Gulf Journal of Mathematics
Vol 21, Issue 1 (2025) 80-91

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v21i1.3524

A NONLOCAL SIXTH ORDER PARABOLIC PROBLEM WITH NONLINEAR SOURCE

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Abstract. The purpose of this article is to study the existence of weak solutions for a class of nonlinear nonlocal Dirichlet parabolic problem involving the $p(x)$-Kirchhoff type triharmonic operator with a nonlinear source. We apply degree theory to operators of the type $T+S+C$, where T is maximal monotone, S is bounded pseudomonotone, and C is compact with $D(T) \subseteq D(C)$ and satisfies a sublinearity condition, to establish our result within the context of Sobolev spaces with variable exponents.

Keywords. Global existence, sixth order parabolic problem, nonlinear source, degree theory

2020 Mathematics Subject Classification. Primary 34A60; Secondary 34G25, 49J52.

1. Introduction and preliminaries

In this research, we focus on the following nonlocal parabolic problem originated from a capillary phenomena

$$\begin{aligned}
&u_t - M(L(u))\Delta^3_{p(x)}u + f(x,u,\nabla u,\Delta u,\nabla\Delta u) = h(x,t) \quad \text{in } \Omega,\\
&u = \Delta u = \Delta^2 u = 0 \qquad\qquad \text{on } \partial\Omega, \\
&u(0) = 0,
\end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 3$, $p \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $L(u) = \int_\Omega \frac{1}{p(x)} |\nabla \triangle u|^{p(x)}\, dx$, $\triangle^3_{p(x)}u := \operatorname{div}\left(\triangle\left(|\nabla\triangle u|^{p(x)-2}\,\nabla\Delta u\right)\right)$ is the so-called $p(x)$-triharmonic operator, $p \in C(\overline{\Omega})$ and f, h are functions that satisfy conditions which will be stated later.
We observe that selecting the functional $L(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx$, $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ instead of $\Delta^3_{p(x)}u$, f independent of $(\nabla u, \Delta u, \nabla\Delta u)$ and without the function h, then we encounter the problem:

$$\begin{aligned}
-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad &\text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,&
\end{aligned} \tag{1.2}$$

Date: Received: Sep 10, 2025; Accepted: Oct 4, 2025.
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which is called the $p(x)$-Kirchhoff type equation. The problem (1.2) is a generalization of the stationary problem of a model introduced by Kirchhoff [21], who investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\big|\frac{\partial u}{\partial x}\big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.3}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature is that the problem (1.3) contains a nonlocal coefficient $\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L\big|\frac{\partial u}{\partial x}\big|^2\,dx$, and hence the equation is no longer a pointwise equation. The parameters in (1.3) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young's modulus of the material, ρ is the mass density and P_0 is the initial tension.

Also, when $p(x)\equiv p(\text{constant})$ and $f(x,u,\nabla u,\Delta u,\nabla\Delta u)\equiv f(x,t,u)$, problem (1.1) can be used to describe the slow rotational flow of highly viscous fluids within small cavities, and the nonlocal term M can describe a possible change in the global state of the fluid caused by its motion in the considered medium, see [23]. In astrophysics, the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary value problems like the one in our problem [35]; to measure the shape and size of limbs and venous ulcers in medical modelling, a mathematical model similar to (1.1) was implemented in [36]. Also, in mathematical biology, one way to conceptualize problem (1.1) is as a model of bacterial spreading, where u represents the population density at the point (x,t) and the external source f represents the processes of birth and death, whose rates depend on the population, see [11].

Owing to their numerous physical applications, PDEs with variable exponents have garnered increased attention in recent years due to their various physical applications. Indeed, they are capable of modeling a wide range of phenomena that come up in the study of thermo-rheological and electro-rheological fluids [30, 31, 32] and image restoration [10, 24]. Numerous researchers have dedicated a significant amount of time to studying parabolic problems involving variable exponent growth condition, we refer the readers to [12, 16, 20, 34].

