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A FRACTIONAL MODEL FOR ELECTRO-MECHANICAL VISCOELASTIC CONTACT WITH NORMAL COMPLIANCE CONDITION

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Abstract. This article addresses the analysis of a quasistatic contact problem between an electro-viscoelastic body and a rigid foundation, considering a gap function. The constitutive relation is constructed using a fractional Kelvin-Voigt model, while the contact is described by a version of the normal compliance condition. After establishing the variational formulation of the model, we prove the existence of a weak solution based on monotone operators, the Caputo derivative, the Faedo-Galerkin method, and Banach's fixed-point theorem.

Keywords. Fractional Kelvin-Voigt, fixed-point, weak solution, Galerkin method, quasistatic contact
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1. Introduction

The field of contact mechanics has been extensively developed through classical approaches, producing robust models for a wide range of physical interactions. These encompass the behavior of flexible and plastic bodies [13], systems with thermal characteristics [6], and those under electrical influences [2]. Understanding has also been significantly advanced for problems involving corrosion and friction between bodies [14].

Recent scientific inquiry has pivoted towards non-classical mechanics, an emerging area of study closely linked to the use of fractional partial derivatives. This mathematical tool has substantially changed concepts across mathematics and its branches. Due to their significant applied value, fractional derivatives have enabled the modeling of contact mechanics problems through modified relations, offering a more accurate description of memory-dependent processes and viscoelastic behavior [1, 5, 4, 3].

In this research, we generalize key findings from [5, 4, 3] to investigate materials with combined viscoelastic properties and electrical effects. These were previously

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modeled using classical mechanics by the relation:

$$\boldsymbol{\sigma}(t) = \mathbb{A}\varepsilon\left(\dot{\mathbf{w}}(t)\right) + \mathbb{B}\varepsilon(\mathbf{w}(t)) + \mathbb{E}^{*}\nabla\psi(t) \quad \text{in } D \times (0,T).$$

For the purpose of this analysis, we adopt a fractional derivative-based Kelvin-Voigt model, characterized by the relation:

$$\boldsymbol{\sigma}(t) = \mathbb{A}\varepsilon\left({}_{0}^{C}D_{t}^{\alpha}\mathbf{w}(t)\right) + \mathbb{B}\varepsilon(\mathbf{w}(t)) + \mathbb{E}^{*}\nabla\psi(t) \quad \text{in } D \times (0,T)$$

In this model, the fractional order is denoted by the parameter α, where $0 < \alpha < 1$. The frictional contact is governed by a normal damped response condition alongside Coulomb's law of friction. This paper is structured as follows: the model and its mechanical formulation are introduced in Section II. Section III establishes the necessary notation, presents preliminary results, states the underlying hypotheses, and derives the problem's weak formulation. Section IV states the principal result on the existence and uniqueness of the solution, formulated in Theorem 4.1. The proof employs a fixed-point argument, combining the properties of monotone operators derived from the Caputo fractional derivative with the Faedo-Galerkin approximation method. Finally, Section V provides a numerical example and discusses its practical application.

2. The model

We define the physical setting as follows, a body with electro-viscoelastic properties occupies a bounded domain $D \subset \mathbb{R}^d$ and has a smooth boundary denoted by ∂D. ∂D is divided into 3 disjoint measurable parts ∂D_1, ∂D_2, and ∂D_3 such that $meas(\partial D_1)$ is positive. Let $(0,T); T > 0$ be the time interval of interest. The body is clamped on $\partial D_1 \times (0,T)$, where the displacement field is zero. A surface traction of density f_1 is applied on ∂D_2. We study the motion of the body when it is subjected to the action of body forces with a density of f_0. Finally, for the mechanical contact on ∂D_3, there can be a frictional contact of a body with a rigid foundation. This frictional contact model is described by the Coulomb's law of dry friction and normal damped response conditions. For the electrical conditions, represented by the potential exchange between the body and the foundation, we have the linear function, $H\left(\psi\left(t\right)\right) = \psi\left(t\right) - \psi_0$. Now we define ν as the unit outward normal vector and $\mathbf{S}^d$ as the space of tensors which are second-order and symmetric on $\mathbb{R}^d$. The inner products and its norms in $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted by

$$w.\tilde{w} = w_i.\tilde{w}_i, \quad \|\tilde{w}\|\mathbb{R}^d = (\tilde{w},\tilde{w})^{\frac{1}{2}} \quad \forall w = (w_i), \tilde{w} = (\tilde{w}_i) \in \mathbb{R}^d,$$

$$\sigma.\tau = \sigma_{ij}.\tau_{ij}, \quad \|\tau\|\mathbb{S}^d = (\tau,\tau)^{\frac{1}{2}} \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d$$

Moreover, the classical notations for the normal and tangential components for the displacement vector and the stress tensor,

$$\tilde{w}_\nu = \tilde{\mathbf{w}} \cdot \nu, \quad \tilde{\mathbf{w}}_\tau = \tilde{\mathbf{w}} - \tilde{w}_\nu \nu. \quad \sigma_\nu = (\sigma\nu)\cdot\nu, \quad \tilde{\mathbf{w}}_\tau = \sigma\nu - \sigma_\nu\nu,$$

We introduce the unitary vector $\delta : \partial D_3 \to \mathbb{R}^d$ defined by $\delta = \frac{\mathbf{w}_\tau(t)}{\|\mathbf{w}_\tau(t)\|}$. There is a gap between ∂D_3 of the two bodies, measured along the direction of ν, denoted by g. Let $\mathbf{w} : D \times [0,T] \to \mathbb{R}^d$ be the displacement field, $\boldsymbol{\sigma} = (\sigma_{ij}) : D \times [0,T] \to \mathbb{S}^d$

the stress tensor and $\varepsilon(w) = (\varepsilon_{ij}(w)) = \frac{1}{2}(w_{i.j} + w_{j.i})$ the linearized strain tensor. Here we will introduce the fractional contact problem.

