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OPTIMIZATION METHOD FOR A NONLINEAR ELLIPTIC PROBLEM IN WEIGHTED VARIABLE EXPONENT SOBOLEV SPACES

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Abstract. In this paper, we prove some existence results of weak solutions for some nonlinear degenerate elliptic problem in weighted variable exponent Sobolev space with Dirichlet type boundary condition.

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1. Introduction

We consider the nonlinear degenerate elliptic problem

$$\begin{cases} -\operatorname{div}\left(\omega\Phi(\nabla u-\Theta(u))\right)+\omega|u|^{p(x)-2}u=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N\,(N \geq 2)$ is a bounded open set, $p \geq 2$, ∇u is the gradient of u, $f \in (W_0^{1,p(x)}(\Omega,\omega))'$, Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$, ω is a positive function on $\mathbb{R}^N$, and Φ is defined by

$$\Phi(\mu)=|\mu|^{p(x)-2}\mu, \qquad \forall \mu \in \mathbb{R}^N.$$

In recent years, considerable attention has been devoted to partial differential equations and variational problems involving variable exponents, motivated by applications in mathematical physics such as elasticity, electro-rheological fluids, and image processing [3, 10].

A variety of particular cases of problem (1.1) have been considered in the literature. For example, when $\Theta = 0$, Kim et al. [8] investigated the following elliptic problem with variable exponent:

$$\begin{cases} -\operatorname{div}(\omega(x)|\nabla u|^{p(x)-2}\nabla u)=g(x)|u|^{p(x)-2}u+f(\lambda,x,u,v) \quad \text{in } \Omega, \\ \qquad u=0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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In addition, when μ is not an eigenvalue of the divergence form

$$\begin{cases} -\operatorname{div}(\omega(x)|\nabla u|^{p(x)-2}\nabla u) = \mu g(x)|u|^{p(x)-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the exponent $p : \bar{\Omega} \longrightarrow (1, \infty)$ is continuous, $g \in L^\infty(\Omega)$, ω is a weight function on $\mathbb{R}^N$, and $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfies a Carathéodory condition, important structural properties of weighted variable exponent Sobolev spaces were established. Moreover, the existence of a weak solution to problem (1.2) was demonstrated by applying the Browder–Minty theorem in combination with bifurcation techniques for nonlinear operator equations.

In the case where $p(\cdot)$ is constant, problem (1.1) was studied by Sabri et al. [11] using the variational method combined with the theory of weighted Sobolev spaces, and the existence and uniqueness of weak solutions were established. Moreover, when $\omega = 1$ with Neumann-type boundary conditions, problem (1.1) has been analyzed by many authors under various assumptions; see, for example, [2, 5].

A well-known approach to nonlinear elliptic problems is the Leray–Lions method, which requires the operator to satisfy three fundamental properties: monotonicity, coercivity, and boundedness. In our problem, however, the perturbation $\Theta(u)$ inside the gradient term destroys the natural monotonicity of the p-Laplacian structure. This loss of monotonicity prevents the direct use of the Leray–Lions framework. To overcome this difficulty, we employ a variational method: the problem is reformulated through an appropriate energy functional, and the existence of weak solutions is established via the direct method of the calculus of variations.

The main purpose of this paper is to study the existence of weak solutions to the nonlinear degenerate elliptic problem (1.1) in weighted variable exponent Sobolev spaces, using the optimization method together with the properties of these spaces. Variational methods have been applied by several authors in related contexts; see [12, 13, 7]. The rest of the paper is organized as follows. In Section 2, we present definitions and fundamental properties of weighted variable exponent Sobolev spaces, along with some preliminary well-known results. In Section 3, we present the main assumptions. In Section 4 we employ variational methods to establish the existence of weak solutions for problem (1.1).

