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STABILITY AND SYMMETRY ANALYSIS OF THE TIME-FRACTIONAL MCKENDRICK EQUATION

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Abstract. In this paper, we apply the Lie symmetries analysis to obtain the infinitesimal generators admitted by the time fractional Riemann-Liouville McKendrick equation. We will show that the reduction via the Erdélyi-Kober fractional operator allows us to construct some exact solutions by using the power series approach. Moreover, the exploitation of the stability of the symmetry algebra allowed us to construct a new family of exact solutions. Based on the obtained infinitesimal generators and the use of Ibragimov's method, we build conservation quantities of the studied equation. Finally, 3D plots are provided to illustrate the behavior of the derived solutions for specific parameter values.

Keywords. McKendrick equation, Lie symmetries, Fractional derivative, Erdélyi-Kober fractional operator, Power series approach, Exact solutions, Conservation laws
2020 Mathematics Subject Classification. Primary 35B40; Secondary 35L70

1. Introduction

Fractional calculus [13, 12] extends the concepts of differentiation and integration to orders that are not limited to integers. This extension provides a more nuanced tool for modeling systems with memory and complex dynamics. Over the last decades, fractional calculus is applied in various fields [1, 11, 8], offering deeper insights into processes that traditional calculus may not fully capture.

Fractional differential equations often lack exact solutions due to their inherent complexity. To address this, researchers use a combination of analytical methods and numerical techniques. These approaches provide practical solutions and insights even when exact answers are elusive.

Lie symmetries analysis is a key method for investigating the invariance properties of ordinary and partial differential equations [14]. Gazizov et al. [7], expanded this technique to fractional differential equations. Their work has since been proven to be an important resource for subsequent research in the area [6, 2, 3, 4, 5].

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This work is mainly concerned with the Lie symmetries analysis and the construction of some exact solutions of the following time fractional Riemann-Liouville McKendrick equation:

$$u_t^\alpha + u_x + f(x)u = 0, \tag{1.1}$$

where u corresponds to the population density depending on the age x and time t, u_t^α the Riemann-Liouville fractional partial derivative $(0 < \alpha \leq 1)$, and $f(x)$ denotes the age-specific mortality rate at age x is determined by the ratio of deaths among individuals aged x to the total number of individuals at that specific age. The particular expression of the mortality rate $f(x)$ allows for the selection of different mortality models, each suited to how mortality varies with age. Choosing the right model helps in better understanding and predicting mortality trends within a population.

This article is structured as follows: Section 2 constructs the Lie symmetry algebra admitted by the time fractional Riemann-Liouville McKendrick equation. Section (3) explores symmetry reductions and employs the Erdélyi-Kober fractional operator method along with power series techniques to obtain some solutions. Section (4) focuses on deriving an infinite family of exact solutions by applying the stability of the Lie symmetry algebra, followed by 3D plots of the solutions for different parameter values. Finally, we use the conservation laws from the Ibragimov theorem, to extract conserved quantities for the equation.

2. Lie symmetry analysis

To begin with, we should recall that the classical McKendrick equation finds application in multiple areas of mathematical biology, such as demography, cell cycle studies, and modeling cell proliferation, especially when age structure plays a fundamental role in the model [10].

In this work, we consider its time-fractional extension, namely:

$$u_t^\alpha = F(t, x, u, u_x), \tag{2.1}$$

where

$$F(t, x, u, u_x) = -u_x - f(x)u,$$

and u_t^α is defined by:

$$u_t^\alpha = \begin{cases} \frac{\partial^l u}{\partial t^l}, & \alpha = l, \\ \frac{1}{\Gamma(l-\alpha)} \frac{\partial^l}{\partial t^l} \int_0^t (t-r)^{l-\alpha-1} u(r,x) dr, & l-1 < \alpha < l, \end{cases} \quad l \in \mathbb{N}.$$

Let us investigate a one-parameter group of transformations:

$$\overline{t} = t + \epsilon\tau(t,x,u) + o\left(\epsilon^2\right), \overline{x} \quad = x + \epsilon\xi(t,x,u) + o\left(\epsilon^2\right),$$

and

$$\overline{u} = u + \epsilon\eta(t,x,u) + o\left(\epsilon^2\right),$$

ϵ represents the group parameter, τ, ξ and η are infinitesimal functions. The infinitesimal generator associated with the above group is written as

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}.$$

X generates a symmetry of the equation (2.1) precisely when the invariance condition:

$$X^{(\alpha)}(u_t^\alpha - F)\big|_{u_t^\alpha - F = 0} = 0,$$

is satisfied, where $X^{(\alpha)}$ represent the $\alpha-$prolongation of X which is:

$$X^{(\alpha)} = X + \eta^x\frac{\partial}{\partial u_x} + \eta^{\alpha,t}\frac{\partial}{\partial\left(\partial_t^\alpha u\right)},$$

where η^x and $\eta^{\alpha,t}$ are:

$$\begin{aligned}
\eta^x =& D_x(\eta) - u_x\left(D_x\xi\right) - u_t\left(D_x\tau\right)\\
=& \eta_x + \left(\eta_u - \xi_x\right)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,\\
\eta^{\alpha,t} =& D_t^\alpha(\eta) + \xi D_t^\alpha\left(u_x\right) - D_t^\alpha\left(\xi u_x\right) + D_t^\alpha\left(D_t(\tau)u\right) - D_t^{\alpha+1}(\tau_u) + \tau D_t^{\alpha+1}(u).
\end{aligned}$$