Problem $\mathcal{P}$. Find the displacement $\mathbf{w} : D \times [0,T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma}$: $D \times [0,T] \to \mathbb{S}^d$, the electric potential $\psi : D \times [0,T] \to \mathbb{R}$ and the electric displacement field $\mathbb{D} : D \times [0,T] \to \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}(t) = \mathbb{A}\varepsilon\left({}_0^C D_t^\alpha \mathbf{w}(t)\right) + \mathbb{B}\varepsilon(\mathbf{w}(t)) + \mathbb{E}^*\nabla\psi(t) \quad \text{in } D \times (0,T), \tag{2.1}$$

$$\text{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } D \times (0,T), \tag{2.2}$$

$$\text{div}\boldsymbol{D} + \mathbf{q}_0 = 0 \quad \text{in } D \times (0,T), \tag{2.3}$$

$$\mathbf{w} = \mathbf{0} \quad \text{on } \partial D_1 \times (0,T), \tag{2.4}$$

$$\mathbb{D}(t) = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{w}(t)) - \boldsymbol{P}\nabla(\psi(t)), \quad \text{in } D \times (0,T), \tag{2.5}$$

$$\left\{ \begin{array}{l} \sigma\nu = -p\nu\left(\mathbf{w}_\nu(t) - g\right), \\ \sigma\tau = -p\tau\left(\mathbf{w}_\nu(t) - g\right)\frac{\mathbf{w}_\tau(t)}{||\mathbf{w}_\tau(t)||}, \text{ if } \mathbf{w}_\tau(t) \neq 0, \end{array} \right. \quad \text{on } \partial D_3 \times (0,T), \tag{2.6}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_1 \quad \text{on } \partial D_2 \times (0,T), \tag{2.7}$$

$$\mathbf{D}.\boldsymbol{\nu} = H\left(\psi(t)\right) \quad \text{on } \partial D_3 \times (0,T), \tag{2.8}$$

$$\psi = 0 \quad \text{on } \partial D_a \times (0,T), \tag{2.9}$$

$$\mathbb{D} \cdot \boldsymbol{\nu} = q_2 \quad \text{on } \partial D_b \times (0,T), \tag{2.10}$$

$$\mathbf{w}(0) = \mathbf{w}_0 \qquad \text{in } D. \tag{2.11}$$

Now we describe the problem (1)-(2.11) and provide an explanation of the equations and the boundary conditions. Equations (1) and (2.5) represent the time-fractional electro-viscoelastic constitutive law of the Caputo type [15]. Relations (2.2) and (2.3) are the equilibrium equations for the stress and electric-displacement fields, respectively. Conditions (2.4) and (2.7) are the displacement and traction boundary conditions. Equation (2.6) represents frictional contact, considering normal damped response conditions with Coulomb's law of dry friction, and (2.8) describes contact models of electric conductivity. Next, (2.9) and (2.10) represent the electric boundary conditions for the electrical potential on ∂D_a and the electric charges on ∂D_b respectively. Equation (2.11) represents the initial displacement field.

3. Variational formulation

In this section, we derive a weak formulation for the problem ($\mathcal{P}$) and explore its solvability. Throughout what follows, we use the classical notation for the L^p spaces and the associated Sobolev spaces related to the domain $D \subset \mathbb{R}^d$.

3.1. **Notations and preliminaries.** We present the following Hilbert spaces H, $\mathcal{H}$, $H^1(D)^d$ and $\mathcal{H}_1$ by,

$$H = L^2(D)^d = \left\{\tilde{\mathbf{w}} = (\tilde{w}_i) : \tilde{w}_i \in L^2(D)\right\}, \mathcal{H} = \left\{\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(D)\right\},$$

$$H^1(D)^d = \{\tilde{\mathbf{w}} = (\tilde{w}_i) \in H : \varepsilon(\tilde{\mathbf{w}}) \in \mathcal{H}\}, \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} : \operatorname{Div} \boldsymbol{\tau} \in H\},$$

equipped with the inner products and the associated norms. Also, for each real Hilbert space X, we introduce the spaces $L^p(0,T;X)$, $C(0,T;X)$ and $W^{k,p}(0,T;X)$, for $1 \le p \le +\infty$ and $k = 1, 2, \cdots$. Considering the conditions (2.4), we introduce the closed subspaces of $H^1(D)^d$ defined by

$$\begin{aligned}
\mathcal{J} &= \left\{\tilde{\mathbf{w}} \in H^1(D)^d : \tilde{\mathbf{w}} = 0 \text{ on } \partial D_1\right\}, \\
\mathcal{G} &= \left\{\varphi \in H^1(D)^d : \varphi = 0 \text{ on } \partial D_a\right\}, \\
\mathcal{Q} &= \{\ \mathfrak{D} = (\mathbb{D}_i) : \mathbb{D}_i \in L^2(D), \operatorname{div}\mathbb{D} \in L^2(D)\},
\end{aligned}$$

with the usual inner products and norms. Since $\operatorname{meas}(\partial D_a) > 0$ and $\operatorname{meas}(\partial D_1) > 0$, the Korn's and Friedrichs-Poincaré inequalities hold, for all $C_K, C_F > 0$:

$$\|\varepsilon(\tilde{\mathbf{w}})\|_{\mathcal{H}} \ \ge\ C_K \|\tilde{\mathbf{w}}\|_{H^1(D)^d}, \quad \forall \tilde{\mathbf{w}} \in \mathcal{J}, \tag{3.1}$$

$$\|\nabla\varphi\|_{\mathcal{Q}} \ \ge\ C_F \|\varphi\|_{H^1(D)}, \quad \forall\, \varphi \in \mathcal{G}, \tag{3.2}$$