2. Preliminaries

We begin this section by introducing some notation, definitions, and lemmas. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 2 \right\}.$$

For any $h \in C(\overline{\Omega})$, define

$$h^+ = \sup_{x \in \overline{\Omega}} h(x), \qquad h^- = \inf_{x \in \overline{\Omega}} h(x).$$

Given $p \in C_+(\overline{\Omega})$, the weighted variable-exponent Lebesgue space $L^{p(\cdot)}(\Omega,\omega)$ consists of all measurable functions u such that

$$\int_\Omega \omega(x)\,|u(x)|^{p(x)}\,dx < \infty,$$

and it is endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega,\omega)} = \inf\left\{\lambda > 0 : \int_\Omega \omega(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}, \qquad 2 \leq p(x) < \infty.$$

The weighted Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is the set of all functions $u \in L^{p(x)}(\Omega)$ whose weak gradient satisfies $\nabla u \in \left(L^{p(x)}(\Omega,\omega)\right)^N$, and it is equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)} := \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{\left(L^{p(x)}(\Omega,\omega)\right)^N}.$$

An interesting feature of a generalized Lebesgue-Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent p(x). This was observed by Zhikov [14] in connection with Lavrentiev phenomenon. However, when the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y| < \frac{c}{-\log|x-y|}, \tag{2.1}$$

for every where $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$. Since smooth functions are dense in variable exponent Sobolev spaces, there is no ambiguity in defining the Sobolev space with zero boundary values, $W_0^{1;p(.)}(\Omega,\omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1;p(\cdot)}(\Omega,\omega)}$ (see [6]).

By a weight we mean a locally integrable function ω on $\mathbb{R}^{\mathbb{N}}$ such that $0 < \omega < \infty$, and in this work we assume ω satisfies:

- (H_1) $\omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$,
- (H_2) $\omega^{-s(x)} \in L^1(\Omega)$ where $s \in \left(\frac{N}{p(x)}, \infty\right) \left[\frac{1}{p(x)-1}, \infty\right)$.

$\langle , \rangle$ denotes the duality between $(W_0^{1,p(x)}(\Omega,\omega))'$ and $W_0^{1,p(x)}(\Omega,\omega)$.
We recall the following useful lemmas.

Lemma 2.1. [8] *If (H_1) holds, $W^{1,p(x)}(\Omega,w)$ is a separable and reflexive Banach space. For $p, s \in C_+(\bar{\Omega})$, set*

$$p_s(x) := \frac{p(x)s(x)}{1+s(x)} < p(x),$$

where $s(x)$ is given in (H_2). Assume that we fix the variable exponent restrictions

$$p_s^*(x) := \begin{cases} \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ \text{arbitrary} & \text{if } N \leq p_s(x), \end{cases}$$

for almost all $x \in \Omega$.

Proposition 2.2. [8] *Let $p, s \in C_+(\bar{\Omega})$ satisfy the log-Hölder continuity condition, and let (H_1) and (H_2) be satisfied. If $r \in C_+(\bar{\Omega}))$ and $1 < r(x) \leq p_s^*$. Then, we obtain the continuous imbedding*

$$W^{1,\mathrm{p}(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$.

Lemma 2.3. [4, 9] *(Generalised Hölder-type inequality).*
i) For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2||u\,\|_{L^{p(x)}(\Omega)}\|\,v\|_{L^{p'(x)}(\Omega)}.$$

ii) For all $p, q \in C_+(\bar{\Omega})$ such that $p(x) \leq q(x)$ a.e. in Ω, we have $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}\Omega$ and the embedding is continuous.

Lemma 2.4. [8] *Denote*

$$\rho(u) := \int_\Omega \omega(x)|u(x)|^{p(x)}dx, \quad for\ all \quad u \in L^{p(x)}(\Omega, \omega).$$

Then

(1) $\|u\|_{L^{p(x)}(\Omega,\omega)} < 1(= 1; > 1)$ *if and only if* $\rho(u) < 1(= 1; > 1)$,

(2) *if* $\|u\|_{L^{p(x)}(\Omega,\omega)} > 1$ *then* $\|u\|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)}$,

(3) *if* $\|u\|_{L^{p(x)}(\Omega,\omega)} < 1$ *then* $\|u\|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(x)}(\Omega,\omega)}$.

Lemma 2.5. [1]. *$\forall x, y \in \mathbb{R}^N$ and $1 < p < \infty$*

$$\frac{1}{p}|x|^p - \frac{1}{p}|y|^p \leq |x|^{p-2}x(x-y).$$

Lemma 2.6. *For $m \geq 0,\ n \geq 0$ and $1 \leq p < \infty$, we have*

$$(m+n)^p \leq 2^{p-1}\left(m^p + n^p\right).$$

3. Main result

Before presenting the main result of this section, we first state the assumptions needed for this work and then introduce the notion of weak solutions to problem (1.1).