From the invariance criterion, we obtain:

$$\eta^{\alpha,\mathrm{t}} + f'u\xi + f\eta + \eta^x = 0.$$

The invariance criterion leads to:

$$\begin{aligned}
&\tau_x = \tau_u = \xi_u = \xi_t = 0,\\
&(\eta_u - \xi_x) - (\eta_u - \alpha D_t(\tau)) = 0,\\
&\frac{\partial^\alpha\eta}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha} + (-fu)\left(\eta_u - \alpha D_t(\tau)\right) + f'u\xi + f\eta + \eta_x = 0,\\
&\begin{pmatrix}\alpha\\ l\end{pmatrix}\frac{\partial^l\eta_u}{\partial t^l} - \begin{pmatrix}\alpha\\ l+1\end{pmatrix}D_t^{l+1}(\tau) = 0, \quad l = 0, 1, 2...
\end{aligned}$$

where $\begin{pmatrix}\alpha\\ l\end{pmatrix}$ denotes the generalized binomial coefficient:

$$\begin{pmatrix}\alpha\\ l\end{pmatrix} = \frac{(-1)^{l-1}\alpha\Gamma(l-\alpha)}{\Gamma(1-\alpha)\Gamma(l+1)}.$$

In addition to the condition $\tau = 0$ when $t = 0$, the solution to the above system is given by:

$$\begin{aligned}
&\tau = c_1 t, \quad \xi = \alpha c_1 x + c_2,\\
&\eta = -\alpha x f u c_1 - f u c_2 + u c_3 + h(t,x).
\end{aligned}$$

Here, the constants c_1, c_2 and c_3 are arbitrary and $h(t,x)$ represents an arbitrary solution.

Then, the corresponding Lie symmetry algebra is:

$$\begin{aligned} X_1 &= t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} - \alpha x f u\frac{\partial}{\partial u}, \\ X_2 &= \frac{\partial}{\partial x} - f u\frac{\partial}{\partial u}, \\ X_3 &= u\frac{\partial}{\partial u}, \quad X_h = h\frac{\partial}{\partial u}. \end{aligned} \tag{2.2}$$

Based on the commutator operator $[X_i, X_j] = X_iX_j - X_jX_i$, we present the commutator table:

$[X_i, X_j]$	X_1	X_2	X_3	X_h
X_1	0	$-\alpha X_2$	0	X_{ψ_1}
X_2	αX_2	0	0	X_{ψ_2}
X_3	0	0	0	$-X_h$
X_h	$-X_{\psi_1}$	$-X_{\psi_2}$	X_h	0

Table 1. Lie bracket of the symmetry generators

where

$$\psi_1 = th_t + \alpha x h_x + \alpha x f h, \quad \psi_2 = h_x + f h$$

and

$$X_{\psi_1} = \psi_1\frac{\partial}{\partial u}, \quad X_{\psi_2} = \psi_2\frac{\partial}{\partial u}.$$

3. Construction of some exact solutions

This section is devoted to the construction of similarity variables to reduce the time fractional Riemann-Liouville McKendrick equation.

Reduction with $X_1 = t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} - \alpha x f u\frac{\partial}{\partial u}$.

This is derived from solving the characteristic equation shown below :

$$\frac{dt}{t} = \frac{dx}{\alpha x} = \frac{du}{-\alpha x f u}.$$

Integrating the above system leads to the invariants:

$$u = e^{-F(x)}\varphi(\xi), \text{ and } \xi = xt^{-\alpha}, \tag{3.1}$$

where $F(x)$ is a primitive function of f (we suppose that f is integrable).

Theorem 3.1. *The time fractional Riemann-Liouville McKendrick equation is converted into a fractional ordinary differential equation by the similarity transformation $u = e^{-F(x)}\varphi(\xi)$, where $\xi = xt^{-\alpha}$, yielding:*

$$\left(\mathcal{P}^{1-\alpha,\alpha}_{\frac{1}{\alpha}}\varphi\right)(\xi) + \varphi_\xi = 0, \tag{3.2}$$

where $\mathcal{P}_\sigma^{\beta,\alpha}\varphi$ is the Erdélyi-Kober fractional operator given by:

$$\left(\mathcal{P}_\sigma^{\beta,\alpha}\varphi\right)(\xi) = \prod_{j=0}^{l-1}\left(\beta + j - \frac{1}{\sigma}\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_\sigma^{\beta+\alpha,q-\alpha}\varphi\right)(\xi),$$