Moreover, by Sobolev trace theorem, there exist two positive constants, such that

$$\|\tilde{\mathbf{w}}\|_{L^2(\partial D_3)^d} \ \le\ C_{s_1} \|\tilde{\mathbf{w}}\|_{\mathcal{J}}, \forall \tilde{\mathbf{w}} \in \mathcal{J}, \tag{3.3}$$

$$\|\varphi\|_{L^2(\partial D_3)} \ \le\ C_{s_2} \|\varphi\|_{\mathcal{G}}, \forall \varphi \in \mathcal{G}, \tag{3.4}$$

Keeping in mind that $\mathcal{J}$ is a Banach space and satisfies the property of reflexivity and separability, and the function defined on the $\mathcal{J} \to H$ is necessarily dense, continuous and its injection map ι is also supposed to be compact, the dual mapping $\iota : H \to \mathcal{J}$ is linear, continuous and compact. To simplify the notation, we define the following functions,

$$\begin{aligned}
&a : \mathcal{J} \times \mathcal{J} \to \mathbb{R}, \quad a(w, \tilde{w}) = (\mathbb{A}\varepsilon(w), \varepsilon(\tilde{w}))_{\mathcal{H}}, \\
&b : \mathcal{J} \times \mathcal{J} \to \mathbb{R}, \quad b(w, \tilde{w}) = (\mathbb{B}\varepsilon(w), \varepsilon(\tilde{w}))_{\mathcal{H}}, \\
&M : \mathcal{G} \times \mathcal{J} \to \mathbb{R}, \quad M(\psi, \tilde{w}) = (\mathbb{E}^*\nabla\psi, \varepsilon(\tilde{w}))_{\mathcal{G}}.
\end{aligned} \tag{3.5}$$

With Riesz's representation theorem, we define the elements $\mathbf{f(t)} \in \mathcal{J}$ and $q(t) \in \mathcal{G}$ as follows,

$$(\mathbf{f(t)}, \tilde{w})_{\mathcal{J}} = \int_D f_0(t).\tilde{w}dx + \int_{\partial D_2} f_1(t).\tilde{w}dx \tag{3.6}$$

$$(q(t), \tilde{w})_{\mathcal{G}} = \int_D q_0(t).\psi dx + \int_{\partial D_2} q_2(t).\psi dx \tag{3.7}$$

for all $w, \tilde{w} \in \mathcal{J}$ and $\psi \in \mathcal{G}$. Additionally, the mapping $J : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ is defined as follows,

$$J(w(t), \tilde{w}(t)) = j_\nu(w(t), \tilde{w}) + j_\tau(w(t), \tilde{w})$$

$$J(w(t), \tilde{w}) = \int_{\partial D_3} (p_\nu\left(\mathbf{w}\nu\left(t\right) - g\right)\tilde{w}\nu) ds + \int_{\partial D_3} p_\tau\left(\mathbf{w}_\nu\left(t\right) - g\right)\delta.\tilde{w}_\tau ds \tag{3.8}$$

$$G\left(\psi, \varphi\right) = \int_{\partial D_3} H\left(\psi\left(t\right)\right)\varphi da \quad \text{for all } \psi \in \mathcal{G}. \tag{3.9}$$

To analyze the mechanical problem defined by (1)-(2.11), it is important to make the following hypotheses,

3.2. **Assumptions on the data.** To establish the existence of a weak solution, we now state the necessary hypotheses on the problem data. These assumptions serve two primary purposes: firstly, they provide the minimal regularity required for the application of functional analytic tools. Secondly, they are motivated by the physical context of the problem, ensuring that the material parameters, source terms, and boundary conditions are consistent with the solid mechanics. The specific assumptions are as follows:
The *viscosity operator* denoted $\mathbb{A} : D \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies,

$$\begin{cases} \text{(a) There exists } L_{\mathbb{A}} > 0 \text{ such that} \\ \qquad \|\mathbb{A}(\mathbf{x}, \varepsilon_1) - \mathbb{A}(\mathbf{x}, \varepsilon 2)\| \leq L_{\mathbb{A}} \|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in D. \\ \text{(b) There exists } m_{\mathbb{A}} > 0 \text{ such that} \\ \qquad (\mathbb{A}(\mathbf{x}, \varepsilon_1) - \mathbb{A}(\mathbf{x}, \varepsilon 2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathbb{A}} \|\varepsilon_1 - \varepsilon_2\|^2 \\ \qquad \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in D. \end{cases} \tag{3.10}$$

The *elasticity operator* $\mathbb{B} : D \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies,

$$\begin{cases} \text{(a) There exists } L_{\mathbb{B}} > 0 \text{ such that} \\ \qquad \|\mathbb{B}(\mathbf{x}, \varepsilon_1) - \mathbb{B}(\mathbf{x}, \varepsilon 2)\| \leq L_{\mathbb{B}} \|\varepsilon_1 - \varepsilon_2\| \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. \mathbf{x} \in D. \end{cases} \tag{3.11}$$