- (H_3) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ and satisfying

$$\begin{cases} |\Theta(x) - \Theta(y)| \leq \lambda|x-y| & \forall (x,y) \in \mathbb{R}^2, \\ \Theta(0) = 0, \end{cases} \tag{3.1}$$

  where λ is a constant such that $0 < \lambda < \frac{1}{2}$.
- (H_4) $f \in (W_0^{1,p(x)}(\Omega, \omega))'$.

We next state the notion of a weak solution to (1.1).

Definition 3.1. Let $f \in (W_0^{1,p(x)}(\Omega,\omega))'$. A function $u \in W_0^{1,p(x)}(\Omega,\omega)$ is called a weak solution of the problem (1.1) if, for any $\varphi \in W_0^{1,p(x)}(\Omega,\omega)$, we have

$$\int_\Omega \omega\Phi\left(\nabla u-\Theta\left(u\right)\right)\cdot\nabla\varphi \mathrm{d}x+\int_\Omega \omega|u|^{p(x)-2}u\varphi \mathrm{d}x=\langle f,\varphi\rangle. \tag{3.2}$$

We are now ready to state the main result of this work.

Theorem 3.2. *Assume that (H_1), (H_2), (H_3), and (H_4) hold. Then the problem (1.1) has a weak solution.*

4. Proof of the existence of weak solutions to problem (1.1)

To establish the existence of solutions to problem (1.1), we consider the associated variational problem

$$\min\left\{J\left(u\right)/u\in W_0^{1,p(x)}\left(\Omega,\omega\right)\right\},$$

for

$$J(u):=\int_\Omega \frac{1}{p(x)}\omega|u|^{p(x)}\ \mathrm{d}x+\int_\Omega \frac{1}{p(x)}\omega|\nabla u-\Theta(u)|^{p(x)}\mathrm{d}x-\langle f,u\rangle. \tag{4.1}$$

Let $u \in W_0^{1,p(x)}\left(\Omega,\omega\right)$. By applying Lemma 2.6 and the (3.1), we get

$$\begin{aligned}\frac{\omega}{p(x)2^{p(x)-1}}|\nabla u|^{p(x)}&=\frac{\omega}{p(x)2^{p(x)-1}}|\nabla u-\Theta\left(u\right)+\Theta\left(u\right)|^{p(x)}\\&\le\frac{\omega}{p(x)}|\nabla u-\Theta(u)|^{p(x)}+\frac{\omega}{p(x)}|\Theta\left(u\right)|^{p(x)}\\&\le\frac{\omega}{p(x)}|\nabla u-\Theta(u)|^{p(x)}+\frac{\omega\lambda^{p(x)}}{p(x)}|u|^{p(x)}\\&\le\frac{\omega}{p(x)}|\nabla u-\Theta(u)|^{p(x)}+\frac{\omega}{p(x)2^{p(x)}}|u|^{p(x)}.\end{aligned} \tag{4.2}$$

Taking the integral over Ω in the above inequality, we obtain

$$\int_\Omega \frac{\omega}{p(x)2^{p^+-1}}|\nabla u|^{p(x)}\ \mathrm{d}x\le\int_\Omega \frac{1}{p(x)}\omega|\nabla u-\Theta(u)|^{p(x)}\mathrm{d}x+\int_\Omega \frac{1}{p(x)}\omega|u|^{p(x)}\mathrm{d}x.$$

Thus

$$\int_\Omega \frac{\omega}{p^+2^{p^+-1}}|\nabla u|^{p(x)}\ \mathrm{d}x\le\int_\Omega \frac{1}{p(x)}\omega|\nabla u-\Theta(u)|^{p(x)}\mathrm{d}x+\int_\Omega \frac{1}{p(x)}\omega|u|^{p(x)}\mathrm{d}x. \tag{4.3}$$

