leq |\mathfrak{a}_2|^2 + \frac{\aleph_1(\alpha)}{2\,\aleph_3(\alpha)}. \tag{3.23}$$

Substituting the estimates from (3.19) and (3.21) into (3.23), respectively, we have

$$|\mathfrak{a}_3| \leq \frac{\aleph_1^2(\alpha)}{\aleph_2^2(\alpha)} + \frac{\aleph_1(\alpha)}{2\,\aleph_3(\alpha)},$$

and

$$|\mathfrak{a}_3| \leq \frac{\aleph_1(\alpha)}{\aleph_3(\alpha)}.$$

Accordingly, the desired result is obtained, which completes the proof with clarity. □

Corollary 3.3. *Let* $\mathfrak{g} \in \mathcal{S}^*_{\sin}$. *Then*

$$|\mathfrak{a}_2| \leq \frac{1}{\sqrt{2}}, \quad \textit{and} \quad |\mathfrak{a}_3| \leq 1.$$

Corollary 3.4. *Let* $\mathfrak{g} \in \mathcal{C}_{\sin}$. *Then*

$$|\mathfrak{a}_2| \leq \frac{1}{\sqrt{6}}, \quad \textit{and} \quad |\mathfrak{a}_3| \leq \frac{1}{3}.$$

The concluding theorem of this section establishes the Fekete–Szeg‘ functional for functions in the class $\mathbb{B}^{\alpha}_{\sin}$.

Proposition 3.5. *Let* $\mathfrak{g} \in \mathbb{B}^{\alpha}_{\sin}$ *and* $\eta \in \mathbb{R}$. *Then*

$$|\mathfrak{a}_3 - \eta\mathfrak{a}_2^2| \leq \begin{cases} \dfrac{\aleph_1(\alpha)}{2\,\aleph_3(\alpha)}, & |1-\eta| \leq 1, \\ \dfrac{|1-\eta|\,\aleph_1(\alpha)}{2\,\aleph_3(\alpha)}, & |1-\eta| \geq 1. \end{cases} \tag{3.24}$$

Proof. Assuming that $\mathfrak{g} \in \mathbb{B}^{\alpha}_{\sin}$, it follows from (3.21) and (3.22) that

$$\begin{aligned} \mathfrak{a}_3 - \eta \mathfrak{a}_2^2 &= \frac{(1-\eta)\,\aleph_1(\alpha)(c_2+d_2)}{8\,\aleph_3(\alpha)} + \frac{(c_2-d_2)\,\aleph_1(\alpha)}{8\,\aleph_3(\alpha)} \\ &= \left(\Xi(\alpha,\eta) + \frac{\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}\right)c_2 + \left(\Xi(\alpha,\eta) - \frac{\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}\right)d_2, \end{aligned} \tag{3.25}$$

where

$$\Xi(\alpha,\eta) = \frac{(1-\eta)\,\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}. \tag{3.26}$$

Accordingly, taking the modulus of (3.25) yields

$$|\mathfrak{a}_3 - \eta \mathfrak{a}_2^2| \leq \begin{cases} \dfrac{\aleph_1(\alpha)}{2\,\aleph_3(\alpha)}, & 0 \leq \Xi(\alpha,\eta) \leq \dfrac{\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}, \\ 4\,\Xi(\alpha,\eta), & \Xi(\alpha,\eta) \geq \dfrac{\aleph_1(\alpha)}{8\,\aleph_3(\alpha)}. \end{cases}$$

After straightforward computations, we obtain (3.28). □

Corollary 3.6. *Let* $\mathfrak{g} \in \mathcal{S}^{*}_{\sin}$ *and* $\eta \in \mathbb{R}$. *Then*

$$|\mathfrak{a}_3 - \eta \mathfrak{a}_2^2| \leq \begin{cases} \dfrac{1}{2}, & |1-\eta| \leq 1, \\ \dfrac{|1-\eta|}{2}, & |1-\eta| \geq 1. \end{cases} \tag{3.27}$$

Corollary 3.7. *Let* $\mathfrak{g} \in \mathcal{S}_{\sin}$ *and* $\eta \in \mathbb{R}$. *Then*

$$|\mathfrak{a}_3 - \eta \mathfrak{a}_2^2| \leq \begin{cases} \dfrac{1}{6}, & |1-\eta| \leq 1, \\ \dfrac{|1-\eta|}{6}, & |1-\eta| \geq 1. \end{cases} \tag{3.28}$$

4. Conclusion

In this work, we introduced a new class of bi-univalent functions involving the Riemann-Liouville subordination derivative with the sine function. we demonstrated that this class is non-empty and determined best bounds for the initial coefficients. Moreover, we established the upper bound for the Fekete-Szegö functional. Future research may focus on computing Hankel determinants of higher orders and exploring connections with other families of analytic functions, thus opening new directions in geometric function theory.

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