The *operator* $a : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ satisfies,

$$\begin{cases} \text{(a) The mapping } (w, \tilde{w}) \mapsto a(w, \tilde{w}) \text{ satisfies the symmetry,} \\ \qquad a_{ijkl} = a_{lkig} = a_{jikl} \in L^\infty(D). \\ \text{(b) There exists } L_a > 0 \text{ such that} \\ \qquad |a(w, \tilde{w})| \leq L_a \|w\|_{\mathcal{J}} \|\tilde{w}\|_{\mathcal{J}} \quad \text{for all } w, \tilde{w} \in \mathcal{J}. \\ \text{(c) The mapping } a(w, w) \text{ satisfies the ellipticity,} \\ \qquad \text{There exists } m_a > 0 \text{ such that,} \\ \qquad\qquad |a(w, \tilde{w})| \geq m_a \|w\|^2_{\mathcal{J}} \quad \text{for all } w \in \mathcal{J}. \end{cases} \tag{3.12}$$

The *operator* $b : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ satisfies,

$$\begin{cases} \text{(a) The mapping } (w, \tilde{w}) \mapsto b(w, \tilde{w}) \text{ satisfies the symmetry,} \\ \qquad b_{ijkl} = b_{lkij} = b_{jikl} \in L^\infty(D). \\ \text{(b) There exists } L_b > 0 \text{ such that} \\ \qquad |b(w, \tilde{w})| \leq L_b \|w\|_{\mathcal{J}} \|\tilde{w}\|_{\mathcal{J}} \quad \text{for all } w, \tilde{w} \in \mathcal{J}. \\ \text{(c) The mapping } b(w, w) \text{ satisfies the ellipticity,} \\ \qquad \text{There exists } m_b > 0 \text{ such that,} \\ \qquad\qquad |b(w, \tilde{w})| \geq m_b \|w\|^2_{\mathcal{J}} \quad \text{for all } w \in \mathcal{J}. \end{cases} \tag{3.13}$$

The *operator* $M : \mathcal{G} \times \mathcal{J} \to \mathbb{R}$ satisfies,

$$\begin{cases} \text{(a) The mapping } (\psi, \tilde{w}) \mapsto M(\psi, \tilde{w}) \text{ satisfies the symmetry,} \\ \qquad M_{ij} = M_{ji} \in L^\infty(D). \\ \text{(b) There exists } L_M > 0 \text{ such that} \\ \qquad |M(\psi, \tilde{w})| \leq L_M \|\psi\|_W \|\tilde{w}\|_{\mathcal{J}} \quad \text{for all } w, \tilde{w} \in \mathcal{J}. \end{cases} \tag{3.14}$$

The *contact function* $p_\nu : \partial D_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following hypotheses,

$$\begin{cases} \text{(a) The mapping } \mathbf{x} \mapsto p_\nu(x,w) \text{ is Lebesgue measurable on } \partial D_3 \\ \qquad \text{for any } w \in \mathbb{R}. \\ \text{(b) } \mathbf{x} \mapsto p_\nu(x,w) = 0 \text{ for } w \leq 0 \text{ for all } x \in \partial D_3. \\ \text{(c) There exists } c_\nu > 0 \text{ such that } |p_\nu(x,w)| \leq c_\nu. \\ \text{(d) There exists } L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(.,w) - p_\nu(.,\tilde{w})| \leq L_\nu |w - \tilde{w}| \quad \text{for all } w, \tilde{w} \in \mathbb{R}_+. \end{cases} \tag{3.15}$$

The *contact function* $p_\tau : \partial D_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies,

$$\begin{cases} \text{(a) The mapping } \mathbf{x} \mapsto p_\tau(x,w) \text{ is Lebesgue measurable on } \partial D_3 \\ \qquad \text{for any } w \in \mathbb{R}. \\ \text{(b) } \mathbf{x} \mapsto p_\tau(x,w) = 0 \text{ for } w \leq 0 \text{ for all } x \in \partial D_3. \\ \text{(c) There exists } c_\tau > 0 \text{ such that } |p_\tau(x,w)| \leq c_\tau. \\ \text{(d) There exists } L_\tau > 0 \text{ such that} \\ \qquad |p_\tau(.,w) - p_\tau(.,\tilde{w})| \leq L_\tau |w - \tilde{w}| \quad \text{for all } w, \tilde{w} \in \mathbb{R}_+. \end{cases} \tag{3.16}$$

On the other hand, we need essential conditions for the piezoelectric operator, electric permittivity operator and electrical conductivity function, where their conditions are provided in several studies, see [7]. The density of volume forces, traction, volume electric charges, surface electric charges satisfy

$$\mathbf{f}_0 \in \mathcal{C}(0,T;H), \quad \mathbf{f}_2 \in \mathcal{C}(0,T;L^2(\partial D_2)^d), q_0 \in \mathcal{C}(0,T;L^2(D)), q_2 \in \mathcal{C}(0,T;L^2(\partial D_b)), \tag{3.17}$$

$$p_\tau(.r) \in L^2(\partial D_3), \quad p_\nu(.r) \in L^2(\partial D_3), \forall r \in \mathbb{R} \quad g \in L^2(\partial D_3), g \geq 0 \text{ a.e on } \partial D_3. \tag{3.18}$$

The initial displacement, the potential of the foundation satisfy

$$\mathbf{w}_0 \in \mathcal{J}, \psi_0 \in L^2(\partial D_3). \tag{3.19}$$

The variational formulation of (1)–(2.11) derived using Green's formula, is given by the following problem: Find the solution such that for all $\forall \mathbf{v} \in V$, $\varphi \in W$ and a.e $t \in (0,T)$.