Applying the Cauchy–Schwarz inequality and Young's inequality, we obtain

$$\begin{aligned}
|\langle f,u\rangle| &\leq \|f\|_{(W_0^{1,p(x)}(\Omega,\omega))')}\|u\|_{W_0^{1,p(x)}(\Omega,\omega)}\\
&= \|f\|_{(W_0^{1,p(x)}(\Omega,\omega))'}\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}\\
&\leq \varepsilon\|\nabla u\|^{p-}_{L^{p(x)}(\Omega,\omega)} + C(\varepsilon)\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'}\\
&\leq \varepsilon\left(\int_\Omega \omega|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{\beta p-} + C(\varepsilon)\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'}\\
&\leq \varepsilon\left(\int_\Omega \omega|\nabla u|^{p(x)}\,\mathrm{d}x + 1\right) + C(\varepsilon)\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'},
\end{aligned}$$

where ε is a positive number and

$$\beta := \begin{cases} \dfrac{1}{p^-}, & \text{if} \quad ||\nabla u||_{L^{p(x)}(\Omega,\omega)} \leq 1,\\ \dfrac{1}{p^+}, & \text{if} \quad ||\nabla u||_{L^{p(x)}(\Omega,\omega)} \geq 1.\end{cases}$$

Taking $\varepsilon = \frac{1}{p^+2^{p^+}}$, we get

$$|\langle f,u\rangle| \leq \frac{1}{p^+2^{p^+}}\left(\int_\Omega \omega|\nabla u|^{p(x)}\,\mathrm{d}x + 1\right) + C\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'}. \tag{4.4}$$

Consequently, from (4.3) and (4.4), it follows that

$$J(u) \geq \int_\Omega \left(\frac{1}{p^+2^{p^+-1}} - \frac{1}{p^+2^{p^+}}\right)\omega|\nabla u|^{p(x)}\,\mathrm{d}x - C_1\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'} - \frac{1}{p^+2^{p^+}}.$$

Hence

$$J(u) \geq \frac{1}{p^+2^{p^+}}\int_\Omega \omega|\nabla u|^{p(x)}\mathrm{d}x - C_1\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'} - \frac{1}{p^+2^{p^+}}.$$

Thus

$$J(u) \geq -C_1\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'} - \frac{1}{p^+2^{p^+}}. \tag{4.5}$$

This implies that

$$-C_1\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'} - \frac{1}{p^+2^{p^+}} \leq \inf_{u\in W_0^{1,p(x)}(\Omega,\omega)} J(u) \leq 0.$$

Therefore, one can choose a minimizing sequence $\{u_m\} \subset W_0^{1,p(x)}(\Omega,\omega)$ such that

$$J(u_m) \leq 1, \tag{4.6}$$

and

$$\lim_{m\to\infty} J(u_m) = \inf_{u\in W_0^{1,p(x)}(\Omega,\omega)} J(u).$$

From (4.6) and (4.5), we get

$$\frac{1}{p^+2^{p^+}}\int_\Omega \omega|\nabla u_m|^{p(x)}\mathrm{d}x \leq C_1\|f\|^{p'-}_{(W_0^{1,p(x)}(\Omega,\omega))'} + 1 + \frac{1}{p^+2^{p^+}}. \tag{4.7}$$

This implies that

$$\int_{\Omega} \omega |\nabla u_m|^{p(x)} \mathrm{d}x \leq C_3, \tag{4.8}$$

so that

$$\|u_m\|_{W_0^{1,p(x)}(\Omega,\omega)} \leq 1 + (C_3)^{\frac{1}{p^-}} := C_4. \tag{4.9}$$

Hence, the above inequality implies that (u_m) is bounded in $W_0^{1,p(x)}(\Omega,\omega)$. Since $W_0^{1,p(x)}(\Omega,\omega)$ is reflexive, we can extract a subsequence (still denoted by u_m) and a function u such that $u_m \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega,\omega)$. Therefore by Proposition 2.2, we have

$$\begin{aligned} u_m &\to u \text{ strong in } L^{p(x)}(\Omega,\omega), &(4.10)\\ \nabla u_m &\rightharpoonup \nabla u \text{ weakly in } L^{p(x)}(\Omega,\omega), &(4.11)\\ u_m &\to u \text{ a.e in } \Omega. &(4.12)\end{aligned}$$

Next, we need to show that

$$\liminf_{m\to\infty} J(u_m) \geq J(u).$$

Since $u_m \to u$ a.e in Ω, by Fatou's Lemma, we have

$$\liminf_{m\to\infty} \int_{\Omega} \frac{1}{p(x)} \omega |u_m|^{p(x)} \mathrm{d}x \geq \int_{\Omega} \frac{1}{p(x)} \omega |u|^{p(x)} \mathrm{d}x. \tag{4.13}$$