$$\textit{and} \quad l = \begin{cases} \alpha, & \alpha \in \mathbb{N}, \\ [\alpha]+1, & \alpha \notin \mathbb{N}, \end{cases}$$

with $\mathcal{K}_\sigma^{\beta,\alpha}\varphi$ represents the Erdélyi-Kober integral operator given by:

$$\begin{cases} \left(\mathcal{K}_\sigma^{\beta,\alpha}\varphi\right)(\xi) = \frac{1}{\Gamma(\alpha)}\int_1^\infty (\mu-1)^{\alpha-1}\mu^{-(\beta+\alpha)}\varphi\left(\xi\mu^{\frac{1}{\sigma}}\right)d\mu, & \alpha > 0, \\ \left(\mathcal{K}_\sigma^{\beta,\alpha}\varphi\right)(\xi) = \varphi(\xi). & \alpha = 0, \end{cases}$$

Proof. We have:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^l}{\partial t^l}\left(\frac{1}{\Gamma(l-\alpha)}\int_0^t (t-s)^{l-\alpha-1}e^{-F(x)}\varphi\left(xs^{-\alpha}\right)ds\right).$$

Let $\mu = \frac{t}{s}$, then $ds = -\frac{t}{\mu^2}d\mu$.
Thus:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = e^{-F(x)}\frac{\partial^l}{\partial t^l}\left(\frac{t^{l-\alpha}}{\Gamma(l-\alpha)}\int_1^\infty (\mu-1)^{l-\alpha-1}\mu^{-(l+1-\alpha)}\varphi\left(\xi\mu^\alpha\right)d\mu\right).$$

By using Erdélyi-Kober integral operator, we get:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = e^{-F(x)}\frac{\partial^l}{\partial t^l}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right).$$

After differentiation, we obtain:

$$\begin{aligned} &\frac{\partial^l}{\partial t^l}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right) \\ &= \frac{\partial^{l-1}}{\partial t^{l-1}}\left[\frac{\partial}{\partial t}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right)\right] \\ &= \frac{\partial^{l-1}}{\partial t^{l-1}}\left[(l-\alpha)\,t^{l-\alpha-1}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi) + t^{l-\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right]. \end{aligned}$$

By employing the chain rule:

$$t^{l-\alpha}\frac{\partial}{\partial t}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi) = -\alpha t^{l-\alpha-1}\xi\frac{\partial}{\partial\xi}\left(\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right),$$

then:

$$\begin{aligned} &\frac{\partial^l}{\partial t^l}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right) \\ &= \frac{\partial^{l-1}}{\partial t^{l-1}}\left[t^{l-\alpha-1}\left(l-\alpha-\alpha\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right]. \end{aligned}$$

After $(l-1)$ iteration, we obtain:

$$\begin{aligned}
&\frac{\partial^l}{\partial t^l}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right)\\
&=\frac{\partial^{l-1}}{\partial t^{l-1}}\left[t^{l-\alpha-1}\left(l-\alpha-\alpha\xi\frac{\partial}{\partial\xi}\right)\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right]\\
&\vdots\\
&=t^{-\alpha}\prod_{j=0}^{l-1}\left(1-\alpha+j-\alpha\xi\frac{d}{d\xi}\right)\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi).
\end{aligned}$$

Using the definition of the Erdélyi-Kober fractional operator, we get:

$$\frac{\partial^l}{\partial t^l}\left(t^{l-\alpha}\left(\mathcal{K}_{\frac{1}{\alpha}}^{1,l-\alpha}\varphi\right)(\xi)\right)=t^{-\alpha}\left(\mathcal{P}_{\frac{1}{\alpha}}^{1-\alpha,\alpha}\varphi\right)(\xi).$$

From differentiation of $u=e^{-F(x)}\varphi(\xi)$, where $\xi=xt^{-\alpha}$, we obtain:
$u_x=-f(x)e^{-F(x)}\varphi+t^{-\alpha}e^{-F(x)}\varphi_\xi$.

Substituting in Eq (1.1), we obtain:

$$\left(\mathcal{P}_{\frac{1}{\alpha}}^{1-\alpha,\alpha}\varphi\right)(\xi)+\varphi_\xi=0.$$

□

By applying the power series approach, we find the solution. To begin with, let:

$$\varphi(\xi)=\sum_{n=0}^{\infty}a_n\xi^n,\quad (a_n\in\mathbb{R}) \tag{3.3}$$

Thus:

$$\varphi_\xi=\sum_{n=0}^{\infty}(n+1)a_{n+1}\xi^n$$

and

$$\left(\mathcal{P}_{\frac{1}{\alpha}}^{1-\alpha,\alpha}\varphi\right)(\xi)=\sum_{n=0}^{\infty}a_n\frac{\Gamma(2+n\alpha)}{\Gamma(2-\alpha+n\alpha)}\xi^n.$$