3.3. **Problem $\mathcal{P}_V$.** Find the displacement field $\mathbf{w} : [0,T] \to \mathcal{J}$, the stress field $\boldsymbol{\sigma}$: $[0,T] \to \mathcal{H}_1$, the electric potential $\psi : [0,T] \to W$ and the electric displacement field $\mathbf{D}$:$[0,T] \to H$ such that

$$\boldsymbol{\sigma}(t) = \mathbb{A}\varepsilon\left({}_0^C D_t^\alpha \mathbf{w}(t)\right) + \mathbb{B}\varepsilon(\mathbf{w}(t)) + \mathbb{E}^* \nabla\psi(t) \quad \text{in } D \times (0,T), \tag{3.20}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\tilde{\mathbf{w}}))_{\mathcal{H}} + J(\mathbf{w}(t), \tilde{\mathbf{w}}) = (\mathbf{f}(t), \tilde{\mathbf{w}})_{\mathcal{J}}, \tag{3.21}$$

$$\mathbb{D}(t) = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{w}(t)) - \boldsymbol{B}\nabla\psi(t), \tag{3.22}$$

$$(\mathbb{D}(t), \nabla\varphi)_H + (q(t), \varphi)_W = G_2(\psi, \varphi), \tag{3.23}$$

$$\mathbf{w}(0) = \mathbf{w}_0, \qquad \text{in } D. \tag{3.24}$$

4. Existence of a solution

Theorem 4.1. *Assuming that assumptions (3.10)-(3.19) hold. Then, there exists at least one solution of problem ($\mathcal{P}_V$). Moreover, the solution satisfies*

$$w \in W^{1,2}(0,T;\mathcal{J}),\ \psi \in \mathcal{C}(0,T;\mathcal{G}),\ \text{and } \mathbf{D} \in \mathcal{C}(0,T;\mathcal{Q}).$$

Proof (of Theorem 4.1). The proof of Theorem (4.1) depends on several steps, the first one is using variational inequalities, the second is the Galerkin method, and in the end we apply the Banach fixed-point theorem.
In the first step, let us $\chi \in L^2(0,T;\mathcal{J})$ be given by,

$$(\chi(t),\tilde{w})_{\mathcal{J}} = M(\psi,\tilde{w}), \quad \forall \tilde{w} \in \mathcal{J}, \tag{4.1}$$

□

we can reformulate problem (3.20)-(3.21)-(3.24) as follows,

4.1. **Problem $\mathcal{PV}_\chi$.** Find the $\mathbf{w} \in \mathcal{J}$ such that $\forall t \in (0,T)$,$\tilde{w} \in \mathcal{J}$, $\alpha \in]0,1[$

$$a({}_0^C D_t^\alpha \mathbf{w}_\chi(t),\tilde{w})_{\mathcal{J}} + b(\mathbf{w}_\chi(t),\tilde{w})_{\mathcal{J}} + (\chi(t),\tilde{w})_{\mathcal{J}} + J(\mathbf{w}\chi(t),\tilde{w}) = (\mathbf{f}(t),\tilde{\mathbf{w}})_{\mathcal{J}}, \tag{4.2}$$

$$\mathbf{w}(0) = \mathbf{w}_0. \tag{4.3}$$

For this problem we have this result,

Lemma 4.2. *$\forall$ $w \in \mathcal{J}$ and $t \in (0,T)$. Problem $(\mathcal{PV}_\chi)$ has at least one solution $\mathbf{w}_\chi \in \mathcal{G}^{1,2}(0,T;\mathcal{J})$ and $\sigma_\chi \in L^2(0,T;L^2(\partial D,\mathbb{S}^d)$.*

Proof (of Lemma 4.2). By the representation theorem of Riesz,

$$(\mathbf{f}\chi(t),\tilde{w})_{\mathcal{J}} = (\mathbf{f}(t),\tilde{w})_{\mathcal{J}} - (\chi,\tilde{w})_{\mathcal{J}}, \tag{4.4}$$

then equation (4.3) can be written as,

$$a({}_0^C D_t^\alpha \mathbf{w}_\chi(t),\tilde{w})_{\mathcal{J}} + b(\mathbf{w}_\chi(t),\tilde{w})_{\mathcal{J}} + J(\mathbf{w}_\chi(t),\tilde{w}) = (\mathbf{f}_\chi(t),\tilde{\mathbf{w}})_{\mathcal{J}}. \tag{4.5}$$

Now, we apply the method of Faedo-Galerkin, that is to say we impose the functions β_j, for $j = 1,\cdots,k$, are eigenfunctions $-\Delta$ such that $(\beta_j)_{1\le j\le k}$ form a Hilbert basis of $H^1(D)$. Fix a $k \in \mathbb{N}$ and the purpose is to find the function in the following form,

$$\mathbf{w}_{\chi k} : (0,T) \to H^1(D), \quad \mathbf{w}_{\chi k} = \sum_{l=1}^{k} d_k^l(t)\beta_l(t). \tag{4.6}$$

We denote $F_k = \text{Spam }(\beta_1(t),\beta_2(t),\cdots,\beta_k(t))$ whence: $\mathbf{w}_{\chi k} \in F_k$ and $\mathbf{w}_{\chi k} \to \mathbf{w}_\chi$.