So by (3.1) and (4.10), we get $\Theta(u_m) \to \Theta(u)$ strong in $L^{p(x)}(\Omega,\omega)$, thus $\Theta(u_m) \rightharpoonup \Theta(u)$ weakly in $L^{p(x)}(\Omega,\omega)$.
We also have, from (4.11), $\nabla u_m - \Theta(u_m) \rightharpoonup \nabla u - \Theta(u)$ weakly in $L^{p(x)}(\Omega,\omega)$, which gives

$$\liminf_{m\to\infty} \int_{\Omega} \frac{1}{p(x)} \omega |\nabla u_m - \Theta(u_m)|^{p(x)} \mathrm{d}x \geq \int_{\Omega} \frac{1}{p(x)} \omega |\nabla u - \Theta(u)|^{p(x)} \mathrm{d}x. \tag{4.14}$$

And by (4.10), we obtain

$$\lim_{m\to\infty} \langle f, u_m \rangle = \langle f, u \rangle. \tag{4.15}$$

Combining (4.13), (4.14) and (4.15), we deduce

$$\liminf_{m\to\infty} J(u_m) \geq J(u).$$

Therefore u is a minimizer of the functional J in $W_0^{1,p(x)}(\Omega,\omega)$.
The next result is key to establishing the existence of solutions to problem (1.1).

Lemma 4.1. *Provided that* (H_1), (H_2) *and* (H_3) *hold, then for any* u, v *in* $W_0^{1,p(x)}(\Omega,\omega)$, *we have*

$$\text{(i)}\ \lim_{t\to 0}\int_\Omega \omega\frac{|u+tv|^{p(x)}-|u|^{p(x)}}{p(x)t}\mathrm{d}x=\int_\Omega \omega|u|^{p(x)-2}uv\mathrm{d}x.$$

$$\text{(ii)}\ \lim_{t\to 0}\int_\Omega \omega\frac{|\nabla u+t\nabla v-\Theta(u+tv)|^{p(x)}-|\nabla u-\Theta(u)|^{p(x)}}{p(x)t}\mathrm{d}x$$
$$=\int_\Omega \omega\Phi(\nabla u-\Theta(u))\cdot\nabla v\mathrm{d}x.$$

Proof. (i) Consider

$$M:[0,1]\longrightarrow\mathbb{R}$$
$$\mu\longmapsto \omega\frac{|u+t\mu v|^{p(x)}-|u|^{p(x)}}{p(x)t}.$$

The function M is continuous on [0,1] and differentiable on]0,1[, then by the mean value theorem, there exists $\gamma\in]0,1[$ such that

$$M(1))-M(0))=M'(\gamma).$$

Then

$$\omega\frac{|u+tv|^{p(x)}-|u|^{p(x)}}{p(x)t}=\omega|u+t\gamma v|^{p(x)-2}(u+t\gamma v)v.$$

Since $\gamma,\ t\in[0,1]$,then by Young's inequality and by Lemma 2.6, we get

$$\begin{aligned}\omega|u+t\gamma v|^{p(x)-2}(u+t\gamma v)v &\le \omega|u+t\gamma v|^{p(x)-1}|v|\\ &\le \omega\frac{1}{p'(x)}|u+t\gamma v|^{p(x)}+\frac{1}{p(x)}|v|^{p(x)}\\ &\le \omega\frac{2^{p(x)-1}}{p'(x)}(|u|^{p(x)}+|v|^{p(x)})+\omega\frac{1}{p(x)}|v|^{p(x)}.\end{aligned}$$

On the other hand

$$\lim_{t\to 0}\omega\frac{|u+tv|^{p(x)}-|u|^{p(x)}}{p(x)t}=\omega|u|^{p(x)-2}uv.$$

Hence, by the Dominated Convergence Theorem, we obtain

$$\lim_{t\to 0}\int_\Omega \omega\frac{|u+tv|^{p(x)}-|u|^{p(x)}}{p(x)t}\mathrm{d}x=\int_\Omega \omega|u|^{p(x)-2}uv\mathrm{d}x.$$