It yield:

$$\sum_{n=0}^{\infty}a_n\frac{\Gamma(2+n\alpha)}{\Gamma(2-\alpha+n\alpha)}\xi^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}\xi^n=0.$$

By evaluating the coefficients for $n=0$, we find:

$$a_1=\frac{-a_0}{\Gamma(2-\alpha)}$$

and for $n\geq 0$,

$$a_{n+1}=\frac{-\Gamma(2+n\alpha)\,a_n}{(n+1)\Gamma(2-\alpha+n\alpha)}. \tag{3.4}$$

The equation (3.4) leads to:

$$|a_{n+1}| \leq M|a_n|,$$

with:

$$M = \left|\frac{\Gamma(2+n\alpha)}{\Gamma(2-\alpha+n\alpha)}\right|.$$

Consider the power series defined by:

$$P = P(\xi) = \sum_{n=0}^{\infty} p_n \xi^n,$$

where

$$p_0 = |a_0|$$

and

$$p_{n+1} = M p_n, \quad n = 0, 1, 2, \ldots$$

Hence, P is the majorant power series of (3.3), such as:

$$\begin{aligned} P &= p_0 + \sum_{n=0}^{\infty} p_{n+1} \xi^{n+1} \\ &= p_0 + M \sum_{n=0}^{\infty} p_n \xi^{n+1} \\ &= p_0 + MP\xi. \end{aligned}$$

Let G the functional equation in terms of ξ expressed as:

$$G(\xi, P) = P - p_0 - MP\xi.$$

Since G is analytic in the neighbourhood of $(0, p_0)$ in a way that:

$$G(0, p_0) = 0 \quad and \quad \frac{\partial}{\partial P} G(0, p_0) = 1 \neq 0,$$

Therefore, by the implicit function theorem [15], we establish the convergence. As a result, the power series solution takes the form:

$$\begin{aligned} u(t,x) =& a_0 e^{-F(x)} - \frac{a_0}{\Gamma(2-\alpha)} x e^{-F(x)} t^{-\alpha} \\ & - \sum_{n=1}^{\infty} \frac{\Gamma(2+n\alpha)\, a_n}{(n+1)\Gamma(2-\alpha+n\alpha)} x^{n+1} e^{-F(x)} t^{-\alpha(n+1)}, \end{aligned} \tag{3.5}$$

Reduction with $X_2 = \frac{\partial}{\partial x} - fu\frac{\partial}{\partial u}$.

The corresponding characteristic equation is given by:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{-fu}.$$

Integrating the above system leads to the invariants:

$$u = e^{-F(x)} \varphi(\xi) \text{ and } \xi = t,$$

where $F(x)$ is a primitive function of f.

Substituting the group invariant solution into Eq. (1.1), we get:

$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} = 0.$$

That yields:

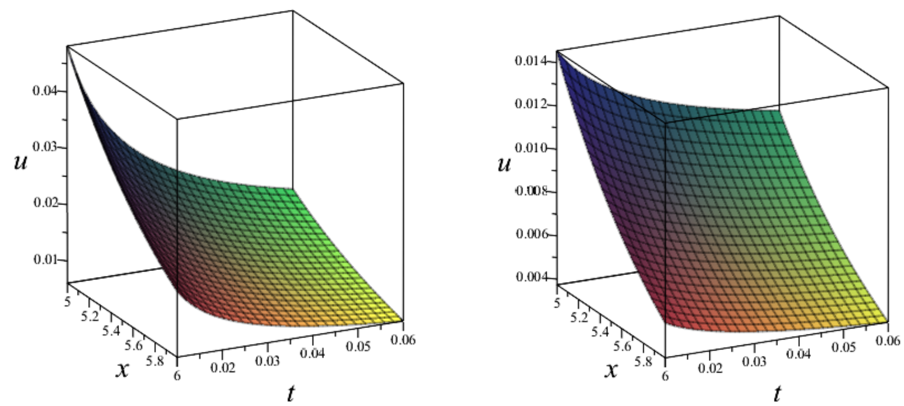
$$\varphi(t,x) = \frac{c}{\Gamma(\alpha)} t^{\alpha-1},$$

where c is an arbitrary constant. Therefore, we obtain:

$$u(t,x) = \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-F(x)}. \tag{3.6}$$

We now examine 3D plots of the above solutions for specific values of the parameters k, c and for two values $\alpha = 0.4$ and $\alpha = 0.8$ in three case: $f(x) = k$, $f(x) = kx$ and $f(x) = \frac{1}{x}$.