For each integer $k > 0$, the approximate problem of $(\mathcal{PV}_\chi)$ given by,

4.2. **Problem $(\mathcal{PVP})$.** Find the $\mathbf{w}_{\chi k} \in L^2(0,T;F_k)$ such that,

$$a({}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t),\beta_i)_{\mathcal{J}} + b(\mathbf{w}_{\chi k}(t),\beta_i)_{\mathcal{J}} + J(\mathbf{w}_{\chi k}(t),\beta_i) = (\mathbf{f}_\chi(t),\beta_i)_{\mathcal{J}}, \tag{4.7}$$

$$\mathbf{w}(0) = \mathbf{w}_0.$$

Considering that $\mathbf{w}_{\chi k}$ defined in (4.6), we observe that,

$$a\left({}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t),\beta_i\right)_{\mathcal{J}} = \mathbb{A}{}_0^C D_t^\alpha d_k^i(t), \qquad b\left(\mathbf{w}_{\chi k}(t),\beta_i\right) = \mathbb{B}d_k^i(t), \tag{4.8}$$

$$J(w_{\chi k}(t),\beta_i) = Jd_k^i(t), \qquad (\mathbf{f}_\chi(t),\beta_i)_{\mathcal{J}} = \mathbf{f}_\chi^i(t). \tag{4.9}$$

$$\tag{4.10}$$

So we can write (4.7) as follows,

$${}_0^C D_t^\alpha d_k^i(t) = \mathbb{A}^{-1}h(t,d_k^i(t)), \qquad d_k^i(0) = \mathbf{w}_{\chi 0}, \tag{4.11}$$

where the function $h(t, d_k^i(t))$ defined by $h(t, d_k^i(t)) = \mathbf{f}_\chi^i(t) - Jd_k^i(t) - \mathbb{B}d_k^i(t)$. Due to (3.11), there exists a positive constant $L_\mathbb{B}$ such that

$$|\mathbb{B}d_{k1}^i(t) - \mathbb{B}d_{k2}^i(t)| \leq L_\mathbb{B}|d_{k1}^i(t) - d_{k2}^i(t)|, \tag{4.12}$$

after that, using the formula of (3.8), hypothesis (3.3), (3.15) and (3.16) we find that,

$$|Jd_{k1}^i(t) - Jd_{k2}^i(t)| \leq C_{s1}^2(L_\tau + L_\nu)|d_{k1}^i(t) - d_{k2}^i(t)|, \tag{4.13}$$

adding the inequality (4.13) and (4.12) we conclude the existence of a $C_1 > 0$ such that,

$$|h(t, d_{k1}^i(t)) - h(t, d_{k2}^i(t))| \leq C_1|d_{k1}^i(t) -, d_{k2}^i(t)|. \tag{4.14}$$

By applying the standard methods for the Cauchy Type and Cauchy problems concerning fractional calculus [10], we establish that there exists a unique absolutely continuous function, $d_k^i(t) = (d_k^1(t), d_k^2(t), \cdots, d_k^k(t))$ on $[0, T_*)$ satisfying this system (4.11).

For the estimation, we use the standard method [9], multiply the equation (4.7) by $d_k^i(t)$, summing for $i = 1, \cdots, k$ and utilizing the fact that $w_{\chi k} \mapsto \frac{1}{2}\|\mathbf{w}\chi k\|\mathcal{J}^2$ is a convex functional, see [5] we have

$$\begin{aligned} &\sup(1, M_a)\, {}_0^C D_t^\alpha \left(\frac{1}{2}\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2\right) \\ &\leq a\left({}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t), \mathbf{w}_{\chi k}(t)\right) + b\left(\mathbf{w}_{\chi k}(t), \mathbf{w}_{\chi k}(t)\right)_\mathcal{J} \end{aligned} \tag{4.15}$$

and also using (3.13) then

$$m_b\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2 \leq |b(\mathbf{w}_{\chi k}(t), \mathbf{w}_{\chi k}(t))|, \tag{4.16}$$

since using (3.15), (3.16), (3.3) we obtain,

$$|J(w_{\chi k}(t), \mathbf{w}_{\chi k}(t))| \leq C_{s1}^2 meas(\partial D_3)(c_\tau + c_\nu)\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2, \tag{4.17}$$

and for all $\epsilon > 0$

$$|(\mathbf{f}_\chi(t), \mathbf{w}_{\chi k}(t))| \leq \epsilon\|\mathbf{f}_\chi(t)\|_\mathcal{J}^2 + \frac{1}{4\epsilon}\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2, \tag{4.18}$$

Then there exist C_2 and C_3 a positive constant such that, for all $\epsilon > 0$ we have

$${}_0^C D_t^\alpha\left(\frac{1}{2}\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2\right) + C_2\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2 \leq C_3\|\mathbf{f}_{\chi k}(t)\|_\mathcal{J}^2. \tag{4.19}$$

Using the results in [11], then we have this estimate,

$$\|\mathbf{w}_{\chi k}(t)\|_\mathcal{J}^2 + \frac{2C_2}{\partial D(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathbf{w}_{\chi k}(s)\|_\mathcal{J}^2\, ds \leq C_4\left(\|\mathbf{f}_\chi(t)\|_{L^2(0,T;F_k)}^2 + \|w_{\chi 0}\|_\mathcal{J}^2\right). \tag{4.20}$$

Now, we take $T_* = +\infty$, and $\tilde{w} \in \mathcal{J}$ with $\|\tilde{w}\|_\mathcal{J} \leq 1$, we can write $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$, where as $\tilde{w}_1 \in F_k$ and $F_k = spam(\beta_1, \cdots, \beta_k)$ in the other side $(\tilde{w}_2, \beta_j) = 0$ for $j = 1, \cdots, k$. Since, the functions $\{\beta_j\}_{j=1}^k$ is orthogonal in $\mathcal{J}$ then : $\|\tilde{w}_1\| \leq \|\tilde{w}\|_\mathcal{J} \leq 1$, using this equation (4.7), we find,

$$a({}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t), \tilde{w}_1)_\mathcal{J} + b(\mathbf{w}_{\chi k}(t), \tilde{w}_1)_\mathcal{J} + J(\mathbf{w}_{\chi k}(t), \tilde{w}_1) = (\mathbf{f}_\chi(t), \tilde{w}_1)_\mathcal{J}.$$