There are limited contributions to the research on triharmonic issues that involve a reaction term $f(x,t,z,y,w)$ influenced by the gradient, the Laplacian, and the gradient of the Laplacian of the solution. References [4, 5, 17, 28, 29, 33] can be mentioned. Recently, in the context of the stationary version of problem (1.1), Mehraban et al. [25] explored the existence and multiplicity of solutions with $p(x)\equiv p(\text{constant})$, $M(t)=1, f(x,u,\nabla u,\Delta u,\nabla\Delta u):=\lambda f(x,u)+\mu g(x,u)$ and, similarly, Zhao et al [40] but in the context of variable exponent $p(x)$ growth.

To solve quasilinear parabolic equations, most authors use techniques such as the nonlinear semigroups theory, the discretization method, De Giorgi iteration technique, subdifferential calculus, theory of monotone operators, fixed point theorems and the classic Galerkin method. Differently from the above mentioned methods, in the present paper, to establish our main result we use the degree theory developed by Asfaw [2] for operators of type $T+S+C$, where T is maximal monotone, S is bounded pseudomonotone and C is compact with $D(T)\subseteq D(C)$ and satisfies a sublinearity condition. It seems that this is the first effort to address a nonlocal $p(x)$-Laplace parabolic problem that involves a gradient term, the Laplacian, and the gradient of the Laplacian, particularly utilizing degree theory.

We point out that the degree theory is one of the main tools for checking the existence of solutions of nonlinear elliptic PDEs, even in spaces of variable exponents, without resorting to usual variational methods. However, recently several scholars have implemented this methodology to solve evolution equations, see [18, 19, 27, 37, 38].

This paper is organized as follows. Section 2 includes some preliminary information on evolution functional spaces and variable exponent Sobolev spaces, as well as the Asfaw topological degree needed for the proof of our main result. Section 3 contains the technical lemmas and fundamental assumptions. The final section focuses on stating and demonstrating our main result about existence of weak solutions for problem (1.1).

2. Preliminaries

To discuss problem (1.1), we need some theory on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ which is called variable exponent Lebesgue–Sobolev spaces (for details, see [13]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\}$$

where $\alpha \in \mathbb{N}^N$, $|\alpha| = \sum_{i=1}^N \alpha_i$, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$, with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \le k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([13])**.** *The spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([13])**.** *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

(1) *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
(2) *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
(3) *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
(4) *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
(5) *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
(6) *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([14, 13])**.** *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p_k^*(x)$ $(q(x) < p_k^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N - kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \ge N. \end{cases}$$

Proposition 2.4 ([15, 13]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$|\int_\Omega uv\,dx| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

We define $W_0^{k,p(x)}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{k,p(x)}(\Omega)}$.

In particular, we denote by

$$X = W_0^{1,p(x)}(\Omega) \cap W^{3,p(x)}(\Omega)$$

and define a norm $\|.\|_X$ by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

Moreover, the norms $\|u\|_X$ and $\|\nabla\triangle u\|_{p(x)}$ are equivalent on X, see [13, 14]. Let

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\nabla\triangle u}{\mu}\right|^{p(x)} dx \le 1\right\}$$

for any $u \in X$. Hence , we see that $\|u\|$ is equivalent to the norms $\|u\|_X$ and $\|\nabla\triangle u\|_{p(x)}$ in X. From now on, we will use $\|.\|$ instead of $\|u\|_X$ on X.

Proposition 2.5 ([15, 13]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$, for a.e.$x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then, we have*

$$i)\textit{For } |u|_{p(x)q(x)} \le 1,\ |u|^{p^+}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^-}_{p(x)q(x)}$$
$$ii)\textit{For } |u|_{p(x)q(x)} > 1,\ |u|^{p^-}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}$$

Proposition 2.6 ([29]). *Set $\Psi_{p(x)}(u) = \int_\Omega |\nabla\triangle u|^{p(x)}\,dx$ for any $u \in X$. Then, we have*

(1) *if $\|u\| \ge 1$, then $\|u\|^{p^-} \le \Psi_{p(x)}(u) \le \|u\|^{p^+}$;*
(4) *if $\|u\| \le 1$, then $\|u\|^{p^+} \le \Psi_{p(x)}(u) \le \|u\|^{p^-}$.*

As in [6], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty[$ to $\overline{Q} \to [1, +\infty[$ by setting $p(x,t) = p(x)$ for all $(x,t) \in \overline{Q}$. Now, we may consider the generalized Lebesgue space

$$L^{p(x)}(Q) = \{u : Q \to \mathbb{R} \quad \text{measurable such that} \int_Q |u(x,t)|^{p(x)}\,dxdt < \infty\}$$

which has similar properties to those of space $L^{p(x)}(\Omega)$.