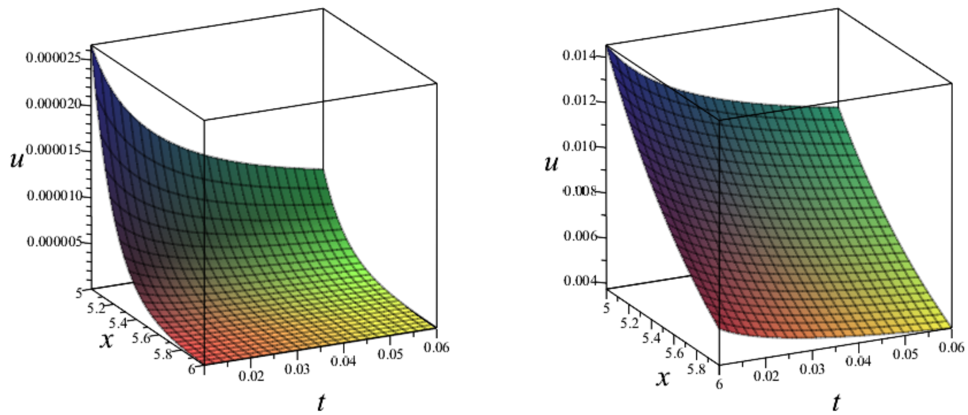
Case 1: $f(x) = k$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 1. Representation of the solutions (3.6) for $k = 1$, $c = 1$ and for two values $\alpha = 0.4$ and $\alpha = 0.8$.

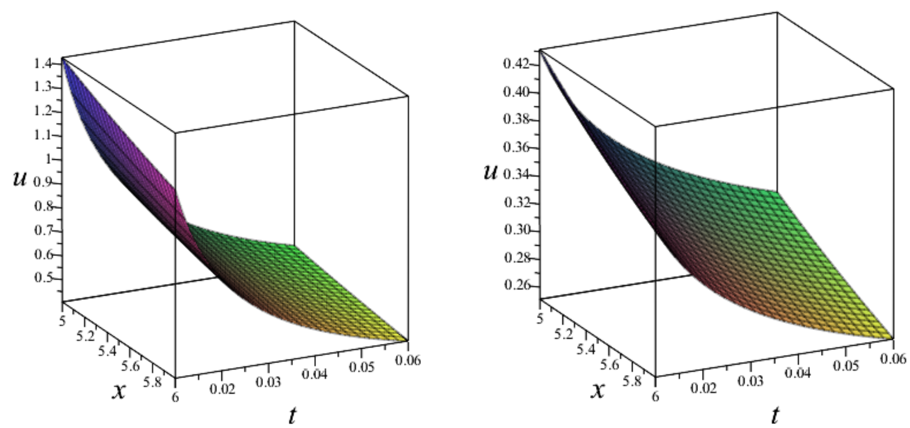
Case 2: $f(x) = kx$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 2. Representation of the solutions (3.6) for $k = 1$, $c = 1$ and for two values $\alpha = 0.4$ and $\alpha = 0.8$.

Case 3: $f(x) = \frac{1}{x}$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 3. Representation of the solutions (3.6) for $c = 1$ and for two values $\alpha = 0.4$ and $\alpha = 0.8$.

4. STABILITY OF THE SYMMETRY ALGEBRA AND NEW FAMILY OF SOLUTIONS

From the commutation table, we have: $[X_2, X_h] = X_{\psi_2}$ and $[X_1, X_h] = X_{\psi_1}$.
As the Lie algebra is stable under the Lie bracket, then if h is a solution of the studied equation, so:

$$\psi_1 = th_t + \alpha x h_x + \alpha x f h \quad and \quad \psi_2 = h_x + fh,$$

are also solutions.

Using the stability of the Lie algebra and taking ψ_2 to be the obtained solution (3.5) corresponding to the reduced fractional ordinary equation (3.2), we find a new exact solution:

$$\psi_2^1 = -\frac{a_0}{\Gamma(2-\alpha)} e^{-F(x)} t^{-\alpha} - \sum_{n=1}^{\infty} \frac{\Gamma(2+n\alpha)\, a_n}{\Gamma(2-\alpha+n\alpha)} x^n e^{-F(x)} t^{-\alpha(n+1)}.$$

Let's apply stability again to find another solution:

$$\psi_2^2 = \frac{\Gamma(2+\alpha)\, a_0}{\Gamma(2-\alpha)} e^{-F(x)} t^{-2\alpha} + \sum_{n=1}^{\infty} \frac{\Gamma(2+(n+1)\alpha)\, a_n}{\Gamma(2-\alpha+n\alpha)} x^n e^{-F(x)} t^{-\alpha(n+2)}.$$

In general, we have:

Theorem 4.1. *For $m \in \mathbb{N}^*$*

$$\begin{aligned}\psi_2^m =& \frac{(-1)^m \Gamma(2+(m-1)\alpha)\, a_0}{\Gamma(2-\alpha)} e^{-F(x)} t^{-m\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^m \Gamma(2+(n+m-1)\alpha)\, a_n}{\Gamma(2-\alpha+n\alpha)} x^n e^{-F(x)} t^{-\alpha(n+m)},\end{aligned}$$

are also exact solutions for the studied equation.

Proof. The proof is easily obtained using the principle of mathematical induction. □

In what follows, we give the solutions for some specific forms of mortality rate $f(x)$.