Now in the same way with (4.19) consequently we can find a positive constant C_5, C_6 such that,

$$\left\|{}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t)\right\|_{\mathcal{J}'} \le C_5 + C_6(\|\mathbf{w}_{\chi k}(t)\|_{\mathcal{J}} + \|\mathbf{f}_\chi(t)\|_{L^2(0,T;\mathcal{J})}), \tag{4.21}$$

and when we use (4.7), we can find a positive constant C such that,

$$\left\|{}_0^C D_t^\alpha \mathbf{w}_{\chi k}(t)\right\|_{\mathcal{J}'} \le C. \tag{4.22}$$

In the last step of thr proof we will use the result of the estimation and the compactness stated in Theorem [11], for the derivative of Caputo, then we conclude the existence of a sub-sequences $\mathbf{w}\chi kl$ and $\mathbf{w}\chi \in L^2(0,T;\mathcal{J})$ which are,

$$\mathbf{w}_{\chi kl} \to \mathbf{w}_\chi, \qquad {}_0^C D_t^\alpha \mathbf{w}_{\chi kl} \rightharpoonup {}_0^C D_t^\alpha \mathbf{w}_{\chi k}, \tag{4.23}$$

for the first we have strong convergence in $L^2(0,T;\mathcal{J})$ and the second weakly convergence in $L^2(0,T;\mathcal{J}')$ then

$$a\left({}_0^C D_t^\alpha \mathbf{w}_{\chi kl}, \tilde{w}\right) \to a\left({}_0^C D_t^\alpha \mathbf{w}_{\chi k}, \tilde{w}\right) \text{ in } \mathbb{R} \qquad b\left(\mathbf{w}_{\chi kl}(t), \tilde{w}\right) \to b(\mathbf{w}_{\chi k}, \tilde{w}) \text{ in } \mathbb{R}. \tag{4.24}$$

By (3.8) and assumptions (3.15)-(3.16), we obtain,

$$|J(\mathbf{w}_{\chi k}(t), \tilde{w})| \le (c_\nu + c_\tau)\|\tilde{w}\|_{\mathcal{J}}, \tag{4.25}$$

keeping in mind that the sequence$\{J(\mathbf{w}_{\chi k}(t), \tilde{w})\}_{k=1}^{+\infty}$ is bounded in $\mathbb{R}$, and for all$v = \mathbf{w}_{\chi k}(t) - \mathbf{w}_{\chi kl}$, then we have,

$$\begin{aligned}&|J\left(\mathbf{w}_{\chi k}(t), \mathbf{w}_{\chi k}(t) - \mathbf{w}_{\chi kl}(t)\right) - J\left(\mathbf{w}_{\chi kl}(t), \mathbf{w}_{\chi k} - \mathbf{w}_{\chi kl}(t)\right)| \\ &\le C_{s1}^2(c_\nu + c_\tau)\left\|\mathbf{w}_{\chi k}(t) - \mathbf{w}_{\chi kl}(t)\right\|_{\mathcal{J}}^2\end{aligned}.$$

Since the trace map $\partial D : \mathcal{J} \to L^2(\partial D_3)$ is a compact operator, we can deduce the existence of a subsequence of $\mathbf{w}_{\chi kl}$ we point it out again by $\mathbf{w}_{\chi kl}$, such that,

$$\mathbf{w}_{\chi kl} \to \mathbf{w}_{\chi k} \text{ strongly in } L^2(0,T;L^2(\partial D_3)), \qquad J(\mathbf{w}_{\chi kl}(t), \tilde{w}) \to J(\mathbf{w}_{\chi k}(t), \tilde{w}) \text{ in } \mathbb{R}.$$

Hence, a weak solution $\mathbf{w}_\chi \in W^{1,2}(0,T;\mathcal{J})$ exists for the Problem $(\mathcal{PV}_\chi)$, accordingly, the proof of Lemma 4.2 is considered complete. □

Finally, the function $\chi \in L^2(0,T;\mathcal{J})$ and $\mathbf{w}_\chi$ acquired in Lemma 4.2 we introduce the operator $\Lambda : L^2(0,T;\mathcal{J}) \to L^2(0,T;\mathcal{J})$ with,

$$(\Lambda\chi(t), \tilde{w})_{\mathcal{J}} = M(\psi_\chi, \tilde{w}) \text{ for all } \tilde{w} \in \mathcal{J} \text{ and } t \in (0,T), \tag{4.26}$$

the existence and uniqueness of the solution in the following lemma.

Lemma 4.3. *For $\chi \in L^2(0,T;)\mathcal{J}$, the function $\Lambda\chi(t) :]0,T[\to W$ is continuous. Moreover, there exists a unique element $\chi^* \in L^2(0,T;\mathcal{J})$ such that $\Lambda\chi^* = \chi^*$.*

Proof. Let $\chi \in L^2(0,T;\mathcal{J})$ and $t_1, t_2 \in (0,T)$. Using (4.1) and the assumption (3.14) we deduce that

$$\|\Lambda\chi(t_1) - \Lambda\chi(t_2)\|_{\mathcal{J}} \le c_s\|\chi(t_1) - \chi(t_2)\|_{\mathcal{J}}. \tag{4.27}$$

As $\mathbf{w}_\chi \in L^2(0,T;\mathcal{J})$, we can deduce that $\Lambda\chi \in C(0,T;\mathcal{J})$. Let now $\chi_1, \chi_2 \in L^2(0,T;\mathcal{J})$, in a similar fashion to (4.27), we have

$$\|\Lambda\chi_1(t) - \Lambda\chi_2(t)\|_{\mathcal{J}} \le c_s\|\chi_1(t) - \chi_2(t)\|_{\mathcal{G}}. \tag{4.28}$$