(ii) To show that

$$\lim_{t\to 0}\int_\Omega \omega\frac{|\nabla u+t\nabla v-\Theta(u+tv)|^{p(x)}-|\nabla u-\Theta(u)|^{p(x)}}{p(x)t}\mathrm{d}x$$
$$=\int_\Omega \omega\Phi(\nabla u-\Theta(u))\cdot\nabla v\mathrm{d}x,$$

we define, for $t\in]0,1[$,

$$G_t(u,v)=\nabla u+t\nabla v-\Theta(u+tv),\quad F_t(u,v)=\nabla u+t\nabla v-\Theta(u),$$

and let M be a function defined as

$$\begin{aligned} M : [0,1] &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \omega \frac{|F_{t\mu}(u,v)|^{p(x)} - |F_0(u,v|^{p(x)}}{p(x)t}. \end{aligned}$$

Since M is continuous on $[0,1]$ and differentiable on $]0,1[$, it follows again from the Mean Value Theorem that there exists $\gamma \in]0,1[$ such that

$$M(1) - M(0) = M'(\gamma).$$

Thus

$$\omega \frac{|F_t(u,v|^{p(x)} - |F_0(u,v|^{p(x)}}{p(x)t} = \omega |F_{t\gamma}(u,v|^{p(x)-2}(F_{t\gamma}(u,v)) \cdot \nabla v.$$

Since $\gamma, t \in [0,1]$, then by Young's inequality and Lemma 2.6, we get

$$\begin{aligned} &\omega |F_{t\gamma}(u,v|^{p(x)-2} \left(F_{t\gamma}(u,v)\right) \cdot \nabla v \\ &\quad \leq \omega |F_{t\gamma}(u,v)|^{p(x)-1} |\nabla v| \\ &\quad \leq \omega \tfrac{|F_{t\gamma}(u,v)|^{p(x)}}{p'(x)} + \omega \tfrac{|\nabla v|^{p(x)}}{p(x)} \\ &\quad \leq 2^{p(x)-1} \left(\omega \tfrac{|\nabla u - \Theta(u)|^{p(x)} + |t\gamma \nabla v|^{p(x)}}{p'(x)} \right) + \omega \tfrac{|\nabla v|^{p(x)}}{p(x)} \\ &\quad \leq 2^{p(x)-1} \left(\omega \tfrac{|\nabla u - \Theta(u)|^{p(x)} + |\nabla v|^{p(x)}}{p'(x)} \right) + \omega \tfrac{|\nabla v|^{p(x)}}{p(x)}. \end{aligned}$$

On the other hand

$$\lim_{t \to 0} \omega \frac{|\nabla u + t\nabla v - \Theta(u)|^{p(x)} - |\nabla u - \Theta(u)|^{p(x)}}{p(x)t} = \omega \Phi \left(\nabla u - \Theta(u) \cdot \nabla v.\right.$$

Hence, by the Dominated Convergence Theorem, we obtain

$$\lim_{t \to 0} \int_\Omega \omega \frac{|\nabla u + t\nabla v - \Theta(u)|^{p(x)} - |\nabla u|^{p(x)}}{p(x)t} \mathrm{d}x = \int_\Omega \omega \Phi \left(\nabla u - \Theta(u) \cdot \nabla v \mathrm{d}x.\right. \quad (4.16)$$

To finish the proof of (ii), it suffices to show that

$$\lim_{t \to 0} \int_\Omega \omega \frac{|\nabla u + t\nabla v - \Theta(u+tv)|^{p(x)} - |\nabla u + t\nabla v - \Theta(u)|^{p(x)}}{p(x)t} \mathrm{d}x = 0.$$

Using Lemma 2.5 and by applying Young's inequalities, we get

$$\begin{aligned}
&\int_\Omega \omega \frac{|G_t(u,v)|^{p(x)} - |F_t(u,v))|^{p(x)}}{p(x)t}\mathrm{d}x \\
&\le \int_\Omega \omega \frac{|G_t(u,v)|^{p(x)-2}(G_t(u,v)\cdot(G_t(u,v)-F_t(u,v))}{t}\mathrm{d}x \\
&\le \int_\Omega \omega \frac{|G_t(u,v)|^{p(x)-1}|\Theta(u+tv)-\Theta(u)|}{t}\mathrm{d}x \\
&\le \int_\Omega \varepsilon\omega \frac{|G_t(u,v)|^{p(x)}}{tp'(x)}\mathrm{d}x + \int_\Omega \omega \frac{|\Theta(u+tv)-\Theta(u)|^{p(x)}}{\varepsilon tp(x)}\mathrm{d}x \\
&\le \int_\Omega \varepsilon\omega \frac{G_t(u,v))|^{p(x)}}{tp'(x)}\mathrm{d}x + \int_\Omega \omega \frac{|t|^{p(x)}|v|^{p(x)}}{\varepsilon tp(x)}\mathrm{d}x \\
&\le \int_\Omega \varepsilon\omega \frac{G_t(u,v)|^{p(x)}}{tp'(x)}\mathrm{d}x + \int_\Omega \omega \frac{|t|^{p^-}|v|^{p(x)}}{\varepsilon tp(x)}\mathrm{d}x,
\end{aligned}$$