Consider X a real reflexive Banach space, and X' its dual space; for each $x \in X$ and $x' \in X'$, the value $x'(x)$ is denoted by $\langle x', x\rangle$. We shall use the standard notation for Bochner spaces i.e. $L^r(0,T;X)$ is the space of strongly measurable function $u :]0,T[\to X$ for which $t \to \|u\|_X \in L^r(0,T)$, $r \ge 1$.

The space $W(Q)$ is defined as follows

$$W(Q) = \left\{u : [0,T] \to W_0^{1,p(x)}(\Omega) \cap W^{3,p(x)}(\Omega); u \in L^{p(x)}(Q) : |\nabla\triangle u| \in L^{p(x)}(Q)\right\},$$

where $\nabla\triangle u$ stands for the gradient of $\triangle u$ with respect to the space variable x. It is is a Banach equipped with the norm

$$\|u\|_{W(Q)} := \|u\|_{L^{p(x)}(Q)} + \|\nabla\triangle u\|_{L^{p(x)}(Q)}, \quad \text{for all } u \in W(Q).$$

Suppose that
$$1+\frac{N}{N+1}<p(x)<p^{*}(x). \tag{2.1}$$
Since $1+\frac{N}{N+1}-\frac{2N}{N+2}>0$, then $p(x)>\frac{2N}{N+2}$ which implies
$$W_0^{1,p(x)}(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow W^{-1,p'(x)}(\Omega),$$
where these embeddings are dense.

Remark 2.7. i)$(W_0^{1,p(x)}(\Omega),\|\cdot\|_{W^{1,p(x)}}(\Omega))$ is a reflexive and separable Banach space.
ii) For $q(x)\in C_+(\overline{\Omega})$ such that $q(x)<p^*(x)$ for any $x\in\overline{\Omega}$, we have $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, the embedding is continuous and compact.
iii)The space $W(Q)$ is a separable and reflexive Banach space endowed with the norm $\|u\|=\|\nabla\triangle u\|_{L^{p(x)}(Q)}$. Furthermore $\|.\|$ and $\|.\|_{W(Q)}$ are equivalent.
iv) By (2.1) and Theorem 1.5 in [1] $W(Q)\hookrightarrow L^{\hat{p}(x)}(Q)$ is compact, where $\hat{p}(x):=\max\{2,p(x)\}$.

We need the following concepts to state a fundamental theorem for our work.
For an operator $T:X\to 2^{X'}$ the graph of T, denoted $G(T)$, is defined by
$$G(T)=\{(x,y):x\in D(T),y\in Tx\}$$
where $D(T)$ denotes the domain of T, given by $D(T)=\{x\in X:Tx\notin\emptyset\}$. We shall use "$\rightharpoonup$" ("$\to$") for the weak (strong) convergence.
An operator $T:X\supset D(T)\to 2^{X'}$ is said to be : a) monotone if for every $x,y\in D(T)$ and every $u\in Tx$ and $v\in Ty$ we have $\langle u-v,x-y\rangle\geq 0$, b) maximal monotone if T is monotone and $G(T)$ is maximal in $X\times X'$ when $X\times X'$ is partially ordered by the set inclusion, c) coercive if either $D(T)$ is bounded or there exists a function $\psi:[0,\infty[\to\mathbb{R}$ such that $\psi(t)\to\infty$ as $t\to\infty$ and
$$\langle u,x\rangle\geq\psi(\|x\|)\|x\|,\quad\forall x\in D(T),u\in Tx.$$
We observe that , in this setting, T is maximal monotone if and only if $R(T+\lambda I)=X'$ for all $\lambda\in[0,+\infty[$.

We recall that $T:X\supset D(T)\to Y$, where Y is a Banach space, is said to be: a) bounded if it maps bounded subsets of $D(T)$ onto bounded subsets of Y, b) compact if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y, and c) of type (S_+) if for any $(u_\nu)\subset D(T)$ with $u_\nu\rightharpoonup u$ and $\limsup\langle Tu_\nu,u_\nu-u\rangle\leq 0$, we have $u_\nu\to u$.