Constant mortality rate: In this case, we have $f(x) = k$. Then, for $m \in \mathbb{N}^*$, the solutions are given by:

$$\begin{aligned}\psi_2^m =& \frac{(-1)^m \Gamma(2+(m-1)\alpha)\, a_0}{\Gamma(2-\alpha)} e^{-kx} t^{-m\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^m \Gamma(2+(n+m-1)\alpha)\, a_n}{\Gamma(2-\alpha+n\alpha)} x^n e^{-kx} t^{-\alpha(n+m)}.\end{aligned} \tag{4.1}$$

Linear Mortality Rate: Here the expressions of the mortality rate take the form $f(x) = kx$. Then for $m \in \mathbb{N}^*$, the solutions are:

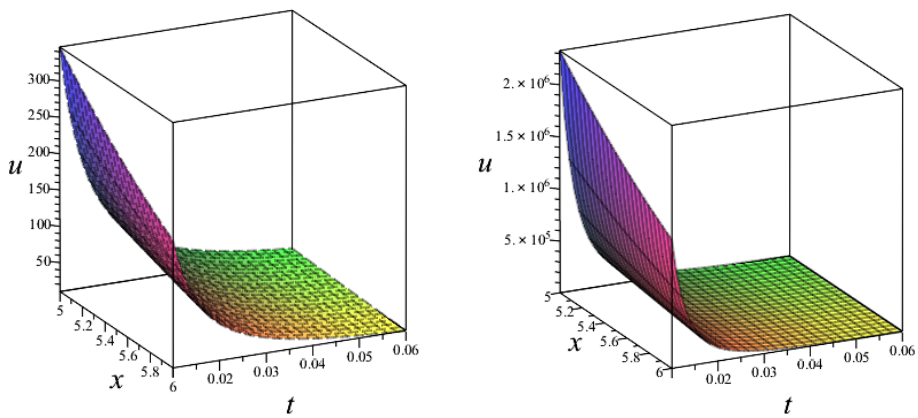
$$\begin{aligned}\psi_2^m =& \frac{(-1)^m \Gamma(2+(m-1)\alpha)\, a_0}{\Gamma(2-\alpha)} e^{-\frac{kx^2}{2}} t^{-m\alpha} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^m \Gamma(2+(n+m-1)\alpha)\, a_n}{\Gamma(2-\alpha+n\alpha)} x^n e^{-\frac{kx^2}{2}} t^{-\alpha(n+m)}.\end{aligned} \tag{4.2}$$

Hyperbolic Mortality Rate: The mortality function is written as $f(x) = \frac{1}{x}$. Hence, for $m \in \mathbb{N}^*$, we obtain the following solutions:

$$
\begin{aligned}
\psi_2^m = & \frac{(-1)^m \Gamma\left(2+(m-1)\alpha\right) a_0}{\Gamma\left(2-\alpha\right)} e^{-\frac{1}{x^2}} t^{-m\alpha} \\
& + \sum_{n=1}^{\infty} \frac{(-1)^m \Gamma\left(2+(n+m-1)\alpha\right) a_n}{\Gamma\left(2-\alpha+n\alpha\right)} x^n e^{-\frac{1}{x^2}} t^{-\alpha(n+m)}.
\end{aligned} \tag{4.3}
$$

Now, we look to 3D plots of the above solutions for some particular cases of the parameters m, k, n, a_0 and for two values $\alpha = 0.4$ and $\alpha = 0.8$.

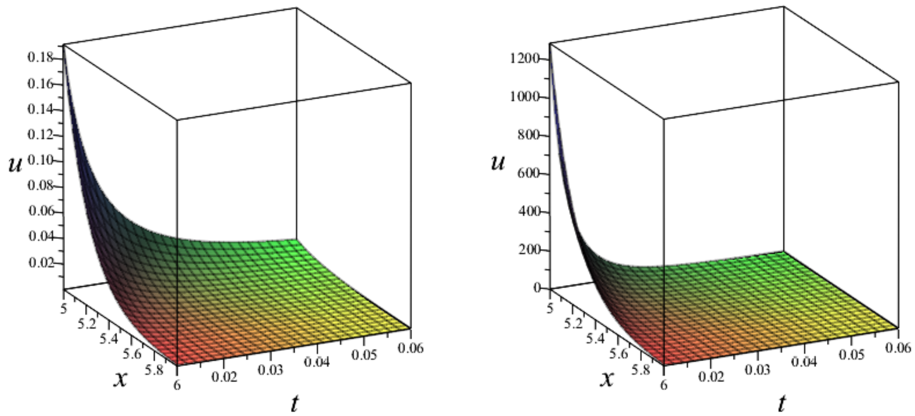
Case 1: $f(x) = k$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 4. Representation of the solutions (4.1) for $m = 2$, $k = 1$, $a_0 = 1$ and $n = 3$.