Then use (4.5), we find

$$\begin{aligned} a\left({}_0^C D_t^\alpha w_{\chi_1}(t) - {}_0^C D_t^\alpha\ w_{\chi_2}(t), w_{\chi_1}(t) - w_{\chi_2}(t)\right) \\ + b\left(w_{\chi_1}(t) - w_{\chi_2}(t), w_{\chi_1}(t) - w_{\chi_2}(t)\right) \\ + J\left(w_{\chi_1}(t), w_{\chi_1}(t)\right) - J\left(w_{\chi_1}(t), w_{\chi_2}(t)\right) \\ + J\left(w_{\chi_2}(t), w_{\chi_2}(t)\right) - J\left(w_{\chi_2}(t), w_{\chi_1}(t)\right) = 0. \end{aligned} \tag{4.29}$$

Consequently, combining (4.5), (3.8) and hypothesis (3.15), (3.16) we have,

$$\begin{aligned} \mid J\left(w_{\chi_1}(t), w_{\chi_1}(t)\right) - J\left(w_{\chi_1}(t), w_{\chi_2}(t)\right) + J\left(w_{\chi_2}(t), w_{\chi_2}(t)\right) \\ - J\left(w_{\chi_2}(t), w_{\chi_1}(t)\right) \mid \leq c_{s1}^2\left(c_\tau + c_\nu\right)\|w_{\chi_1}(t) - w_{\chi_2}(t)\|_{\mathcal{J}}^2. \end{aligned} \tag{4.30}$$

Then we use the definition of Caputo derivatives (see [8, 12]), we deduce

$$\left\|{}_0^C D_t^\alpha u_{\chi_1}(t) - {}_0^C D_t^\alpha u_{\chi_2}(t)\right\|_V \leq \frac{T^{1-\alpha}}{\partial D(\alpha)}\|\dot{u}_{\chi_1}(t) - \dot{u}_{\chi_2}(t)\|_V \tag{4.31}$$

then combining (4.30) we apply the integration from 0 to T and we use the Gronwall inequality, then for $c' > 0$, (4.30) we get:

$$\|w_{\chi_1}(t) - w_{\chi_2}(t)\|_{L^2(0,T;\mathcal{J})} \leq c'\|\chi_1(t) - \chi_2(t)\|_{L^2(0,T;\mathcal{J})}, \tag{4.32}$$

add to (4.28), (4.32), we have

$$\|\Lambda\chi(t_1) - \Lambda\chi(t_2)\|_{L^2(0,T;\mathcal{J})} \leq C_s\|\chi(t_1) - \chi(t_2)\|_{L^2(0,T;\mathcal{G})}. \tag{4.33}$$

To confirm that the operator Λ contracts, we repeat this process n times, resulting in,

$$\|\Lambda_{\chi 1}^n(t) - \Lambda_{\chi 2}^n(t)\|L^2(0,T;\mathcal{J}) \leq \frac{(C_s)^n}{n!}\|\chi_1(t) - \chi_2(t)\|_{L^2(0,T;\mathcal{J})}, \tag{4.34}$$

such that for n sufficiently large, the power Λ^n of Λ is a contraction in $L^2(0,T;\mathcal{J})$. Therefore, we have a unique $\chi^* \in L^2(0,T;\mathcal{J})$ such that $\Lambda\chi^* = \chi^*$. □

We now prove Theorem (4.1).

Proof (of Theorem 4.1). Let $\chi^* \in L^2(0,T;\mathcal{J})$ be a fixed point of the operator Λ. Denote by $\mathbf{w}\chi^*$ the solution of Problem $(\mathcal{PV}\chi)$, for $\chi^* = \chi$, using the definition of Λ (4.3), we find the solution of Problem $(\mathcal{P}_V)$. □

Finally, to prove the existence of a solution $(\psi, \mathbb{D})$, let $\chi \in \mathcal{C}(0,T;\mathcal{J})$ and we consider the following auxiliary problem,

4.3. **Problem $\mathcal{PV}_\psi$.** For this problem we are interested in $\psi : [0,T] \to \mathcal{G}$, $\mathbb{D} : [0,T] \to \mathcal{Q}$ an electrical potential, such that

$$\mathbb{D}_\chi(t) = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{w}_\chi(t)) - \boldsymbol{B}\nabla\psi(t), \tag{4.35}$$

$$(\boldsymbol{B}\nabla\psi(t), \nabla\varphi)_H - (\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{w}\chi(t)), \nabla\varphi)H + G(\psi,\varphi) = (q(t),\varphi)_{\mathcal{G}}, \tag{4.36}$$

for $\psi, \varphi \in \mathcal{G}$, $t \in (0,T)$. For Problem $(\mathcal{PV}_\psi)$ we have the following results.

Lemma 4.4. *Problem* $(\mathcal{PV}_\psi)$ *has a unique weak solution*

$$\{\psi_\chi, \mathbb{D}_\chi\}, \forall\chi \in \mathcal{C}(0,T;\mathcal{J})$$

such that

$$\psi_\chi \in \mathcal{C}(0,T;\mathcal{G}),\ \mathbb{D}_\chi \in \mathcal{C}(0,T;\mathcal{Q}),\ . \tag{4.37}$$

Furthermore, the solutions of Problem $(\mathcal{PV}_{\chi_i})$*, Furthermore, the solution satisfies*

$$\|\psi_1(t) - \psi_2(t)\|_{\mathcal{G}} \leq C\,\|\mathbf{w}_1(t) - \mathbf{w}_2(t)\|_{\mathcal{J}}\,. \tag{4.38}$$

Proof. See [7] □

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