The following theorem is a generalization of the existence result due to Asfaw and Kartsatos [3] for the sum of two operators T and S, and yields the surjectivity of operators of type $T+S+C$.

Theorem 2.8 (Asfaw T. [2])**.** *Let X be real Banach space. Let $T:X\supseteq D(T)\to X'$ be maximal monotone with $0\in T(0)$, $S:X\to X'$ be bounded and of type (S_+), and $C:D(C)\to X'$ be compact with $D(T)\subseteq D(C)$ and belonging to the class Γ_σ^τ (i.e there exist $\sigma\geq 0$ and $\tau\geq 0$ such that $\|Cx\|\leq\tau\|x\|+\sigma$ for all $x\in D(C)$). Assume, further that $T+S+C$ is coercive. Then $T+S+C$ is surjective.*

3. Hypotheses and technical Lemmas

Firstly, we impose the conditions that allow us to achieve the solutions of problem (1.1). Concerning the function M, we suppose that

(M_0) $M : [0, +\infty[\to]m_0, m_1[$ is continuous and nondecreasing function with $m_0, m_1 > 0$.

Now, we can give the properties of the nonlocal $p(x)$-Laplace operator

$$\mathcal{L}(u) = -M\Big(L(u)\Big) \operatorname{div}\Big(\Delta\Big(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\Big)\Big).$$

Consider the following functional $S : W(Q) \to W'(Q)$, ($W'(Q)$ is the dual of $W(Q)$) given by

$$\langle S(u), v\rangle = \int_Q M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}\,dx\right)|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\cdot\nabla\Delta v\,dxdt,$$

then $\mathcal{L} : W(Q) \to W'(Q)$ and

$$\langle\mathcal{L}(u), v\rangle := \langle S(u), v\rangle,\ \forall u, v \in W(Q). \tag{3.1}$$

Define the following functional

$$\Phi(u) = \int_0^T \widehat{M}(L(u)\,dt, \quad \forall u \in W(Q)$$

where $\widehat{M}(s) = \int_0^s M(\lambda)\,d\lambda$ for all $s \in [0, +\infty[$. Then, it is easy to prove that Φ is well defined and continuously Gâteaux differentiable, and its Gâteaux derivative at point $u \in W(Q)$ is given by

$$\langle\Phi'(u), v\rangle = \langle S(u), v\rangle \quad \text{for all } v \in W(Q).$$

Hence, the nonlocal $p(x)$-Laplace operator $\mathcal{L}$ is the derivative operator of Φ in the weak sense. Also, it is obvious that Φ' is continuous.

Remark 3.1. By Theorem 3.1 in [8] we know that, for every $t \in]0, T[$ fixed, the operator $S : X \to X'$ is a strictly monotone bounded homeomorphism and of type (S_+).

Lemma 3.2. *If M satisfies (M_0), then*

(i) *$\mathcal{L} : W(Q) \to W'(Q)$ is a continuous, bounded and strictly monotone operator;*

(ii) *$\mathcal{L}$ is of type (S_+), i.e. if $u_\nu \rightharpoonup u$ in $W(Q)$ and*

$$\limsup_{\nu\to+\infty}\langle\mathcal{L}(u_n) - \mathcal{L}(u), u_\nu - u\rangle = 0,\ \textit{then } u_\nu \to u \textit{ in } W(Q).$$

Proof. i) We see that $\mathcal{L}$ is continuous because $\mathcal{L} = \Phi'$. Now, we prove that $\mathcal{L}$ is bounded. Let $\mathcal{B}$ be a bounded subset in $W(Q)$. It is obvious that $\{M(L(u)) : u \in \mathcal{B}\}$ is bounded, since M is continuous. Also if $u, v \in \mathcal{B}$, we have

$$\begin{aligned}
|\langle\mathcal{L}u, v\rangle| &= |\int_Q M(L(u))|\nabla\triangle u|^{p(x)-2}\nabla\triangle u\cdot\nabla\triangle v\,dxdt| \\
&\le m_1\int_Q |\nabla\triangle u|^{p(x)-1}|\nabla\triangle v|\,dxdt \\
&\le m_1\||\nabla\triangle u|^{p(x)-1}\|_{p'(x),Q}\|\nabla\triangle v\|_{p(x),Q} \le C\|u\|^{p^+}\|v\|
\end{aligned}$$