where ε is a positive number. Choosing $\varepsilon = t^{\frac{p}{2}}$, we have

$$\int_\Omega \omega \frac{|G_t(u,v)|^{p(x)} - |F_t(u,v))|^{p(x)}}{p(x)t}\mathrm{d}x \le t^{\frac{p^-}{2}}\int_\Omega \omega|G_t(u,v)|^{p(x)}\mathrm{d}x + t^{\frac{p^-}{2}}\int_\Omega \omega|v|^{p(x)}\mathrm{d}x. \tag{4.17}$$

In the same manner, we prove that

$$\int_\Omega \omega \frac{|G_t(u,v)|^{p(x)} - |F_t(u,v)|^{p(x)}}{tp(x)}dx \ge -t^{\frac{p^-}{2}}\int_\Omega \omega|F_t(u,v))|^{p(x)}\mathrm{d}x - t^{\frac{p^-}{2}}\int_\Omega \omega|v|^{p(x)}\mathrm{d}x. \tag{4.18}$$

Then we obtain from (4.17) and (4.18), as $t \to 0$,

$$\lim_{t\to 0}\int_\Omega \omega \frac{|G_t(u,v)|^{p(x)} - |F_t(u,v)|^{p(x)}}{tp(x)}\mathrm{d}x = 0.$$

Therefore

$$\lim_{t\to 0}\int_\Omega \omega \frac{|\nabla u + t\nabla v - \Theta(u+tv)|^{p(x)} - |\nabla u + t\nabla v - \Theta(u)|^{p(x)}}{p(x)t}\mathrm{d}x = 0. \tag{4.19}$$

Combining (4.16) and (4.19), we obtain

$$\begin{aligned}
&\lim_{t\to 0}\int_\Omega \omega \frac{|\nabla u + t\nabla v - \Theta(u+tv)|^{p(x)} - |\nabla u - \Theta(u)|^{p(x)}}{p(x)t}\mathrm{d}x \\
&\quad = \int_\Omega \omega\Phi(\nabla u - \Theta(u))\cdot\nabla v\mathrm{d}x.
\end{aligned}$$

□

We will show that u is a weak solution of the problem (1.1). Since u is a minimizer of the functional $J(u)$ in $W_0^{1,p(x)}(\Omega,\omega)$, then for any $v \in W_0^{1,p(x)}(\Omega,\omega)$

we have

$$
\begin{aligned}
0 &\leq \frac{J(u+tv)-J(u)}{t}\\
&= \int_\Omega \omega \frac{|u+tv|^{p(x)}-|u|^{p(x)}}{p(x)t}\mathrm{d}x\\
&\quad + \int_\Omega \omega \frac{|\nabla u + t\nabla v - \Theta(u+tv)|^{p(x)} - |\nabla u - \Theta(u)|^{p(x)}}{p(x)t}\mathrm{d}x - \langle f, v\rangle. \qquad (4.20)
\end{aligned}
$$

By letting $t \to 0$ and using Lemma 4.1, we deduce from (4.20)

$$
0 \leq \int_\Omega \omega |u|^{p(x)-2}uv\mathrm{d}x + \int_\Omega \omega\Phi(\nabla u - \Theta(u) \cdot \nabla v \mathrm{d}x - \langle f, v\rangle.
$$

Thus

$$
\int_\Omega \omega |u|^{p(x)-2}uv\mathrm{d}x + \int_\Omega \omega\Phi(\nabla u - \Theta(u)) \cdot \nabla v \mathrm{d}x = \langle f, v\rangle,
$$

which proves that u is a weak solution of (1.1).

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