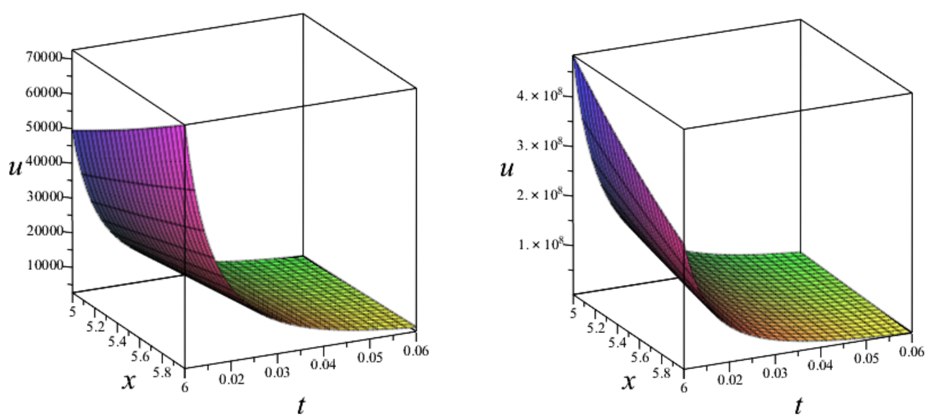
Case 2: $f(x) = kx$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 5. Representation of the solutions (4.2) for $m = 2$, $k = 1$, $a_0 = 1$ and $n = 3$.

Case 3: $f(x) = \frac{1}{x}$



(A) $\alpha = 0,4$ (B) $\alpha = 0,8$

FIGURE 6. Representation of the solutions (4.3) for $m = 2$, $k = 1$, $a_0 = 1$ and $n = 3$.

Remark. 1: Effect of the mortality rate
If $f(x)$ decreases, individuals live longer, leading to a slower decrease in $u(t, x)$ and potentially altering the age structure by allowing more individuals to remain in older age groups.

Remark. 2: Effect of order α
We observe a notable effect of varying the fractional order α on the population

density. Specifically, our results demonstrate that increasing α leads to an increase in population density. This finding suggests that the fractional calculus incorporated into the model significantly influences the dynamics of population evolution.

Other solutions can be obtained by applying the stability of the symmetry algebra for the solution $h(t,x) = \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-F(x)}$. Then ψ_1 is also an arbitrary solution for the studied equation.
We obtain a new exact solution as:

$$\psi_1^1 = (\alpha - 1) \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-F(x)}.$$

Theorem 4.2. *for $m \in \mathbb{N}^*$, we have:*

$$\psi_1^m(t,x) = (\alpha - 1)^m \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-F(x)},$$

are exact solutions of the studied equation (1.1).

Proof. The proof can be readily derived by applying the principle of mathematical induction. □

Below, we discuss some particular case of the mortality rate.

Constant Mortality Rate: Then for $m \in \mathbb{N}^*$, the solutions are given by:

$$\psi_1^m(t,x) = (\alpha - 1)^m \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-kx}.$$

Linear Mortality Rate: Hence, for $m \in \mathbb{N}^*$, the solutions take the form:

$$\psi_1^m(t,x) = (\alpha - 1)^m \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-\frac{kx^2}{2}}.$$

Hyperbolic Mortality Rate : The solutions are (for $m \in \mathbb{N}^*$):

$$\psi_1^m(t,x) = (\alpha - 1)^m \frac{c}{\Gamma(\alpha)} t^{\alpha-1} e^{-\frac{1}{x^2}}.$$

5. Constructions of conservation Laws

Here, since the equation does not derive from a Lagrangian, we will adopt the Ibragimov's method [9] to build the conservation laws.
A conserved vector $C = (C^t, C^x)$ associated with the time fractional Riemann-Liouville McKendrick equation fulfils the conservation equation:

$$D_t\left(C^t\right) + D_x\left(C^x\right)\big|_{(1.1)} = 0.$$

The formal Lagrangian corresponding to the time fractional Riemann-Liouville McKendrick equation is:

$$\mathcal{L} = v(t,x)\left(u_t^\alpha + u_x + f(x)u\right), \tag{5.1}$$

the dependent variable v is newly introduced.
Its adjoint equation is obtained from:

$$\frac{\delta \mathcal{L}}{\delta u} = 0. \tag{5.2}$$

Here, $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator

$$\begin{aligned}\frac{\delta}{\delta u} =& \frac{\partial}{\partial u} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial\left(D_t^\alpha u\right)} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} \\ &+ \sum_{k=3}^{\infty} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u_{t_1 \ldots i_k}},\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots$$

and $\left(D_t^\alpha\right)^*$ represents the adjoint operator of D_t^α, and D_x denotes the total derivative with respect to x.
We have:

$$\left(D_t^\alpha\right)^* = (-1)^n \mathcal{P}_T^{n-\alpha}\left(D_t^n\right),$$

where

$$\mathcal{P}_T^{n-\alpha} g(t, x) = \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{g(\tau, x)}{(\tau - t)^{1+\alpha-\pi}} d\tau, \quad n = [\alpha] + 1.$$