So, $\mathcal{L}$ is a bounded operator. Next, we will prove that $\mathcal{L}$ is a strictly monotone operator. Let $t \in]0, T[$, then from Remark 3.1 it follows that

$$\int_\Omega (M(L(u))|\nabla\triangle u|^{p(x)-2}\nabla\triangle u - M(L(v))|\nabla\triangle v|^{p(x)-2}\nabla\triangle v)\cdot(\nabla\triangle u - \nabla\triangle v)\,dx \ge 0$$

$\forall u(.,t), v(.,t) \in X.$

From this inequality, integrating over $[0,T]$ we obtain

$$
\begin{aligned}
&\langle \mathcal{L}(u)-\mathcal{L}(v),u-v\rangle = \langle S(u)-S(v),u-v\rangle \\
&= \int_0^T \int_\Omega \big[(M(L(u))|\nabla\triangle u|^{p(x)-2}\nabla\triangle u - M(L(v))|\nabla\triangle v|^{p(x)-2}\nabla\triangle v)\cdot(\nabla\triangle u-\nabla\triangle v)\,dxdt \\
&\geq 0, \qquad \forall u,v\in W(Q).
\end{aligned}
$$

Next, we show that the operator $\mathcal{L}$ is of type (S_+). Let $(u_\nu)_\nu$ be a sequence in $W(Q)$ such that $u_\nu \rightharpoonup u$ and $\limsup\limits_{\nu\to\infty}\langle \mathcal{L}(u_\nu)-\mathcal{L}(u),u_\nu-u\rangle \leq 0$. Then

$$
\lim_{\nu\to\infty}\langle \mathcal{L}(u_\nu)-\mathcal{L}(u),u_\nu-u\rangle = 0. \tag{3.2}
$$

That is

$$
\lim_{\nu\to\infty}\int_Q M\Big(L(u_\nu)\Big)|\nabla\triangle u_\nu|^{p(x)-2}\nabla\triangle u_\nu.(\nabla\triangle u_\nu-\nabla\triangle u)\,dx\,dt=0 \tag{3.3}
$$

On the other hand, we deduce that there exists a positive constant C such that

$$
|L(u_\nu)| \leq \frac{1}{p^-}\int_Q |\nabla\triangle u_\nu|^{p(x)}\,dx\,dt \leq \frac{1}{p^-}(\|u_\nu\|^{p^+}+1)\leq C. \tag{3.4}
$$

Then , since M is continuous, up to a subsequence there is $t_0\geq 0$ such that

$$
M\Big(L(u_\nu)\Big)\to M(t_0)\geq m_0 \qquad \text{as } \nu\to\infty
$$

This and (3.3) imply

$$
\lim_{\nu\to\infty}\int_Q |\nabla\triangle u_\nu|^{p(x)-2}\nabla\triangle u_\nu.(\nabla\triangle u_\nu-\nabla\triangle u)\,dx\,dt=0
$$

By proceeding similarly as [15], Theorem 3.1, ii), one can obtain

$$
\lim_{\nu\to\infty}\|\nabla\triangle u_\nu\|_{p(x),Q}=\|\nabla\triangle u\|_{p(x),Q}.
$$

Therefore, by the uniform convexity of $W^{3,p(x)}(\Omega)$ and the equivalence of norms on X, we conclude that $u_\nu\to u$ in $W(Q)$ as $\nu\to\infty$. □

Now we deal with the properties for the superposition operator induced by the functions f and g in (1.1). We assume that

(F_1) $f:\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a Carathéodory function such that
i)$|f(x,t,\eta,\xi,s,\zeta)|\leq c_1(|\eta|+|\xi|+|s|+|\zeta|)+k(x,t)$, $\quad k\in L^\infty(0,T;L^{p(x)}(\Omega))$;
ii) $f(x,t,\eta,\xi,s,\zeta)\eta\geq|\eta|^{r(x)}$, for all $(x,t)\in\Omega\times(0,T);\eta,s\in\mathbb{R}$ and $\xi,\zeta\in\mathbb{R}^n$, $1\leq r(x)\leq 2$.