As

$$\frac{\partial \mathcal{L}}{\partial u} = f v, \quad \frac{\partial \mathcal{L}}{\partial u_x} = v, \quad \left(D_t^\alpha\right)^* \frac{\partial \mathcal{L}}{\partial\left(D_t^\alpha u\right)} = \left(D_t^\alpha\right)^* v.$$

From the equation (5.1) and equation (5.2), we get:

$$\left(D_t^\alpha\right)^*(v) + f v - v_x = 0.$$

Given that Ibragimov's approach for deriving conservation laws in fractional partial differential equations is analogous to the method for integer-order partial differential equations, the adjoint equation follows the same structure as in the integer-order case. Therefore, we have:

$$X^{(\alpha)}(\mathcal{L}) + D_t(\tau)\mathcal{L} + D_x(\xi)\mathcal{L} = W_i \frac{\delta \mathcal{L}}{\delta u} + D_t\left(C^t\right) + D_x\left(C^x\right),$$

where W_i denotes the characteristic functions, expressed as:

$$W_i = \eta_i - \tau_i u_t - \xi_i u_x.$$

The components C^t and C^x are defined by:

$$C^t = \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}\left(W_i\right) D_t^k \frac{\partial \mathcal{L}}{\partial\left(D_t^\alpha u\right)} - (-1)^n \mathcal{J}\left(W_i, D_t^n \frac{\partial \mathcal{L}}{\partial\left(D_t^\alpha u\right)}\right),$$

$$C^x = W_i \frac{\delta \mathcal{L}}{\delta u_x} + D_x\left(W_i\right) \frac{\delta \mathcal{L}}{\delta u_{xx}},$$

where $\mathcal{J}$ is given by:

$$\mathcal{J}(g_1, g_2) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{g_1(\tau, x) g_2(\theta, x)}{(\theta-\tau)^{1+\alpha-n}} \mathrm{d}\theta \mathrm{d}\tau.$$

Then:

$$C^t = D_t^{\alpha-1}(W_i) \frac{\partial \mathcal{L}}{\partial (D_t^\alpha V)} + \mathcal{J}\left(W_i, D_t\left(\frac{\partial \mathcal{L}}{\partial (D_t^\alpha V)}\right)\right)$$

and

$$C^x = W_i \frac{\partial \mathcal{L}}{\partial u_x}.$$

Case 1. In the case of the generator $X_1 = t\frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} - \alpha x f u \frac{\partial}{\partial u}$.
The characteristic function

$$\begin{aligned} W_1 &= \eta_1 - \tau_1 u_t - \xi_1 u_x \\ &= -\alpha x f u - t u_t - \alpha x u_x. \end{aligned}$$

Thus, the conserved quantities are given by:

$$C_1^t = v D_t^{\alpha-1}(-\alpha x f u - t u_t - \alpha x u_x) + \mathcal{J}(-\alpha x f u - t u_t - \alpha x u_x, v_t)$$

and

$$C_1^x = (-\alpha x f u - t u_t - \alpha x u_x) v.$$

Case 2. In the case of the generator $X_2 = \frac{\partial}{\partial x} - f u \frac{\partial}{\partial u}$.
This gives:

$$W_2 = -fu - u_x.$$

Consequently:

$$C_2^t = v D_t^{\alpha-1}(-fu - u_x) + \mathcal{J}(-fu - u_x, v_t)$$

and

$$C_2^x = (-fu - u_x) v.$$

Case 3. For the generator $X_3 = u\frac{\partial}{\partial u}$.
From this, we derive the corresponding Lie characteristic function:

$$W_3 = u.$$

Here, the conserved vectors are as follows:

$$C_3^t = v D_t^{\alpha-1}(u) + \mathcal{J}(u, v_t)$$

and

$$C_3^x = uv.$$

Case 4. For the generator $X_h = h\frac{\partial}{\partial u}$.
The corresponding Lie characteristic function is:

$$W_h = h.$$

Therefore, the following are the conserved vectors in this case:

$$C_h^t = v D_t^{\alpha-1}(h) + \mathcal{J}(h, v_t)$$

and

$$C_h^x = hv.$$

6. Conclusion

In this work, we gave particular attention to the Lie group analysis technique to solve the time fractional Riemann-Liouville McKendrick equation. Using the reduction by the first generator and applying the Erdélyi-Kober fractional operator and power series approach, we construct some exact solutions. By reducing the second generator and applying the stability of the Lie bracket, we obtain a family of other solutions. For different parameter values, we plotted the 3D graph and we have observed the effects of the mortality rate and the α-parameters on the solutions. Moreover, based on the fractional version of Ibragimov's nonlocal conservation theorem conservatives rules are built. In conclusion, we trust that the ideas and methods developed in this work will be proven to be valuable to researchers working on the theory of the Lie symmetries of fractional partial differential equations.

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