Lemma 3.3. *Under assumption (F_1) the operator $C: W(Q)\supseteq D(C)\to W'(Q)$ given by*

$$
\langle Cu,v\rangle=\int_Q f(x,t,u,\nabla u,\Delta u,\nabla\Delta u)v\,dxdt \tag{3.5}
$$

where $u\in D(C)=\{z\in W(Q): z_t\in W'(Q)\}$, is continuous and compact.

Proof. For any $u \in W(Q)$, let $u_\nu \to u$ in $W(Q)$. Thus, there exists a subsequence $\{u_\nu\}$ such that

$$\begin{aligned}
&u_\nu \to u, \nabla u_\nu \to \nabla u, \triangle u_\nu \to \triangle u \text{ and } \nabla\triangle u_\nu \to \nabla\triangle u \text{ a.e. in } \quad Q\\
&|u_\nu(x)| \le k(x), |\partial u_\nu(x)/\partial x_j| \le w_j, |\partial^2 u_\nu(x)/\partial x_j^2| \le z_j,\\
&|\partial^3 u_\nu(x)/\partial x_i \partial x_j^2| \le l_{ij} \text{ a.e.} \quad x \in \Omega,\\
&\text{for some } k, w_j, z_j, l_{ij} \in L^{p(x)}(Q)
\end{aligned} \tag{3.6}$$

Thus, since f is a Carathéodory function, we get

$$f(x,t,u_\nu,\nabla u_\nu,\Delta u_\nu,\nabla\Delta u_\nu) \to f(x,t,u,\nabla u,\Delta u,\nabla\Delta u) \qquad \text{a.e in } Q \tag{3.7}$$

From (F_1) $i)$ it follows

$$|f(x,t,u_\nu,\nabla u_\nu,\Delta u_\nu,\nabla\Delta u_\nu)| \le c_1 \sum_{i=1}^{4} |l_i(x,t)| + |k(x,t)| \qquad \text{a.e in } Q \tag{3.8}$$

Since

$$c_1 \sum_{i=1}^{4} |l_i(x,t)| + |k(x,t)| \in L^{p(x)}(Q),$$

we conclude with the dominated convergence theorem that

$$C(u_\nu) \to C(u) \quad \text{in } L^{p(x)}(Q).$$

Therefore, the entire sequence $\{C(u_\nu)\}$ converges to $C(u)$ in $L^{p(x)}(Q)$. Thus, C is contimuous on $W(Q)$.

Recall that the embedding $I : W(Q) \to L^{p(x)}(Q)$ is continuous and compact (See Remark 2.7 iii)) and so the adjoint operator $I^* : L^{p'(x)}(Q) \to W'(Q)$ given by

$$\langle I^*(f), \varphi\rangle = \int_Q f\varphi \, dxdt$$

is also compact. Therefore $C = I^* \circ C : W(Q) \supseteq D(C) \to W'(Q)$ is continuous and compact. ☐

Lemma 3.4. *Let $T : W(Q) \supseteq D(T) \to W'(Q)$ be defined by*

$$\langle Tu, v\rangle = -\int_Q uv_t \, dxdt, \quad \textit{for all } u \in D(T), v \in W(Q), \tag{3.9}$$

where $D(T) = \{z \in W(Q) : z_t \in W'(Q), z(0) = 0\}$. The linear operator T is generated by $(.)_t \equiv \partial/\partial t$ via the relation

$$\langle Tu, v\rangle = \int_0^T \langle u_t(t), v(t)\rangle_X \, dt \quad \textit{for all } u \in D(T), v \in W(Q).$$

Then T is a closed, densely defined, maximal monotone operator.

Proof. The proof can be established by mimicking the arguments of [39] with little modifications. ☐

4. Main result

In this section, we give the notion of weak solution for the nonlinear parabolic problem (1.1) and we state the main result of this work.

Definition 4.1. A function $u \in W(Q)$ with $u_t \in W'(Q)$ is called a weak solution of problem (1.1) if:

$$\int_Q u_t\varphi\, dxdt + \int_Q \langle \mathcal{L}(u), \varphi\rangle\, dxdt + \int_Q f(x,t,u,\nabla u,\Delta u,\nabla\Delta u)\varphi\, dxdt = h,$$

for all $\varphi \in W(Q)$.

For the existence of a weak solution, we will use Theorem 2.8 to achieve our goal. Then, we only need to verify that all the conditions in this Theorem are fulfilled.

Theorem 4.2. *Let $h \in W'(Q)$. Suppose $(M_0)$and(F_1) hold. Then problem (1.1) admits at least one weak solution.*

Proof. In order to prove that problem (1.1) has a weak solution in $W(Q)$ it is sufficient to show that, for $h \in W'(Q)$. the equation

$$T + \mathcal{L} + C = h \quad \text{in } W'(Q) \tag{4.1}$$

is solvable, where $T, \mathcal{L}$ and C are given in (3.9), (3.1) and (3.5) respectively. To get this, we apply Asfaw's abstract theorem.

In view of lemmas 3.2, 3.3 and 3.4, it remains to show that C lies in Γ_σ^τ and $T+\mathcal{L}+C$ is coercive. This is done in the following two claims.

Claim 1: C lies in Γ_σ^τ

By applying condition (F_1) and Holder's inequality, we have

$$\begin{aligned}
&|\langle Cu, v\rangle| = |\int_0^T \int_\Omega f(x,t,u,\nabla u,\Delta u,\nabla\Delta u)v\, dxdt| \\
&\le c_1' \int_Q (|u| + |\nabla u| + |\Delta u| + |\nabla\Delta u|)|v|\, dxdt + c_2' \int_0^T |k(t)|_{L^{p(x)}(\Omega)}|v(t)|_X\, dt \\
&\le c_3'\|u\|\|v\| + c_4'\|k\|_{L^\infty(0,T;L^{p(x)}(\Omega))}\|v\|,
\end{aligned}$$

for all $u, v \in W(Q)$.

Therefore, taking supremum over all $v \in W(Q)$ with $\|v\| \le 1$ we get $\|Cu\|_{W'(Q)} \le \tau\|u\| + \sigma$. So, $C \in \Gamma_\sigma^\tau$.

Claim 2: $T + \mathcal{L} + C$ is coercive

Now, by using (M_0), (F_1) and monotonicity of T, ($\langle Tu, u\rangle \ge 0, \quad$ for all $u, v \in D(T)$), we have for $\|u\| > 1$

$$\begin{aligned}
&\langle Tu + Su + Cu, u\rangle \\
&\ge \int_0^T \int_\Omega M(L(u))\big(\frac{1}{p(x)}|\nabla u|^{p(x)}\big)\, dxdt + \int_0^T \int_\Omega f(x,t,u,\nabla u,\Delta u,\nabla\Delta u)u\, dxdt \\
&\ge \frac{m_0}{p^+}\|u\|^{p^-} + \|u\|^{r^-} \to +\infty, \text{ as } \|u\| \to +\infty.
\end{aligned}$$

Noting that $\left|\int_Q hu\, dxdt\right| \le c_5'\|h\|_{L^{p'(x)}(Q)}\|u\|, \quad$ for all $u \in W(Q)$, we get $R = R(h) > 0$ such that

$$\langle Tu + Su + Cu - h, u\rangle > 0$$

for $u \in D(T) \cap \{v \in W(Q) : \|v\| = R\}$.

We conclude that the equation $Tu + Su + Cu = h$ has a weak solution in $D(T)$ since the criteria of Theorem 2.8 are verified. Therefore, problem (1.1) admits at least one weak solution. □

It is important to note that for the existence of weak solutions for (1.1), the monotonicity assumption on f with respect to u is not required. See the works [7, 9, 22, 26] and their references, for existence of weak solutions in elliptic as well as parabolic problems under monotone nonlinearities independent of ∇u.

It seems to be interesting to study problem (1.1) and the properties of its solutions for $\frac{2N}{N+1} < p_m(x) < 2$. We plan to address these questions in a future research.

Acknowledgement. The author would like to thank the anonymous referees for for their helpful suggestions. This work was financially supported by the VRIP- UNMSM, Project Number B25143881, R.R 005446-2025-R.

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